COURSE ANNOUNCEMENT
MATH 286A 1988
TOPICS IN DIFFERENTIAL GEOMETRY
Prof. R. Schoen

We will discuss analytic constructions of Riemannian metrics, and the nonlinear PDE's which arise in such constructions. Some emphasis will be given to conformal deformation of metrics to constant scalar curvature and the related problems arising in conformed geometry. We will also discuss the heat equation method developed by R. Hamilton, and the (Riemannian) Einstein equations. We will place an emphasis on variational method throughout much of the course, but will occasionally use other methods from nonlinear functional analysis. Some background in Differential Geometry such as 217 would be very helpful for students as would a basic knowledge of second order linear elliptic and parabolic PDE including such topics as the maximum principle, Harmach inequality, elliptic regularity theory, Schauder estimates, basic facts about eignvalues. Some familiarity with finite dimensional Morse theory would be helpful but not essential. Level of prerequisite material assumed will depend to some extent on the audience.

First meeting Sept 29th, Thursday, 2:15-3:30 380-383P.

Autumn 1988
Lecture Notes by
Dan Pollack

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Topics in Differential Geometry — R. Schoen — 9/29/88

2 dim'l case: conformal deformation of metrics
— the P.D.E approach

(M^2, g_0) 2 dim'l compact, closed smooth manifold with a given Riemannian metric.

K_0 - Gauss curvature of g_0

Assume the Gauss-Bonnet Theorem $\int_M K_0 \, d\omega_{g_0} = 2\pi \chi(M)$

To deform g_0 within its conformal class, i.e. look for a new metric $g = e^{2u} g_0$, $u \in C^\infty(M)$

Exercise: $K(g) = e^{-2u}(K_0 - \Delta_{g_0} u)$

¿ Given a function $R(x)$ on M find u so that $K(g) = R(x)$?

i.e. to solve:

$$* \quad \Delta_{g_0} u = K_0 - R(x) e^{2u}$$

Main Ref: Kazdan-Warner Ann. of Math (1974) 99, 14-47

The Character of the problem depends highly on the sign of $\chi(M)$.

Case 1 $\chi(M) < 0$: (analytically, this is the easiest case.)

Use the Method of of Upper and Lower Solutions.

M^n, g_0 compact, closed

Look at equations of the form $\Delta u = f(x, u)$

Theorem: If we can find functions $u_\pm$ satisfying

$\Delta u_- \geq f(x, u_-)$ (u_- is a subsolution)

$\Delta u_+ \leq f(x, u_+)$ (u_+ is a supersolution)

and $u_- \leq u_+$ on M, then $\exists$ a solution u, $\Delta u = f(x, u)$

with $u_- \le u \le u_+$ on M. ∎ (Proof later)

How does this apply to our problem?

$\underline{\chi(M) < 0}$

① May choose $K < 0$ by first solving a linear equation. let v be a solution of

$$\Delta_{g_0} v = K_0 - \frac{2\pi\, \chi(M)}{\mathrm{Vol}(g_0)} = K_0 + c$$

(c a positive constant)

We can solve this sinze Laplacian is nagative definate and the r.h.s. has zero average value. and this is orthog. to the constants. so $\int_M v \, d\omega_g = 0$

Now replace g_0 by $e^{2v} g_0$

$$K(e^{2v} g_0) = e^{-2v}(K_0 - \Delta_{g_0} v) = -c e^{-2v} < 0 \text{ on } M$$

② If $k(x)$ is any negative function on M, then Upper and Lower solution exist (Poincaré Thm)

$$f(x,u) = K_0(x) - k(x) e^{2u}$$

Then let u_- be ~~any~~ large negative constant

$$\Delta u_- = 0 \ge f(x, u_-)$$

let u_+ be a large positive constant

$$\Delta u_+ = 0 \le f(x, u_+)$$

So that the problem is solved modulo the proof of the method.

<u>Proof</u>: Solve Sequence of linear solutions.

$$Lu = g(x,u)$$

$$Lu = \Delta u - cu \qquad c \gg 0 \qquad \text{for } c \text{ large}$$

$\frac{\partial g}{\partial u} < 0$ on $M \times [\text{Min of } u_-, \text{Max of } u_+]$

Let u_1 be the solution of $Lu_1 = g(x, u_-)$

L^{-1} exists (Δ neg. def. and we've subtracted a negative const.)

$Lu_1 = g(x, u_-) \leq Lu_- \Rightarrow u_1 \geq u_-$ by Max. princ.

$Lu_1 = g(x, u_-) \geq g(x, u_+) \geq Lu_+$

$\Rightarrow u_1 \leq u_+$ by Max. princ.

Define a sequence of functions sucessively:

Let u_{i+1} be the solution of $Lu_{i+1} = g(x, u_i)$ $i = 1, 2, 3 \ldots$

Can show: $u_- \leq u_1 \leq u_2 \leq \ldots \leq u_K \leq \ldots \leq u_+$

Monotone bounded sequence $\Rightarrow$ converges

$u = \lim u_i$ $\qquad Lu = g(x, u)$

<u>Ex</u>: To derive the Equation $K(g) = e^{-2u}(K_0 - \Delta_{g_0} u)$

let $\{\omega_1, \omega_2\}$ be a local oriented orthonormal co-frame field on M for the metric g_0, set $\tilde{\omega}_i = e^u \omega_i$ then $\{\tilde{\omega}_1, \tilde{\omega}_2$ is a local oriented O.N. co-frame field for $g = e^{2u} g_0$.

Let φ_{12} be the Riemannian connection one-forms (φ_{12} is uniquely determined by $\varphi_{12} + \varphi_{21} = 0$ and $d\omega_i = -\varphi_{ij} \wedge \omega_j$), then the Gaussian curvature K_0 of the metric g_0 is determined by

$$K_0 \, dA = K_0 \, \omega_1 \wedge \omega_2 = d\varphi_{12}$$

Compute $\tilde{\varphi}_{12}$; Let $du = u_1\omega_1 + u_2\omega_2$, $*du = u_1\omega_2 - u_2\omega_1$

$$d\tilde{\omega}_1 = e^u (du \wedge \omega_1 - \varphi_{12} \wedge \omega_2)$$

$$= e^u ((u_1\omega_1 + u_2\omega_2) \wedge \omega_1 - \varphi_{12} \wedge \omega_2)$$

$$= e^u (u_1 \underbrace{\omega_1 \wedge \omega_1}_{0} + u_2 \omega_1 \wedge \omega_2 - \varphi_{12} \wedge \omega_2)$$

$$= e^u (u_1 \omega_2 \wedge \omega_2 - u_2 \omega_1 \wedge \omega_2 - \varphi_{12} \wedge \omega_2)$$

$$d\tilde{\omega}_1 = (u_1\omega_2 - u_2\omega_1 - \varphi_{12}) \wedge e^u\omega_2$$
$$d\tilde{\omega}_1 = (u_1\omega_2 - u_2\omega_1 - \varphi_{12}) \wedge \tilde{\omega}_2$$

so $\tilde{\varphi}_{12} = u_2\omega_1 - u_1\omega_2 + \varphi_{12}$
$= \varphi_{12} - *du$

$$d\tilde{\varphi}_{12} = d\varphi_{12} - d*du$$
$$\Rightarrow K(g)\,\tilde{\omega}_1\wedge\tilde{\omega}_2 = K_0\,\omega_1\wedge\omega_2 - \Delta u\,\omega_1\wedge\omega_2$$
$$K(g)e^{2u}\,\omega_1\wedge\omega_2 = K_0\,\omega_1\wedge\omega_2 - \Delta u\,\omega_1\wedge\omega_2$$
$$\therefore\ K(g)e^{2u} = K_0 - \Delta u \qquad \boxtimes$$

10/4/88

(M^2, g_0) $\quad g = e^{2u}g_0$, $K(g) = k(x)$ given $\quad (*)\ \Delta_{g_0}u = K(g_0) - k(x)e^{2u}$

Case 2: $\chi(M) = 0$ $\quad \int_M K(g_0)\,d\omega_{g_0} = 0$

so $\exists\, v$ with $\Delta v = K(g_0)$, so $K(e^{2v}g_0) \equiv 0$

$\therefore$ Assume (M^2, g_0) has $K(g_0) \equiv 0$

Theorem: $(*)$ is solvable for k $(\not\equiv 0)$ if and only if $\quad (*)\ \Delta_{g_0}u = -k(x)e^{2u}$

(i) k changes sign

(ii) $\int_M k\,d\omega_{g_0} < 0$

Proof:

"$\Rightarrow$" (i) ✓ Gauss-Bonnet

(ii) $e^{-2u}\Delta u = -k(x)$

$$0 < 2\int_M e^{-2u}|Du|^2 = \int_M e^{-2u}\Delta u = -\int_M k(x) \Rightarrow \int_M k(x) < 0 \quad ✓$$

"$\Leftarrow$" Variational Approach:

k given satisfying (i) and (ii),

Consider $\mathcal{Z} = \{u \in W^{1,2}(M) : \int_M u = 0,\ \int_M k(x)e^{2u}d\omega_g = 0\}$ $\quad$ see Kazdan-Warner p. 22

(Borderline Sobolev inequality $\Rightarrow \int_M e^{u^2} \le c\int_M |\nabla u|^2$ if $\int_M u = 0$

So the Integral $\int k(x)e^{2u}d\omega_g$ makes sense for $u \in W^{1,2}(M)$)

The fact that k changes sign guarantees that $\mathcal{X}$ is non-empty, ((i) $\Rightarrow$ $\mathcal{X} \neq \{\emptyset\}$) Since we can weight u so that $\int_M k(x) e^{2u} d\omega_g = 0$ and then subtract off $\bar{u}$, preserving the equality.

We use the direct method to minimize the Dirichlet Integral: i.e. We minimize $\int_M |\nabla u|^2 d\omega_{g_0}$ for $u \in \mathcal{X}$ by taking a minimizing sequence.

Let $I = \inf \{ \int_M |\nabla u|^2 d\omega_{g_0}, u \in \mathcal{X} \}$, $I \geq 0$ hence bounded from below

Say $\{u_i\} \in \mathcal{X}$ is a minimizing sequence, $\int_M |\nabla u_i|^2 d\omega_{g_0} \searrow I$

$\mathcal{X}$ nonempty, $\exists$ $u_0 \in \mathcal{X}$, let $b = \int_M |\nabla u_0|^2 d\omega_{g_0}$, Then we can assume $\int_M |\nabla u_i|^2 d\omega_{g_0} \leq b$ $\forall i$

Now the Poincaré inequality states that there is a constant c_3 such that if $u \in C^\infty(M)$ with $\bar{u} = 0$ then $\|u\|_2 \leq c_3 \|\nabla u\|_2$ ~~which~~ This implies

$$\|u\|_{1,2} \leq c_2 \|\nabla u\|_2 \qquad (1)$$

So that $\|u_i\|^2_{1,2} \leq c \int_M |\nabla u_i|^2 d\omega_{g_0} \leq cb$ $\forall i$.

Thus $\{u_i\}$ is a bounded sequence in $W^{1,2}$, so that since the unit ball in any Hilbert space is weakly compact, $\exists$ a weak limit $u \in W^{1,2}$, i.e. $\exists$ a subsequence, call it $\{u_i\}$ s.t. $u_i \rightharpoonup u$. $\Rightarrow \int u = 0$ $(\langle u_i, 1 \rangle_{W^{1,2}} = \int u_i = 0$ $\forall i$, $\langle u_i, 1 \rangle_{W^{1,2}} \to \langle u, 1 \rangle_{W^{1,2}}$ $1 \in W^{1,2}(M)$ M cpct

Now the functional $\int_M |\nabla u|^2 d\omega_{g_0}$ is not continuous under weak convergence but it is lower semicontinuous, i.e. $\|u\|^2 = \lim \langle u, u_i \rangle \leq \liminf \|u\| \|u_i\|$

So $\|u\| \leq \liminf \|u_i\|$.

Also need to show that u satisfies the criterion $\int_M k(x) e^{2u} d\omega_g = 0$, so that then we'd have $u \in \mathcal{X}$

Prop: Assume $\dim M = 2$. If $u_j \in W^{1,2}(M)$ and $u_j \to u$ weakly in $W^{1,2}(M)$, then $e^{u_j} \to e^u$ strongly in $L^2(M)$.

Proof: 1st Note $|e^t - 1| \leq |t| e^{|t|}$ (use Power series).

2nd: The Rellich-Kondrashov compactness theorem asserts that if $u_j \to u$ weakly in $W^{1,2}(M)$ the $u_j \to u$ strongly in $L^4(M)$ (see Adams P. 144 (4)) Then

$$\int_M |e^{u_j} - e^u|^2 \, d\omega_{g_0} = \int_M e^{2u} |e^{u_j - u} - 1|^2 d\omega_{g_0} \leq \int_M e^{2u} e^{2|u_j - u|} |u_j - u|^2 d\omega_{g_0}$$

$$\leq \left(\int_M e^{8u} d\omega_{g_0}\right)^{1/4} \left(\int_M e^{8|u_j - u|} d\omega_{g_0}\right)^{1/4} \|u_j - u\|_4^2$$

by Cauchy-Schwarz (twice)

The first two integrals are bounded by Trudinger inequality on the exceptional case of the sobolev inequalities (see Kaz-Warn. 3-5)

$\therefore$ since $u_j \to u$ strongly in L^4 we have $e^{u_j} \to e^u$ strongly in $L^2(M)$

$\therefore$ $u \in \mathcal{Z}$

Inequality (1) implies that $\left(\int_M |\nabla u|^2 d\omega_{g_0}\right)^{1/2}$ is a norm equivalent $\| \ \|_{1,2}$ on the subspace $\mathcal{Z}$ of $W^{1,2}(M)$; therefore

$$\int_M |\nabla u|^2 d\omega_g \leq \int_M |\nabla u_i|^2 d\omega_g \quad \forall i.$$

Therefore u minimizes the Dirichlet integral in $\mathcal{Z}$.

So u satisfies the Euler-Lagrange equations for this functional with constraints to work in $\mathcal{Z}$, i.e. using Lagrange multiplier theory we find that there are constants λ_1, λ_2 such that u is a critical point for

$$\int_M (|\nabla u|^2 + \lambda_1 K(x) e^{2u} + \lambda_2 u) \, d\omega_g \quad \text{on } W^{1,2}(M).$$

From this we get the weak form of the Euler-Lagrange equations, i.e. $\Delta u - 2(\lambda_1 K e^{2u} + \lambda_2) = 0$ is weakly satisfied by $u \in W^{1,2}(M)$. Using the Trudinger result we can

show that $2\lambda_1 k e^{2u} + \lambda_2$ belongs to $L^p(M)$ for some $p > 2$ fixed.
L^p regularity $\Rightarrow$ $u \in W^{2,p}(M)$ and hence $u \in C^1(M)$ (Sobolev inequality: $p > \dim M \Rightarrow \exists\, c$ s.t. $\forall u \in W^{1,p}(M)$, $\|u\|_\infty \le c\,\|u\|_{1,p}$ see K-W 3.8)
Then $2\lambda_1 k e^{2u} + \lambda_2 \in C^1(M)$, so by the Schauder estimates $u \in C^2$. Continuing inductively (turn the crank) $\Rightarrow u \in C^\infty(M)$.

Integrate the equation $\Rightarrow$ $\lambda_2 = 0$ since $u \in \mathcal{L}$

$$\Delta u = 2\lambda_1 k(x)\, e^{2u}$$

$$0 < \int_M e^{-2u} \Delta u \, d\omega_{g_0} = 2\lambda_1 \int_M k(x)\, d\omega_{g_0}\ , \quad (ii) \Rightarrow \lambda_1 < 0$$

hen $u + c$ (where $e^{2c} = -2\lambda_1$) is a solution of $(*)$
i.e. $\Delta(u+c) = -k\, e^{2(u+c)}$ ∎

$\underline{\chi(M) > 0}$: From the P.D.E. point of view this case is alot harder.

$$(*) \qquad \Delta u = K(g_0) - k(x)\, e^{2u}$$

As before we can assume $k(x)$ is positive.

ake a Standard S^2  , g_0 - standard metric

$$\Delta u = 1 - k(x) e^{2u}$$
$$= 1 - e^{2u}$$

ot's of Solutions exist (in some sense too many solutions),
Je can use the Conformal group of S^2 (non-isometric transformations!
$F : S^2 \to (S^2, g_0)$ $\quad F^*(g_0) = e^{2u_F} g_0 \Rightarrow u_F$ is a solution!
Since the space of solution is non-compact we don't get any good estimates on solutions.

Method for when $R(x) \equiv 1$ and $\int_M K(g_0) = 4\pi$ (i.e. $M \simeq S^2$)

Construct a conformal diffeomorphism from (M, g_0) i.e.

$\Phi : (M, g_0) \longrightarrow (S^2, g_1)$ g_1 = standard round metric.

$\Phi^*(g_1) = e^{2u} g_0$ for some u which is then a solution.

and $K(\Phi^*(g_1)) \equiv 1$

Look at the linear equation $\Delta u = K(g_0)$ (by integrating we see that this cannot have solutions.),

but we can solve $\Delta u = K(g_0) - 4\pi \delta_{p_0}$ where δ_{p_0} is a delta function at $p_0 \in M$.

Namely: $\exists$ solution of $\Delta u = K(g_0) - 4\pi \delta_{p_0}$ where δ_{p_0} is a delta function at $p_0 \in M$, $u \in C^\infty(M \setminus \{p_0\})$ and $K(e^{2u} g) \equiv 0$ on $M \setminus \{p_0\}$. The singularity of u at p_0 is comparable to the singularity of the fundamental solution of Δu, i.e.

$u(x) = c \log|x|$ + l.o. terms, where x are coordinates about p_0, taking $c = -2$ (comparing to the fund. sol'n).

We consider $(M \setminus \{p_0\}, e^{2u} g_0)$, the metric looks like $|x|^{-4} g_0$ near p_0, so this is a complete, noncompact flat manifold with 1-end (ie not a cylinder); therefore it is isometric to $\mathbb{R}^2$.

Choose $q \in M \setminus \{p_0\}$, then the isometry is given by

flat $\mathbb{R}^2 \overset{\text{Isom}}{\cong} T_q(M \setminus \{p_0\}) \xrightarrow{\exp_q} M \setminus \{p_0\}$.

To get the required conformal diffeomorphism, Φ we take the composition

$$\underbrace{M \setminus \{p_0\} \xrightarrow{\exp_q^{-1}} \mathbb{R}^2 \xrightarrow[\text{Proj.}]{\text{Stereo.}} S^2}_{\Phi} \longrightarrow \{P_0\}$$

Φ is a continuous conformal transformation, show Φ extends smoothly across p_0

he Prescribed curvature on (S^2, g_1) problem is still open.
$\Delta u = 1 - k(x)e^{2u}$, $k(x)$ positive is not enough, in fact even if $k(x)$ is very close to 1, the equation need not have solutions. There is lots of literature on this problem. The $\mathbb{RP}^2$ case was resolved by J.Moser ($k(x)$ positive somewhere is necessary and sufficient.)

Higher Dimensions (M^n, g), $n \geq 3$ Backround

Riemann Curvature Tensor R_{ijkl}

Symmetries: ① (Anti-symmetry) $R_{ijkl} = -R_{jikl} = -R_{ijlk} = 0$

② (1^{ST} Bianchi Identity) $R_{ijkl} + R_{iklj} + R_{iljk} = 0$

③ (①,② ⇒ ③) $R_{ijkl} = R_{klij}$

① and ③ are encoded in the curvature operator;

$$\mathcal{R} : \Lambda^2(T_pM) \longrightarrow \Lambda^2(T_pM)$$

① + ③ ⟺ $\mathcal{R}$ is a well defined self-adjoint linear transformation

If $e_1, \ldots, e_n$ is an orthonormal basis for T_pM then $\{e_i \wedge e_j : i<j\}$ is a O.N. basis for $\Lambda^2(T_pM)$ then:

$$\langle \mathcal{R}(e_i \wedge e_j), e_k \wedge e_l \rangle = R_{ijkl}$$

so ① ⇒ $\mathcal{R}$ is well defined and ③ ⇒ Matrix is symmetric.

Let's take some traces:

$$R_{ik} = \sum_{j,l} g^{jl} R_{ijkl} \equiv \text{Ricci tensor, a symmetric two tensor}$$

$$R = \sum_{i,k} g^{ik} R_{ik} \equiv \text{Scaler curvature function}$$

$$R_{ij} = T_{ij} + \frac{1}{n} R g_{ij}$$

T_{ij} = trace free Ricci tensor : $tr_g T_{ij} = 0$

(M is Einstein $\Leftrightarrow T_{ij} \equiv 0$)

Suppose we're given H, a symmetric trace-free 2-tensor
Want to construct $\mathcal{H}: \Lambda^2 \to \Lambda^2$, satisfying ①,②,③ whose first trace is H.

$$\mathcal{H}_{ijkl} = \frac{1}{n-2}\left(H_{ik}g_{jl} - H_{il}g_{jk} + H_{jl}g_{ik} - H_{jk}g_{il}\right)$$

This tensor satisfying ①, ②, ③ and its trace is H, such four-tensors are called trace-tensors, since they're determined by their trace.

Similarly, given a function $f(x)$, we construct $\mathcal{Z}: \Lambda^2 \to \Lambda^2$

$$\mathcal{Z}_{ijkl} = \frac{1}{n(n-1)} f(x)\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)$$

Take the Riemann Curvature tensor and subtract off it traces.
let: $\mathcal{T}$ = trace tensor formed from T_{ij}, the trace-free Ricci tens
$\mathcal{S}$ = trace tensor formed from R, the scaler curvature.

$$\mathcal{R} - \mathcal{T} - \mathcal{S} = \mathcal{W} \equiv \text{Weyl tensor}$$

All traces of $\mathcal{W}$ are zero.

~~Exam~~

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}\left(T_{ik}g_{jl} - T_{il}g_{jk} + T_{jl}g_{ik} - T_{jk}g_{il}\right) - \frac{1}{n(n-1)} R\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)$$

$$\mathcal{R} = W + \mathcal{T} + \mathcal{S}$$

scalar curv =

Note: $n=2$: $\mathcal{W} \equiv \mathcal{T} \equiv 0$, $R_{ijkl} = K(x)\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)$
$n=3$: $\mathcal{W} \equiv 0$

Pf: 2 case ① 2 distinct indices
② 3 distinct indices

② 3 distinct $j=k$ W_{ijjl} i, j, l distinct

$\sum_{k=1}^{3} W_{ikkl} = W_{ijjl}$ by working in normal coordinates at the point $(\delta_i^j = g_{ij})$, where i, j and l are distinct.

$W_{ijjl} = R_{ijjl} + T_{il} = -R_{ijlj} + R_{il} - \frac{1}{3}\cancel{R g_{il}}^{0}$

$= -R_{il} + R_{il} = 0$

2 distinct: ⓐ $W_{1212} = -W_{1313} = -W_{3232}$ (use Def'n of W_{ijkl}, R_{ij}, R

3 distinct: ⓑ $W_{1212}, W_{1313}, W_{2323}$, ⓐ + ⓑ $\Rightarrow$ All are $\equiv 0$.

$\therefore$ $n = 3 \Rightarrow W \equiv 0$

<u>Def'n</u> g is <u>Einstein</u> if $T(g) = 0$

i.e. $Ric(g) = \frac{1}{n} R g$

<u>$n=2$</u> All metrics are Einstein

<u>$n=3$</u> g is Einstein $\Leftrightarrow R_{ijkl} = \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk})$

i.e. g is a metric of constant curvature.

<u>Th Symmetry</u>: <u>2^{nd} (differential) Bianchi Identity</u>

④ $R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0$

Take a trace in i and k with respect to an orthonormal basis.

$R_{il;m} + \sum_i R_{ijlm;i} - R_{jm;l} = 0$

Take a trace in j and l.

$R_{;m} - 2\sum_i R_{im,i} = 0$

i.e. $\sum_j (R_{ij} - \frac{1}{2} R g_{ij})_{;j} = 0$ $\quad$ ($g_{ij,j} = 0 \Leftrightarrow$ connection is torsion free)

If g is Einstein $\Rightarrow R_{ij} = \frac{1}{n} R g_{ij}$

so we get $\left(\frac{1}{n} - \frac{1}{2}\right) \sum_j (R g_{ij})_{;j} = 0$

$\Leftrightarrow$ $n \geq 3$ $\nabla R = 0$

$\Rightarrow R$ is constant.

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<u>A</u> more concrete construction of the Weyl tensor

5T) Multi-Linear Algebra.

V^n - real n-dim vector space , g an inner product.

$\mathcal{V}$ - space of 4-tensors satisfying the symmetries ① and ②

$= \Lambda^4 V / I$, I is the ideal generated by ① and ②

$\mathcal{V} \simeq \mathrm{Hom}_{(Sym)}(\Lambda^2 V, \Lambda^2 V) \Rightarrow \dim \dfrac{\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2}+1\right)}{2}$

<u>Trace maps</u>:

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & \\ S^2V & & \mathbb{R} \end{tikzcd}$$

(Symmetric 2-tensors)

$\tau_1(R_{ijk\ell}) = \sum_{j,\ell} g^{j\ell} R_{ijk\ell} \in S^2(V)$

$\tau_2(R_{ijk\ell}) = \sum_{i,j,k,\ell} g^{ik} g^{j\ell} R_{ijk\ell} \in \mathbb{R}$

$\hat{\mathcal{V}} =$ ~~xxxxx~~ Kernel τ_2 , $\mathcal{V} = \hat{\mathcal{V}} \oplus \mathcal{V}_2$

$\mathcal{V}_2 = 1$ dim'l subspace orthogonal to $\hat{\mathcal{V}}$

Je construct the inverse of τ_2

$$\tau_2^{-1} : \mathbb{R} \to \mathcal{V}_2 \quad , \quad \tau_2^{-1}(c) = \frac{c}{n(n-1)}(g_{ik}g_{j\ell} - g_{i\ell}g_{jk})$$

$\hat{\mathcal{V}} = \mathcal{V}_0 \oplus \mathcal{V}_1$, where $\mathcal{V}_0 = (\mathrm{Ker}\,\tau_1) \cap \hat{\mathcal{V}}$, $\mathcal{V}_1$ = ortog. compliment

$$\begin{tikzcd} \hat{\mathcal{V}} \arrow[d] \\ (S^2V)_{\text{TraceFree}} \arrow[u, bend right, dashed, "\tau_1^{-1}"'] \end{tikzcd}$$

So for h_{ij}, a symmetric trace-free 2-tensor i.e. $h_{ij} \in \mathcal{V}_1$

$$\tau_1^{-1}(h_{ij}) = \frac{1}{n-2}(h_{ik}g_{j\ell} - h_{i\ell}g_{jk} + h_{j\ell}g_{ik} - h_{ji}g_{k\ell})$$

So $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2$

$\dim \mathcal{V}_1 = \frac{n(n+1)}{2} - 1$, $\dim \mathcal{V}_2 = 1$.

<u>onformal</u> <u>deformations</u> : $\hat{g} = \Omega^2 g$, $\Omega > 0$

$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z$

$E_i = \frac{\partial}{\partial x^i}$, $\nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k E_k$

$$\Gamma_{ij}^k = \frac{1}{2} \sum_\ell g^{k\ell}(g_{ik,j} + g_{j\ell,i} - g_{ij,\ell})$$

$$R(E_k, E_l) E_j = \sum_i R^i_{jkl} E_i$$

$$R^i_{jkl} = \underbrace{\Gamma^i_{jl,k} - \Gamma^i_{jk,l}}_{2^{nd}\text{ order in } g} + \underbrace{\sum_m \left(\Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}\right)}_{\text{quadratic nonlinearity in } g.}$$

$$\hat{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \Omega^{-1}\Omega_j + \delta^k_j \Omega^{-1}\Omega_i - \Omega^{-1}\Omega_j^{\ k} g_{ij}$$

where $\Omega_j^{\ k} = \sum_l g^{kl}\Omega_{jl}$

$$\hat{R}_{ijkl} = \Omega^2 R_{ijkl} \mp (h_{ik}g_{jl} - h_{il}g_{jk} + h_{jl}g_{ik} - h_{jk}g_{il})$$

$$h_{ik} = \Omega^3 (\Omega^{-1})_{;ik} - \tfrac{1}{2}\Omega^4 |\nabla\Omega^{-1}|^2 g_{ik}$$

rst observations:

1) $\hat{W}_{ijkl} = \Omega^2 W_{ijkl}$ so $\hat{W}^i_{jkl} = W^i_{jkl}$

This is the conformal invariance of Weyl tensor.

2) $\hat{R}_{ik} = R_{ik} + (n-2)\Omega(\Omega^{-1})_{;ik} - (n-2)^{-1}\Omega^{2-n}\Delta(\Omega^{n-2})g_{ik}$

$\hat{T}_{ik} = T_{ik} + (n-2)\Omega\left[(\Omega^{-1})_{;ik} - \frac{1}{n}\Delta(\Omega^{-1})g_{ik}\right]$

3) $\hat{R} = \Omega^{-n}\left[R - \frac{4(n-1)}{n-2}\Omega^{-3/n-2}\Delta(\Omega^{2/n-2})\right]$

let $u = \Omega^{2/n-2}$ $\quad \hat{g} = u^{4/n-2} g$

$\hat{R} = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} Lu$, where $Lu = \Delta u - \frac{n-2}{4(n-1)} R(g) u$

his is The Conformal Laplacian, it is a conformally invariant perator in the sense that if $\hat{g} = u^{4/n-2} g$, then we have for any function φ: $L(u\varphi) = u^{\frac{n+2}{n-2}} \hat{L}(\varphi)$.

<u>riational</u> <u>Principle</u>: $\mathcal{R}(g)$ (not the curvature operator)

M^n compact, closed

$$\mathcal{R}(g) = \int_M R(g)\, d\omega_g, \quad \mathrm{Vol}(g) = 1$$

is behaves like an invariant Dirichlet integral for the metric. recalling $\int_M |\nabla u|^2 = -\int_M u\Delta u$ has Euler-Lagrange equation $\Delta u = 0$ though the r.h.s. has an integrand also involveing 2^{nd} order derivatives of u. The total scaler curvature function is simi

In that the integrand, $R(g)$ is 2^{nd} order in g and the Euler-agrange equations (the Einstein equations : $Ric(g) = \frac{1}{n} R(g)\, g$) also are second order in g.

Let $\mathcal{M}_1 = \{g :$ Riemannian metrics on M; $Vol(g) = 1\}$

<u>First Variation formula:</u> (Euler-Lagrange Equations)

g given Riemannian metric $\in \mathcal{M}_1$, h - symmetric $(0,2)$ tensor

(The tangent space to $\mathcal{M}_1$ at g is $S^2(T_pM)$)

equire an additional condition on h, that $\frac{\partial}{\partial t} Vol(g+th)\Big|_{t=0} = 0 = \int_M d\dot{\omega}_g$

$d\omega_g = \sqrt{\det g}\, dx$

$d\dot{\omega}_g = \frac{1}{2}\, trace_g(h) \sqrt{\det g}\, dx = \frac{1}{2} Tr_g(h)\, d\omega_g$

Using $(\log \dot{\det} g) = Tr_g(h)$, since $\frac{\partial \log(\det g)}{\partial g_{ij}} = g^{ij}$ which can be derive from the cofactor expansion for $\det g$)

So $\int_M d\dot{\omega}_g = 0 \iff \int_M Tr_g(h)\, d\omega_g = 0$ $\qquad g(t) = g + th$

<u>Observation</u>: $R = \sum_{i,j} g^{ij} R_{ij}$, $\quad \dot{R} = -\sum h^{ij} R_{ij} + \sum g^{ij} \dot{R}_{ij}$

Now $R_{ij} = \Gamma^k_{ki;j} - \Gamma^k_{ij;k} + \{$Quadratic term in $\Gamma\}$,

We can work in normal coordinates where $\Gamma \equiv 0$ at a point, and can therefore ingnore the Quadratic terms when computing $\dot{R}_{ij}$ (we'll always have a Γ as a factor.)

ow $\dot{\Gamma}^k_{ij}$ is a tensor, since in the difference quotient it is the difference of ~~of~~ two Γ's.

$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (g_{il,j} + g_{jl,i} - g_{ij,l})$

$\dot{\Gamma}^k_{ij} = \frac{1}{2} g^{kl} (h_{il;j} + h_{jl;i} - h_{ij;l})$ at a point, but since it is a tensor, we can replace the coordinate differentiation by covariant differentiation.

i.e. $\dot{\Gamma}^k_{ij} = \frac{1}{2} g^{kl} (h_{il;j} + h_{jl;i} - h_{ij;l})$

So we can now observe that $\dot{R}_{ij}$ is a linear combination of 2^{nd} derivatives of h.

Now $\dot{R} = -h^{ij}R_{ij} + g^{ij}(\dot{\Gamma}^k_{ij;k} - \dot{\Gamma}^k_{ki;j})$

$\quad = -h^{ij}R_{ij} +$ divergence terms

(since covariant differentiation commutes with contraction)

i.e. we can, upon integrating, ignore the $\sum g^{ij}\dot{R}_{ij}$ term as it is a sum of divergence terms.

So upon integration we get

$$\frac{d}{dt}\int_M R(g(t))\, d\omega_{g(t)} = -\int_M \langle h, Ric(g(t))\rangle_{g(t)}\, d\omega_{g(t)} + \frac{1}{2}\int_M R(g(t))\, Tr_{g(t)}(h)\, d\omega_{g(t)}$$

$$= -\int_M \langle h, Ric(g(t)) - \tfrac{1}{2} R(g(t))\, g(t)\rangle_{g(t)}\, d\omega_{g(t)}$$

(since $\dot{\omega}_{g(t)} = \frac{1}{2} Tr_{g(t)}(h)\, \omega_{g(t)}$ and $Tr_{g(t)}(h) = \sum_{i,j} g(t)^{ij} h_{ij}$)

o evaluating at $t=0$, g is a critical point $\mathcal{R}(g)$ on $\mathcal{M}_1$

$$\Longrightarrow \int_M \langle h, Ric(g) - \tfrac{1}{2} R g\rangle_g\, d\omega_g = 0 \quad \forall h \text{ s.t } \int_M Tr_g(h)\, d\omega_g = 0$$

e want to incorporate the total trace condition into the Euler-Lagrange quation, let $V(t) = \underline{Vol(g+th)}$, then the normalized family $\overline{g(t)} = V(t)^{-2/n} g(t)$ s a path in $\underline{\mathcal{M}_1}$.

$$\mathcal{R}(\overline{g(t)}) = \int_M R(\overline{g(t)})\, d\omega_{\overline{g(t)}} = V(t)^{\frac{2-n}{n}} \int_M R(g(t))\, d\omega_{g(t)} \quad \left(\text{since } R(\overline{g(t)}) = V(t)^{n/2} R(g(t))\right.$$

$$\left.\text{and } d\omega_{\overline{g(t)}} = V(t)^{-1} d\omega_{g(t)}\right)$$

d $\dot{V}(t) = \int_M d\dot{\omega}_{g(t)} = \frac{1}{2}\int_M Tr_{g(t)}(h)\, d\omega_{g(t)} = \frac{1}{2}\int_M \langle h, g(t)\rangle_{g(t)}\, d\omega_{g(t)}$

o $\displaystyle \frac{d}{dt}\mathcal{R}(\overline{g(t)}) = -V(t)^{\frac{2-n}{n}} \int_M \langle h, F(g(t))\rangle_{g(t)}\, d\omega_{g(t)}$

where $F(g) = \mathrm{Ric}(g) - \frac{1}{2}R(g)\,g + \frac{n-2}{2n}\mathcal{R}(g)\,g$.

o g is a critical point of $\mathcal{R}(g)$ on $\mathcal{M}_1$ $\Leftrightarrow$

$$\mathrm{Ric}(g) - \tfrac{1}{2}R(g)\,g + \tfrac{n-2}{2n}\mathcal{R}(g)\,g \equiv 0$$

$$\Rightarrow \quad R(g) - \tfrac{n}{2}R(g) + \tfrac{n-2}{2}\mathcal{R}(g) \equiv 0$$

$$\Rightarrow R(g) \equiv \mathcal{R}(g)$$

and so: $\mathrm{Ric}(g) - \frac{R}{n}g = T(g) = 0$ i.e. g is Einstein.

One would like some sort of existence theory for this.
First difficulty is that $\mathcal{R}(\cdot)$ is unbounded above and below.
Critical points do not tend to be local maxima or local minima
et g be a critical point, say a round sphere,  S^n, g
by Linearizing $F(g)$ about g (i.e. computing the second variation)
t can be shown (see R.M. Schoen: Variational Theory for Total Scaler
urvature Functional for Riemannian Metrics and Related Topics) that $\mathcal{R}(\cdot)$
s minimized among metrics conformally equivalent to g, i.e. in
$[g]_1 = \{e^{2u}g : u \in C^\infty(M)\} \cap \mathcal{M}_1$, and $\mathcal{R}(\cdot)$ is maximized among
variations orthogonal to $[g]_1$

We will proceed, having separated the problem to the two directions; the conformal directions and the orthogonal direction bu will keep the total variation in mind.

<u>Restricted</u> <u>Problem</u>: g_0 given on M

$[g_0]_1 = \{e^{2u}g : u \in C^\infty(M)\} \cap \mathcal{M}_1$

Extremize $\mathcal{R}(\cdot)$ on $[g_0]_1$

$h = \eta\, g$, $\eta \in C^\infty(M)$ and $\int \eta\, d\omega_g = 0$ (trace condition)

First variation formula $\Rightarrow \int_M (1-\frac{n}{2}) R \eta \, d\omega_g = 0 \quad \forall \eta$

$n \geq 3 \implies R \equiv K$, a constant (*)

$g \in [g_0]_1, \quad g = u^{\frac{4}{n-2}} g_0, \quad u > 0$

$$R(g) = -c(n)^{-1} u^{-\frac{n+2}{n-2}} Lu \quad \text{(Yamabe's Equation)}$$

where $Lu = \Delta_{g_0} u - c(n) R(g_0) u$, $\quad c(n) = \frac{n-2}{4(n-1)}$

(*) $\Rightarrow Lu + c(n) K u^{\frac{n+2}{n-2}} = 0 \quad u > 0$ on M, $K \equiv$ constant.

Prescribed scaler curvature problem is $K = r(x)$ a given smooth function on M, instead of $K \equiv$ constant.

$$\mathcal{R}(g) = -c(n)^{-1} \int_M (Lu) u^{-\frac{n+2}{n-2}} u^{\frac{2n}{n-2}} d\omega_{g_0} = -c(n)^{-1} \int_M u Lu \, d\omega_{g_0}$$

$$= c(n)^{-1} \int_M (|\nabla_{g_0} u|^2 + c(n) R(g_0) u^2) \, d\omega_{g_0}$$

with the constraint $\int_M u^{2n/n-2} d\omega_{g_0} = 1$

($\Rightarrow \int u^2 d\omega_{g_0}$ bounded

So $\mathcal{R}(g)$ is bounded below on $[g_0]_1$

let $I([g_0]) = \inf \{\mathcal{R}(g) : g \in [g_0]\}$.

we have: $-\infty < I([g_0])$ and $I([g_0])$ is a conformal invariant (this uses the conformal invariance of L).

Note: $I([g_0])$ is not a topological invariant.

The behavior of this problem depends highly on $\text{Sign}(I[g_0])$ in an analogous way to which the sign of $\mathcal{K}(M)$ influenced the Probem of prescribed Gauss curvature in dim. 2.

Lemma: The following are equivalent (TFAE)

(i) $I([g_0]) > 0$ (resp $= 0$) (resp < 0)

(ii) $\exists g \in [g_0]$ with $R(g) > 0$ (resp $\equiv 0$) (resp < 0) everywhere (or for some, hence every)

(iii) $\lambda_0(-L_g) > 0$ (resp 0) (resp < 0) for any $g \in [g_0]$.

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<u>Proof</u> (In the positive case; the others follow similarly)

(iii) $\Rightarrow$ (ii) Suppose $\lambda_0(-L_g) > 0$

We may take $v > 0$, to be a lowest eigenfunction for $-L_g$, i.e. : $-L_g v = \lambda_0 v > 0$

look at the scaler curvature of the new metric $v^{4/n-2} g$

$$R(v^{4/n-2} g) = -c(n)^{-1} \frac{L_g v}{v^{\frac{n+2}{n-2}}} > 0 \text{ on } M \checkmark$$

(ii) $\Rightarrow$ (i) Let $g \in [g_0]$ with $R(g) > 0$.

$$L_g u = \Delta_g u - c(n) R_g u$$

$$\mathcal{R}(u^{4/n-2} g) = c(n)^{-1} \int_M (|\nabla_g u|^2 + c(n) R_g u^2)\, d\omega_g \quad \text{with } \int_M u^{2n/n-2} d\omega_g = 1$$

Since $R_g > 0$ $\exists$ $c > 0$ s.t. $c\|u\|_{1,2}^2 \leq \mathcal{R}(u^{4/n-2} g)$.

Recall the Sobolev embedding theorem, that the inclusion $W^{1,2}(M) \hookrightarrow L^{2n/n-2}(M)$ is a bounded linear operator.

so $\exists$ $c_1 > 0$ s.t. $c_1 \|u\|_{\frac{2n}{n-2}}^{\frac{n-2}{n}} \leq c\|u\|_{1,2}^2 \leq \mathcal{R}(u^{4/n-2} g)$,

by our constraint on u

$$\mathcal{R}(u^{4/n-2} g) \geq c_1 > 0 \quad \text{for all such } u$$

hence $I([g_0]) \geq c_1 > 0 \checkmark$

(i) $\Rightarrow$ (iii)

Let $v > 0$ be the lowest eigenfunction normalized to have $L^{\frac{2n}{n-2}}$-norm $= 1$, $L_{g_0} v = -\lambda_0 v$.

$$\int_M v^{\frac{2n}{n-2}} = 1 \Rightarrow \mathcal{R}(v^{4/n-2} g_0) \geq I([g_0]) > 0$$

$$R(v^{4/n-2} g_0) = c(n)^{-1} E(v) = c(n)^{-1} \int_M -v L v \, d\omega_{g_0} = c(n)^{-1} \lambda_0 \int_M v^2 \, d\omega_{g_0}$$

$$\Rightarrow \lambda_0 > 0 \checkmark$$

Uniqueness Theorems for the Problem

1) Easy case (non-positive)

Prop. $g_1, g_2 \in [g_0]$ & $R_1 \leq R_2 < 0$ on M, then $g_1 \leq g_2$

Note: This is true with $R_2 \leq 0$ only, and R_2 not identically zero, One should use a more delicate maximum principle.)

Proof $g_i = u_i^{4/n-2} g_0$, $i = 1, 2$

$$g_1 = \left(\frac{u_1}{u_2}\right)^{4/n-2} g_2 = v^{4/n-2} g_2$$

we must show $v \leq 1$

$$R_1 = -c(n)^{-1} L_2 v \cdot v^{-\left(\frac{n+2}{n-2}\right)}$$

$$\Delta_{g_2} v = c(n) R_2 v - c(n) R_1 v^{\frac{n+2}{n-2}}$$

$$\geq c(n) R_1 \left(v - v^{\frac{n+2}{n-2}}\right)$$

let $v(p) = \max_M v$, then $\Delta_{g_2} v \leq 0$

so $R_1(p)\left(v(p) - v(p)^{\frac{n+2}{n-2}}\right) \leq 0$

$\Rightarrow \left(v(p) - v(p)^{\frac{n+2}{n-2}}\right) \geq 0$

$n \geq 3 \Rightarrow \frac{n+2}{n-2} > 1$, so $\Rightarrow v(p) \leq 1$ ∎

This immediately gives a general uniqueness theorem;

Corollary: If $R(g_1) \equiv R(g_2) < 0$, then $g_1 \equiv g_2$

Proof: $\Delta_{g_2} v = c(n) R \left(v - v^{\frac{n+2}{n-2}}\right)$

let q be s.t. $v(q) = \min_M v$

if $v(q) < 1$ Then $v(q) - v(q)^{\frac{n+2}{n-2}} > 0$

so $\Delta_{g_2} v(q) < 0$

but $v(q) = \min v \Rightarrow \Delta_{g_2} v(q) \geq 0$ a contradiction

$\therefore v \geq 1$ so by the proposition $v \equiv 1$

and $g_1 \equiv g_2$ ∎

Theorem If $I([g_0]) \le 0$, then there is a unique normalized metric $g \in [g_0]_1$ satisfying (*).

Proof: If $I([g_0]) = 0$, then the Lemma implies

$\exists!\, g \in [g_0]_1$ with $R(g) \equiv 0$ with $g = u^{4/n-2} g_0$.

i.e. $L_{g_0} u = 0$. The uniqueness follows since u being a positive first eigenfunction is unique up to scale and the scale is provided by requiring g have unit volume

If $I([g_0]) < 0$, we prove a stronger result, take any $r(x) \in C^\infty(M)$ such that $r(x) < 0$ on M.

Consider the prescribed scaler equation.

$$Lu + c(n)\, r(x)\, u^{\frac{n+2}{n-2}} = 0 .$$

By the Lemma we may assume $R(g_0) < 0$, we have

$$\Delta_{g_0} u = c(n)\left(R(g_0)\, u - r(x)\, u^{\frac{n+2}{n-2}}\right) = f(x,u).$$

Use the method of sub and supersolutions, let ε be a small positive constant and set

$\underline{u} = \varepsilon$ is a subsolution: $0 = \Delta_{g_0} \underline{u} \ge f(x, \underline{u})$

$\overline{u} = \varepsilon^{-1}$ is a supersolution: $0 = \Delta_g \overline{u} \le f(x, \overline{u})$

$\therefore\ \exists$ a solution u, $\varepsilon \le u \le \varepsilon^{-1}$.

Uniquenes follows from the previsus corollary. ∎

A Harder Uniquemess Theorem

i) Theorem (Obata): If $\exists$ an Einstein metric in $[g_0]_1$ $(I([g_0]) > 0)$, Then any constant scaler curvature metric in $[g_0]$ is Einstein. Secondly, there is a unique such metric in $[g_0]_1$ unless $(M, g_0) \underset{\text{equivalent}}{\overset{\text{conformally}}{\approx}} (S^n, \text{standard})$ and in this case, all constant scaler curvature metrics have constant sectional curvature.

Proof: Choose g_0 to be Einstein, say $R(g) \equiv K$, constant, $g_0 = \Omega^2 g$

g_0 Einstein $\Rightarrow 0 = T_{ij}(g_0) = T_{ij}(g) + (n-2)\Omega\left[(\Omega^{-1})_{;ij} - \frac{1}{n}\Delta_g(\Omega^{-1})\, g_{ij}\right]$

(covariant derivative is taken w.r.t. g.)

Twice contracted 2nd Bianchi identity $\Rightarrow \sum_j (R_{ij} - \frac{1}{2}R g_{ij})_{;j} = 0$

$R_{ij} = T_{ij} - \frac{1}{n}R g_{ij}$ so $\sum_j (T_{ij} - (\frac{1}{n}+\frac{1}{2})R g_{ij})_{;j} = 0$

$R(g)$ constant $\Rightarrow \sum_j (T_{ij})_{;j} = 0$ i.e. $\operatorname{div} T = 0$

$$\int_M \langle \Omega^{-1} T_{ij}(g), T_{ij}(g_0)\rangle_g \, d\omega_g = \int_M \Omega^{-1} \langle T(g), T(g_0)\rangle_g \, d\omega_g = 0 \quad (T(g_0) = 0)$$

$\overset{*}{\|}$

$$\int_M \Omega^{-1} \|T(g)\|_g^2 \, d\omega_g + (n-2)\int_M \langle T_{ij}(g), (\Omega^{-1})_{;ij}\rangle d\omega_g + \frac{(n-2)}{n}\int_M \langle T_{ij}(g), \Delta_g(\Omega^{-1}) g_{ij}\rangle d\omega_g$$

Now $\Delta_g(\Omega^{-1}) g_{ij} = \operatorname{div}_g(\operatorname{grad}(\Omega^{-1})\cdot g_{ij})$, so by integrating by parts on the 2nd and 3rd integrals and using $\operatorname{div} T = 0$, we get $0 = \Omega^{-1}\int_M \|T(g)\|_g^2 \, d\omega_g \Rightarrow T(g) = 0$

So g_0 Einstein $\Rightarrow g$ Einstein.

Note: by applying the Schwarz inequality

$$\int_M \langle \Omega^{-1} T_{ij}(g), T_{ij}(g_0)\rangle_g \, d\omega_g \le \left(\int \Omega^{-1}\|T(g)\|_g^2\right)^{1/2}\left(\int \Omega^{-1}\|T(g_0)\|_g^2\right)^{1/2}$$

so the equality $* \Rightarrow \int_M \Omega^{-1}\|T(g)\|_g^2 \, d\omega_g \le \int_M \Omega^{-1}\|T(g_0)\|_g^2 \, d\omega_g$

for any constant scaler curvature g and for any $g_0 = \Omega^2 g$.

i.e. Constant scaler curvature metrics minimize $T(g)$ within their conformal class.

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For the 2^{nd} part we suppose g is Einstein and $g_0 = \Omega^2 g$ is also Einstein, To prove $\Omega \equiv 1$ unless (M, g) is conformally diffeomorphic to (S^n, g_1); (g_1 - round metric).

First Note: Cor. of 1^{st} Part of thm: (S^n, g_1) any $g \in [g_1]$ with constant $R(g)$ has constant sectional curvature.

Proof: (S^n, g_1) is Einstein so g is Einstein, (S^n, g) is locally conformally flat $\Rightarrow W \equiv 0$ so g has constant sectional curvatures.

Proof: (of 2^{nd} Part): $\underset{0}{\underset{\shortparallel}{T(g_0)}} = \underset{0}{\underset{\shortparallel}{T(g)}} + (n-2)\Omega\,[\text{Trace free Hessian}_g\,(\Omega^{-1})]$

set $v = \Omega^{-1}$, we get (M, g) with

$$v_{;ij} = \frac{1}{n} \Delta_g v \; g_{ij} \quad \text{satisfied.}$$

(Note: on Standard Sphere the $n+1$ dim'l space of 1^{st} order spherical harmonics are solutions

Interprete the equation in terms of conformal Killing vector fields
let $X = \nabla v$ be the gradiant vector field

$$X_{i;j} + X_{j;i} - \frac{2}{n}(\operatorname{div} X)\, g_{ij} = 0$$

If $\operatorname{div} X = 0$, then the equation say that X is a Killing vector field, if $\operatorname{div} X \neq$ then X is a conformal vector field.

(Recall: An infinitesimal transformation X of M is said to be conformal if $L_X g = \sigma g$, $\sigma \in C^\infty(M)$, σ = constant is a homothety, $\sigma = 0$ is an isometry. The local 1-param. group of diffeo's generated by X (flow along integral curves) is conformal iff X is conformal).

Example: On (S^n, g_1) consider the gradiant of the height function this fixes two poles $(0,\infty)$ and the flow is centered dilations.

We want to use this conformal flow to show that M is locally conformally flat.

1st take the divergence of the equation:

$$\sum_j v_{;ijj} = \frac{1}{n}\sum_i v_{;jji}$$

Now $\sum_j v_{;ijj} = \sum_j v_{;jij} = \sum_j v_{;jji} + \sum R_{ij} v_{;j} = (\Delta v)_{;i} + \frac{1}{n} R v_{;i}$

(g is Einstein)

so we get $\quad \frac{1}{n}(\Delta v)_{;i} = (\Delta v)_{;i} + \frac{1}{n} R v_{;i}$

or $\quad \nabla\left(\Delta v + \frac{R}{n-1} v\right) \equiv 0$

Scale the metric so $R(g) \equiv n(n-1)$,

get $\quad \nabla(\Delta v + n v) \equiv 0$.

Now we can change v by an additive constant and the equation will still be satisfied, so we may assume $\int_M v \, d\omega_g = 0$

$$\Delta v + n v = c$$

integrate: $\int_M \underbrace{\Delta v \, d\omega_g}_{0} + n \int_M \underbrace{v \, d\omega_g}_{0} = c \, \mathrm{Vol}_g(M)$

$$\Longrightarrow \quad c = 0.$$

So v is an eigenfunction for Δ_g

$$\Delta v + n v = 0$$

$$v_{;ij} = -v g_{ij}$$

in Normal Coodinates; if we multiply by $v_{;j}$, we get

$$v_{;j} v_{;ij} + v_{;j} v \, \delta_{ij} = 0$$

Summing $\quad \frac{1}{2}\nabla(|\nabla v|^2) + \frac{1}{2}\nabla(v^2) = 0$

$\therefore \quad |\nabla v|^2 + v^2 = c$, a positive constant.

This implies that at every critical point of v, the hessian is either positive definate, or negative definate.

Hence v has only 2 two critical pt's.

It is not difficult to see that this implies M is diffeomorphic to S^n. (see, for example Milnor's Morse Theory pg. 25; the diffeomorphism is provided by the gradiant flow).

Claim: $W \equiv 0$ on M (i.e. M is conformally flat)

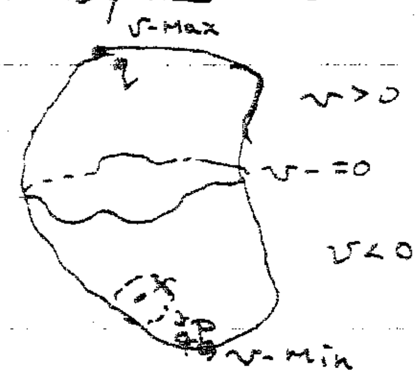
(This makes sense since the gradiant flow is conformal).

let F_t = Group of diffeo's generated by $X = \nabla v$

For $x \in M$ near $p (= \min - v)$

$$\lim_{t \to -\infty} F_t(x) = p$$

let $F_t^*(g) = \Omega_t^2 g$



Under the Flow F_t, as $t \to -\infty$ a small ball about x tend to an infinitessimally small (tiny) ball near p so

as $t \to -\infty$, $\Omega_t(x) \to \infty$

Compute $\|W(F_t^* g)\|^2(x) = \|W(\Omega_t^2 g)\|^2(x) = \Omega_t^4 \|W(g)\|^2(x)$

On the Other hand,

$$\|W(F_t^* g)\|^2(x) = \|W(g)\|^2(F_t(x)).$$

(Since computing curvatures commutes with pull backs.

So $\|W(g)\|^2(x) = \Omega_t^{-4} \|W(g)\|^2(F_t(x))$

So letting $t \to -\infty$ we see $\|W(g)\|^2(x) = 0$.
since $\|W(g)\|^2(p)$ is a bounded constant.
Since a similar argument work for any $x \in M$
we have $W \equiv 0$
Thus M is conformally diffeomorphic to S^n. ■

Two exceptional cases

1) (S^n, g_1). What $g \in [g_1]$ satisfy $R(g) \equiv n(n-1)$?
$S^n \setminus \{\infty\}$ is conformally equivalent to $\mathbb{R}^n$ with coodinates $\{x^1, \dots, x^n\}$ through stereographic projection.

$$g_1 = \frac{4}{(1+|x|^2)^2} \sum (dx^i)^2$$

So if $g = u^{4/n-2}\delta$ (δ - the Euclidean metric)
the equation: $\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0$, $u > 0$ $(R_\delta = 0)$
has the basic solution $u_1(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}}$

All other solutions are gotten by conformal transformations, specifically:

Dilations: $F_\lambda(y) = \lambda y = x$

$$F_\lambda^*(g_1) = \frac{4\lambda^2}{(1+\lambda^2|y|^2)^2} \sum (dy^i)^2$$

So we get a 1-parameter family of solutions

$$g_\lambda = F_\lambda^*(g_1) = (u_\lambda)^{4/n-2}\,\delta$$

where $$u_\lambda = \left(\frac{2\lambda}{1+\lambda^2|y|^2}\right)^{\frac{n-2}{2}}$$

All other solutions are gotten by changing points

i.e. $\left(\dfrac{2\lambda}{1+\lambda^2|x-x_0|^2}\right)^{\frac{n-2}{2}}$

There is more geometric way of seeing the space of solutions: namely, the solutions are parametrized by the interior of the unit ball in $\mathbb{R}^{n+1}$, as follows.

center - corresponds to the identity solution.

Given any other point $p \in B^{n+1}(1)$, we draw the line through p and the center, then there exists a unique centered dilation fixing the endpoints on S^n and taking p to the center, F_p, $F_p(p)=0$

then $F_p^*(g_0)$ is the corresponding solution.

All the solutions are minima for the variational problem.

---

② $(S^1(T)\times S^{n-1}(1), g_0)$ $\quad g_0$ = the product metric.

Where, T is the length of the S^1 factor, let $M_T = S^1(T)\times S^{n-1}(1)$

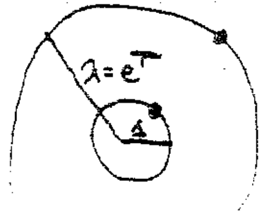
The product metric g_0 determines a locally conformally flat structure.

The universal cover of M_T is $\mathbb{R}\times S^{n-1}(1)$, which is conformally equivalent to $\mathbb{R}^n - \{0\} \approx S^n - \{0,\infty\}$.

$$(*) \quad R(g) = n(n-1)$$

<u>Symmetry Theorem</u>: Any solution of $(*)$ on $S^n \setminus \{0,\infty\} \approx \mathbb{R}^n \setminus \{0\}$ is rotationally symmetric (we assume the solution is not regular at 0 and ∞)

So $u(x) = u(|x|)$



let $(t, \xi) \in S^1(T)\times S^{n-1}(1)$

Then any solution $u^{4/n-2} g_0$ on $S^1(T) \times S^{n-1}(1)$, then satisfies $u(t, \xi) = u(t) \quad \forall (t, \xi)$.

So the PDE reduces to an O.D.E. which we can analyze

$$R(g_0) = R(S^{n-1}(1), g_0|_{S^{n-1}(1)}) = (n-1)(n-2)$$

So $\Delta_{g_0} u - \frac{(n-2)^2}{4} u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0$

becomes

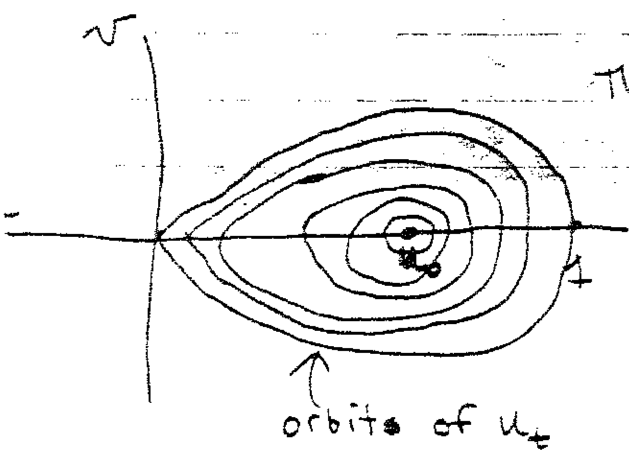
$$\frac{d^2 u}{dt^2} - \frac{(n-2)^2}{4} u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0$$

This is autonomous 2^{nd} order system of O.D.E's. We can't solve these explicitely (except in dim 4), but we can understand properties of solutions as orbits of vector fields in the (u,v) plane.

$$X(u,v) = \left(v, \frac{(n-2)^2}{4} u - \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}\right)$$

ie $\Rightarrow$ $u' = v$

$$v' = \frac{(n-2)^2}{4} u - \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}$$



The solution u_0 corresponds to the product metric.

The second explicit solution is $u(t) = (\cosh t)^{-\frac{n-2}{2}}$

You can show all solutions are periodic and still has to under stand the periods.

$\exists$ a magic number $T_0 = (n-2)^{-1/2} 2\pi$ s.t.

For $T < T_0$ $\exists!$ solution of (*) u_0

For $T_0 < T < 2T_0$ $\exists$ 2 geometrically distinct solutions
1 of Morse index 2 an
1 which is Minimizing (Index 0)

similarly

For $2T_0 < T < 3T_0$ $\exists$ 3 geometrically distinct solutions
1 index 4
1 index 2
1 minimizing.

From the ODE analysis we get that any soluton on $S^n \setminus \{0, \infty\}$ which is complete, is invariant under a discrete subgroup of dilations.

10/27/88 Solutions on domains in S^n

$\Omega \subset S^n$, open g_0 - the standard metric on S^n

consider (Ω, g) where $g = u^{4/n-2} g_0$

There are three natural boundary conditions one can assume on the metric g.

1) the domain Ω with metric g is Complete (this "geometric" boundary condition arose in the case $S^1(T) \times S^{n-1}(1)$, whose universal cover is $S^n \setminus \{0, \infty\}$, and the solution lifts to one which blows up at $\{0, \infty\}$.

2) u is a Proper function on Ω. this implies $u \to \infty$ on $\partial\Omega$

3) Λ has measure zero, $u \in L^{\frac{n+2}{n-2}}(S^n)$, $Lu + u^{\frac{n+2}{n-2}} \le 0$ on S^n, in the weak sense.

ote: 1) $\Rightarrow$ 3) with Hausdorff dim $\Lambda \le \frac{n-2}{2}$.

the following we'll assume any of these three boundary conditions on (Ω, g)

u satisfies $Lu + u^{\frac{n+2}{n-2}} \le 0$ weakly if $\forall \varphi \in C^\infty(S^n)$, $\varphi \ge 0$

$$\int_{S^n} u L\varphi + u^{\frac{n+2}{n-2}}\varphi \le 0.$$

onsider the equation for u: (*) $R(g) =$ constant

$$Lu + u^{\frac{n+2}{n-2}} = 0 \quad \text{on } \Omega,\ u > 0$$

Main Tool is:

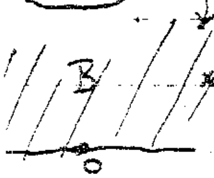
<u>Alexandrov Reflection Method</u> (devious application of the Maximum princ

(This application is due to Gidas-Ni-Nirenberg)

Let $B \subset \Omega$ be a round ball which avoids the singular set Λ

choose $p \in \partial B$, and use stereographic projection to take B to a half plane, p going to ∞ (will normalize)

A solution has a Taylor expansion



$x^1, \dots, x^n$ coordinates

$$y(y) = a \sum b^i y^i + O(|y|^2)$$

$g = v^{4/n-2} \sum (dx^i)^2$, where v is the Kelvin transform of u

$\sim v \sim$: $-\Lambda$

ie the expansion at ∞ in the half-plane is the inversion of the expansion at origin in the ball, B.

$$v(x) = |x|^{2-n}\left(a + \sum b^i \frac{x^i}{|x|^2} + O(|x|^{-2})\right)$$

¿ <u>How to choose an origin?</u> (ie we have the freedom of translations and dilations.)

Consider the effect of a translation: $x = \hat{x} + \hat{c}$

$$v(x) = \hat{v}(\hat{x}) = |\hat{x} + \hat{c}|^{2-n}\left(a + \frac{\vec{b}\cdot\hat{x}}{|\hat{x}|^2} + O(|\hat{x}|^{-2})\right)$$

$$[(\hat{x}+\hat{c})\cdot(\hat{x}+\hat{c})]^{\frac{2-n}{2}} = |\hat{x}|^{2-n}\Big(\underbrace{\left(\frac{\hat{x}}{|\hat{x}|} + \frac{\vec{c}}{|\hat{x}|}\right)\cdot\left(\frac{\hat{x}}{|\hat{x}|} + \frac{\vec{c}}{|\hat{x}|}\right)}_{1 + \frac{2\vec{c}\cdot\hat{x}}{|\hat{x}|^2} + O(|\hat{x}|^{-2})}\Big)^{\frac{2-n}{2}}$$

So $|\hat{x}+\vec{c}|^{2-n} + |\hat{x}|^{2-n}\left(1 + (2-n)\dfrac{\vec{c}\cdot\hat{x}}{|\hat{x}|^2} + O(|x|^{-2})\right)$

$\therefore\ \hat{V}(\hat{x}) = |\hat{x}|^{2-n}\left[a + \dfrac{b\cdot\hat{x}}{|\hat{x}|^2} - \dfrac{(n-2)\,ac\,\hat{x}}{|\hat{x}|^2} + O(|\hat{x}|^{-2})\right]$

Now we can set $\overset{(\text{divide})}{a=1}$ and then there exists a unique c s.t.

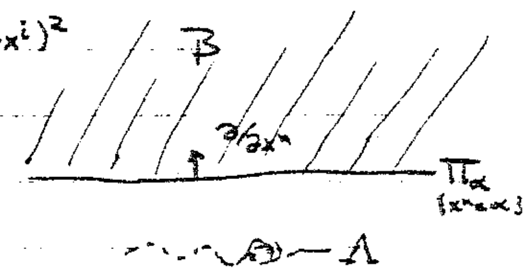
$\hat{v}(\hat{x}) = |\hat{x}|^{2-n}\left(1 + O(|\hat{x}|^{-2})\right)$

This gives, having choosen p, a unique prefered origin, q.

<u>What is the relation between these points $\overset{(p)}{\infty}$, q, Λ ?</u>

we have $B = \{x : x^n \geq \alpha\}$, we know B lies above Λ here.

$g = v^{4/n-2}\sum(dx^i)^2$



<u>Some Conclusions</u>:

1) Any ball $B\subset\Omega$ is geodesically convex w.r.t. g.
(Provided $\Lambda\neq\phi$)

<u>Proof</u>:

To interpret geodesic convexity we need to calculate the 2^{nd} fundamental form of Π_α w.r.t. g.

– choose the inward (upword) normal $\partial/\partial x^n$

<u>Exercise</u> 2^{nd} F.F. of Π_α w.r.t. g $= -c(n)\,v^{-1}\dfrac{\partial v}{\partial x^n}\,g$

<u>Proof</u>: $g = v^{4/n-2}\sum(dx^i)^2$, The second fundamental form of Π_α w.r.t $\sum(dx^i)^2$ is zero. So $\mathrm{II}_g(X,Y) = -g(X,Y)\,N\,D\left(\frac{2}{n-2}D\log v\right)$

Where N is the normal component, so

$= -g(X,Y)\,\dfrac{2}{n-2}\cdot v^{-1}\dfrac{\partial v}{\partial x^n}$ ▫

So ∂B is geodesically convex $\Leftrightarrow \dfrac{\partial v}{\partial x^n} < 0$

i.e. we want to show the solution v is decreasing up to Π_α

$v(x) = |x|^{2-n}\left(1 + O(|x|^{-2})\right)$

$\Delta_0 v + v^{\frac{n+2}{n-2}} = 0$

Take $\lambda \gg \alpha$ and look at Π_λ. For $x = (x', x^n)$, $x^n \le \lambda$

set: $x^* = (x', 2\lambda - x^n)$ i.e. reflect across Π_λ.

Let $v^*(x) \equiv v(x^*)$

$\cdot\, x^*$

Π_λ

$\cdot\, x$

For x^n large we have $\frac{\partial v}{\partial x^n} < 0$

so $v^* < v$ if λ is sufficiently large Π_α

(for $x = (x', x^n)$ with $x^n < \lambda$)

Prop: Two possibilities occur: (Note $\Lambda \neq \{\phi\} \Rightarrow$ ① can not occur for $\lambda > 0$)

① Either $\exists\ \lambda \ge \alpha$ such that Π_λ is a plane of symmetry for v (i.e. $v^* \equiv v$)

or ② For $\lambda = \alpha$ we have $v^* < v$ for $x^n < \alpha$ and $\frac{\partial v}{\partial x^n} < 0$ for $x^n = \alpha$.

(we also conclude $\alpha > 0$).

The proof involves two parts: the interior maximum principle and the boundary maximum principle (the Hopf boundary point lemma).

Claim: The set of λ where the proposition holds is open and closed.

openess: This is almost immediate, follows from the strict inequalities and the decay conditions (provided $\alpha > 0$)

closed: Take a limit to get $v^* \le v$, $\frac{\partial v}{\partial x^n} \le 0$ at some point. Now v^* satisfies same equation as v and we have the ordering $v^* \le v$.

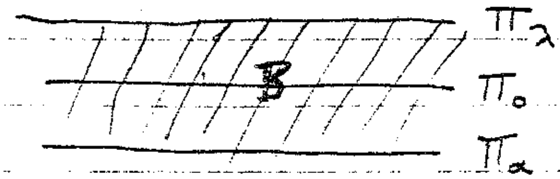
Strong Maximum principle $\Rightarrow$ either the inequality ($v^* \le v$) holds strictly ($v^* < v$) or they are identical ($v^* \equiv v$).

Hopf boundary point $\Rightarrow$ If we have the inequality in the interior but they agree at some pt in the boundary, then they either agree identically or we have strict inequality at the boundary point

(For a more detailed argument see Gidas-Ni-Nirenberg - section 4).

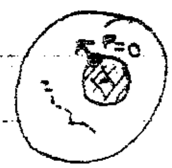
Can actually prove $\alpha > 0$.

Assume not:



Expansions of v, v^* agree near ∞, $v^*(x) = |x|^{2-n}(1+O(|x|^{-2}))$

Compactify back to S^n:



$u^* < u$

$u(y) = a + O(|y|^2)$

$u^*(y) = a + O(|y|^2)$

$\frac{\partial(u-u^*)}{\partial v} = 0$ at origin ($p=0$) this contradicts the Hopf boundary point lemma.

This establishes Prop. and thus Conclusion 1.

Recall having choosen p we've determined a unique q, as the preferred asymtotic center of expansion. To get back to the question of the relationship between p, q and Λ.

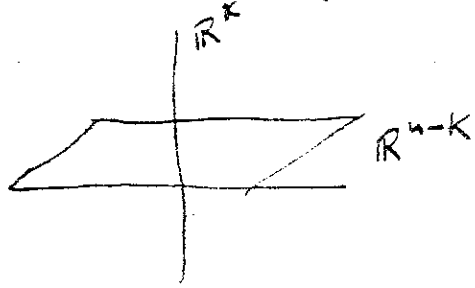
claim: All balls $\subset S^n$ containing p and q, intersects Λ.

Proof: if not then we can treat such a ball as B above so that $Im(q)$ lies above Π_α, but the image of q, $Im(q)$ lies on Π_0 and from the above we know $\alpha > 0$. contradiction.

Conclusion ② If $\Lambda \overset{\text{fully}}{\subset} S^k$ for $0 \le k < n$ (i.e. $\Lambda \subset S^k$, $\Lambda \not\subset S^\ell \ \forall \ell < k$)

Then g is symmetric about S^k. ($\Lambda \neq \emptyset$)

Proof: $S^n \setminus S^k \approx \mathbb{H}^{k+1} \times S^{n-k-1}$ (conformally diffeomorphic)



verify :

$(x,y) \in \mathbb{R}^n$, $x \in \mathbb{R}^{n-k}$, $y \in \mathbb{R}^k$: $\rho = |x|$, $\xi = \frac{x}{|x|} \in S^{n-k-1}$

Generalized cylindrical coordinates

(ρ, ξ, y) with the Euclidean metric $d\rho^2 + \rho^2 d\xi^2 + dy^2$

$$d\rho^2 + \rho^2 d\xi^2 + dy^2 = \rho^2 [\rho^{-2}(d\rho^2 + dy^2) + d\xi^2]$$

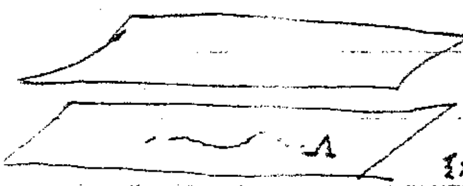
for $(y, \rho) \in \mathbb{H}^{k+1}$, within the upper half space model $\rho^{-2}(d\rho^2 + dy^2)$ is the hyperbolic metric.

$d\xi^2$ is the Euclidean metric on S^{n-k-1}

$\therefore$ we have established the conformal equivalence.

g symmetric about S^k means that g is invariant under all rotations in S^n which fix S^k.

Look at any $S^{n-1} \supset S^k \supset \Lambda$, by applying the Alexandroff reflection argument we can show g is symmetric with respect to reflection ~~configurations~~ across S^{n-1}:



$\{x : x^n = \varepsilon\}$ We can reflect across $x^n = \varepsilon$

$\therefore$ Take a limit as $\varepsilon \searrow 0$.

Λ $\{x : x^n = 0\} = S^{n-1}$ Conclude $v(x', -x^n) \le v(x', x^n)$

for $x^n \ge 0$. We can also use the Alexandroff reflection principle from the complementary ball : i.e. consider reflections from the other side. $\Rightarrow v(x', -x^n) \ge v(x', x^n)$

$\therefore$ $v(x', -x^n) = v(x', x^n)$, $\therefore$ g is symmetric w.r.t. to reflections across S^{n-1}

These generate all rotations fixing S^k.

<u>Conclusion</u> ③ A solution regular on $S^n \setminus \{p\}$ is regular on all of S^n

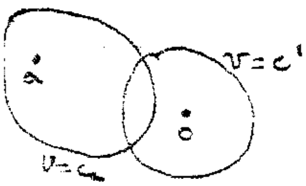
<u>Proof</u>: This follows almost immediately from ②.

use stereographic projection to take $S^n \setminus \{p\}$ to $\mathbb{R}^n$, $p \to \infty$

$g = v^{4/n-2}\,\delta$, δ the Euclidean metric on $\mathbb{R}^n$.

② implies that v is rotationally symmetric about zero. By taking a translation, we see that v is rotationally symmetric about any point $\Rightarrow$ v constant. decay conditions $\Rightarrow$ $v = 0$.

(or equation $\Rightarrow$ $v = 0$)



$\Rightarrow c_1 = c_2$

11/1/88 Recall set up: (Ω, g) $\Omega \subset S^n$, g_0 stand. on S^n

$$g = u^{4/n-2} g_0$$

u satisfies (*) $R(g) = c_1(n)$ or $L_0 u + u^{\frac{n+2}{n-2}} = 0$ where $L_0 u = \Delta u - \frac{n(n-2)}{4} u$

and either (1) g is complete

(2) u is a proper function on Ω.

(3) $\Lambda = S^n \setminus \Omega$ has measure 0, $u \in L^{\frac{n+2}{n-2}}(S^n)$ and $Lu + u^{\frac{n+2}{n-2}} \le 0$ distributionally on S^n

Note: (i) $\Rightarrow$ (iii) with $\dim_H \Lambda \le \frac{n-2}{2}$.

Assuming (1), (2) or (3) we concluded:

1) Any round ball $B \subset\subset \Omega$ is geodesically convex w.r.t. g.

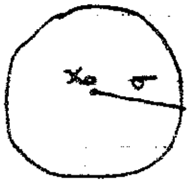
2) If $\Lambda \neq \{p\}$, and $\Lambda \overset{\text{any}}{\subset} S^k$, then g is symmetric about S^k.

3) Any solution of (*) smooth on $S^n \setminus \{p\}$ extends across p.

All these conclusions are very general in the sense that they don't use any Energy bounds on the solutions.

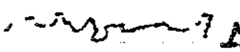
<u>Theorem 1</u>: $u(x) \leq c\, d(x,\Lambda)^{-\frac{n-2}{2}}$, $\forall x \in \Omega$ $(\Omega \neq S^n)$

<u>Proof</u>: Estimate arrived at by contradiction (blow up argument).



$B_\sigma(x_0) \subset\subset \Omega$

$\rho(x) = \mathrm{dist}_{g_0}(x, x_0)$


Λ

let $f(x) = (\sigma - \rho(x))^{\frac{n-2}{2}} u(x)$

Since u is smooth up to the boundary of $B_\sigma(x_0)$, $f(x) \equiv 0$ on $\partial B_\sigma(x_0)$
It suffices to show $f(x) \leq C$ for $x \in B_\sigma(x_0)$. Since if we take $\sigma = \frac{d(x_0,\Lambda)}{2}$ Then $f(x_0) \leq \sigma^{\frac{n-2}{2}} u(x_0) \leq C$

so $u(x_0) \leq C\left(\frac{1}{2}\right)^{-\frac{n-2}{2}} (d(x_0,\Lambda))^{-\frac{n-2}{2}}$.

Let x_1 be a maximum point of $f(x)$ in $B_\sigma(x_0)$
choose coordinates $y^1, \dots, y^n$ which are normal w.r.t. g_0 and are centered at x_1

$$g_0 = \sum g_{0\,ij}\, dy^i dy^j$$

where $g_{0\,ij} = \delta_{ij} + O(|y|^2)$

For $\lambda >> 0$, consider the dilated coordinates $z = \lambda y$

$$g_0 = \lambda^{-2} \sum g_{0\,ij}(z/\lambda)\, dz^i dz^j = \lambda^{-2} \hat{g}_0$$

$$\text{So } g = v_\lambda^{4/n-2}\, \hat{g}_0 \quad \text{where } v_\lambda(z) = \lambda^{-\frac{n-2}{2}} u(z/\lambda)$$

Note: for $\lambda >> 0$, $\hat{g}_0$ is almost Euclidean, ie if we consider $z \in B$, a bounded sets then

$$\hat{g}_0 = \delta + O\left(\frac{|z|^2}{\lambda^2}\right) \longrightarrow \delta \quad \text{as } \lambda \to \infty$$

We assume that no such c exists. Then there exists a sequence g_i, Λ_i, σ_i such that $\forall i$

$$(\sigma_i - \mathrm{dist}(x_{1,i}, x_{0,i}))^{\frac{n-2}{2}} u_i(x_{1,i}) = f_i(x_{1,i}) > i$$

Now $(\sigma_i - \mathrm{dist}(x_{1,i}, x_{0,i}))^{\frac{n-2}{2}}$ is bounded and we can thus conclude $u_i(x_{1,i}) \to \infty$ as $i \to \infty$.

Note u_i is given as $g_i = u_i^{4/n-2} g_0$

Take normal coordinates $y^1, \dots, y^n$ centered at $x_{1,i}$ (As i varies we change the maps but keep the same y coordinate space).

then $\lambda_i = u_i(0)^{2/n-2} \to \infty$ as $i \to \infty$.

let $z = \lambda_i y = u_i(0)^{2/n-2} y$

let $v_i(z) = \lambda_i^{-\frac{n-2}{2}} u_i(z/\lambda_i) = u_i(0)^{-1} u_i(z/u_i(0)^{2/n-2})$

$v_i(z) = 1$ at $z = 0$

The ball in y-coordinates of radius $R/u_i(0)^{2/n-2}$ about $y=0$ Corresponds to the ball of radius R about $z=0$ in $z$coordinates

$g_i = v_i^{4/n-2} \hat{g}_{0,i}$

where $\hat{g}_{0,i} = \sum_{i,j} g_{0\,ij}(u_i(0)^{-2/n-2} z)\, dz^i dz^j$

The $\hat{g}_{0,i}$ converge in C^2 on compact subsets, to the Euclidean metric, δ.

Now on the ball of radius $r_i/2$ in x coordinates, Where $r_i = \sigma_i - \rho(x_{1,i})$, we have

$$u_i(x) \leq c\, u_i(x_{1,i})$$

$\therefore$ we have $v_i(z) \leq C\, v_i(0) = C$ (independent of i) for z in the corresponding ball:

$$|z| \leq C\left[u_i(x_{1,i})(\sigma_i - \rho(x_{1,i}))^{\frac{n-2}{2}}\right]^{2/n-2} \leq C\, i^{2/n-2}$$

$\therefore$ $v_i(z)$ is bounded on a very large ball in the z-space, and have maximum value $v_i(0) = 1$

$v_i(z)$ satisfy $L_{\hat{g}_{0,i}} v_i + v_i^{\frac{n+2}{n-2}} = 0$

We fix a compact set Ω', $v_i(z)$ are uniformly bounded on Ω', We can rewrite the equation as

$$L_{\hat{g}_{0,i}} v_i + v_i^{4/n-2} v_i = 0$$

and view it as a linear equation with uniformly bounded coefficients, Thus by the Schauder interior estimates we get a uniform bound $\|v_i\|_{C^{2,\alpha}(\Omega')} \leq C$ for any $\Omega' \subset\subset \Omega$

The constant is independent of i.

Therefore by Arzela-Ascoli $\exists\, v \in C^2(\Omega')$ s.t.

$v_i \to v > 0$ in $C^2(\Omega')$ (actually we've passed to a subsequence)

and v satisfies $\Delta v + v^{\frac{n+2}{n-2}} = 0$

Now we can do this for any compact set in an arbitrarily large ball in z-coordinates, so we get

$$\Delta v + v^{\frac{n+2}{n-2}} = 0 \quad \text{on } \mathbb{R}^n, \quad v > 0.$$

Note: The upper bound on v_i independent of i, gives via Harnack's inequality a uniform lower bound for $\inf v_i$, ~~i.e.~~ ~~$\inf_{\Omega} v_i$~~

i.e. $\sup_{\Omega'} v_i \leq C \inf_{\Omega'} v_i \Rightarrow \underline{v > 0}$.

Therefore by conclusion ③ we have that v extends across ∞ to a regular solution on S^n, by Obata's theorem, we know v exactly i.e. $v^{4/n-2}\,\delta$ is the standard metric on S^n.

Say z_0 is a maximum point of v.

choose λ so that $v_\lambda(z_0) = v(z_0)$, the

$$v_\lambda(z) = c\lambda^{\frac{n-2}{2}}(1+\lambda^2|z-z_0|^2)^{-\frac{n-2}{2}}.$$

This gives us a "bubble" in the dilation, i.e. a sufficiently large ball will have concave boundary.

This implies that a small ball in Ω_i has concave boundary for g_i, for i sufficiently large, This contradicts conclusion ①. ∎

Another example

(M, g_0) M compact closed

Assume g_0 is locally conformally flat

i.e. locally $g_0 = \lambda^2(x) \sum (dx^i)^2$

(Note for $n \geq 4$, M l.c.f $\Longleftrightarrow$ $W(g_0) \equiv 0$, $W(g_0)$ = Weyl tensor for g_0

Prop (See Schoen-Yau "C.F manifolds, Kleinian groups and Scaler Curvature" Inventiones Math. 92, 47-71 (1988), or later in course for proof.)

M locally conformally flat, $R(g_0) \geq 0$. $\tilde{M}$ the universal covering space of M. Then M is conformally equivalent to a domain $\Omega \subset S^n$.

Theorem 2: look at $g \in [g_0]_1$ s.t. (*) $R(g) \equiv$ constant $g = u^{4/n-2} g_0$

or $L_{g_0} u + c\, u^{\frac{n+2}{n-2}} = 0$ (Note $c = E(u)$ encodes the Vol. 1 condition)

$\exists\, C = C(g_0)$ such that any solution of * satisfies $\operatorname{Max} u \leq C$

(Note: this implies $\|u\|_{C^k} \leq C'$ $\forall k$)

[Also we must make the technical assumption that $|\pi_1(M)| = \infty$, finite $\pi_1(M)$ is trivial via Obata's theorem]

Basic ingredient: B (round) $\hookrightarrow M$ (embeddings)

B is convex w.r.t. g for any g solution of (*).

Work with the universal cover and use the proposition, using a Blow up and rescaling argument as in theorem 1

A detailed proof will be inserted here later.

The main point is that thes estimates are uniform and do not assume minima or an energy bound.

The General theory we are after:

1^{st} Modify equation. (M^n, g_0) $n \geq 3$

$$(*)_p \equiv \quad Lu + E(u)\, u^p = 0 \quad , u > 0 \quad p \in [1, \tfrac{n+2}{n-2}]$$

multiply by u and integrate $\int u Lu + E(u) \int u^{p+1} = 0 \Rightarrow \int u^{p+1} = 1$

(note $p = \frac{n+2}{n-2}$ this means $\int u^{2\chi-2} = 1$ ie. $\mathrm{Vol}(u^{4/n-2} g_0) = \mathrm{Vol}(g_0)$).

Consider solutions $u \in C^{2,\alpha}_+(M)$, for $\Lambda > 0$

let $\Omega_\Lambda = \{ u \in C^{2,\alpha}_+(M) : u > \Lambda^{-1}, \|u\|_{2,\alpha} < \Lambda \}$

This is an open subset of $C^{2,\alpha}_+(M)$.

<u>Main Estimate</u>: (M, g_0), $\exists \Lambda_0 = \Lambda_0(g_0)$ such that <u>any</u> solution of $(*)_p$ for any $p \in [1, \frac{n+2}{n-2}]$ lies in Ω_Λ

(we're assuming $(M, g_0) \not\cong (S^n, \text{standard})$).

Previewing, this give:

<u>Existence Theorem</u>: If we are in the "generic case" ie. if $[g_0]$ is such that all solutions of $(*)$ in $[g_0]_1$ are non-degenerate (i.e. Hess is a non-degenerate quadratic form.) then

<u>Conclusion</u>: $\exists$ a finite # of solutions $g_1, \ldots, g_K$ and if

$\mathrm{Ind}(g_i) = \#$ of eigenvalues of $-\Delta g_i$ in $(0, \frac{R(g_i)}{n-1})$

then $1 = \sum_{i=1}^{K} (-1)^{\mathrm{Ind}(g_i)}$.

11/3/88 The philosophy is to embedd the Yamabe problem into a family of (subcritical) problems:

$$(*)_p \qquad Lu + E(u)\, u^p = 0 \quad , \quad p \in [1, \tfrac{n+2}{n-2}]$$

$$Lu = \Delta_{g_0} u - c(n) R(g_0)\, u.$$

by our Lemma (pg. 17-18) we can assume $R(g_0) > 0$

$\therefore$ L is negative definate and thus invertible

$L^{-1} : C^{k,\alpha}(M) \longrightarrow C^{k+2,\alpha}(M)$ is bounded.

Note: $Lu + E(u)\, u^p = 0 \iff u + E(u) L^{-1}(u^p) = 0$

$\iff u \in F_p^{-1}(0)$

where we define $F_p : C^{2,\alpha}_+ \longrightarrow C^{2,\alpha}$

by $F_p(u) = u + E(u) L^{-1}(u^p)$

Main Estimate we want: [assuming $(M, g_0) \neq (S^n, \text{stand.})$]

$\exists\, \Lambda_0 = \Lambda_0(g_0)$ such that $F^{-1}(0) \subset \Omega_{\Lambda_0}$ $\forall\, p \in [1, \frac{n+2}{n-2}]$

1^{st} We discuss existance.

$F_p(u) = u + E(u) L^{-1}(u^p)$, is a Fredholm map, i.e.

$F_p(u) = I + K$, where K is a compact map as follows:

(Note: we are regarding the domain F_p as Ω_Λ for some Λ.)

If we take $\{u_j\} \subset \Omega_\Lambda$ (i.e. bounded, and bounded away from zero.)

$Ku_j = E(u_j) L^{-1}(u_j^p)$.

Now $E(u_j)$ is bounded since $u_j \in \Omega_\Lambda$, and u_j^p is bounded in $C^{2,\alpha}$

$\Rightarrow L^{-1}(u_j^p)$ is bounded in $C^{4,\alpha}$, but $C^{4,\alpha} \hookrightarrow C^{2,\alpha}$ compactly

$\therefore L^{-1}(u_j^p)$ has a convergent subsequence in $C^{2,\alpha}$.

Thus K is compact.

Degree Theory: If $F_p \neq 0$ on $\partial\Omega_\Lambda$ the $\exists$ integer $\deg(F_p, \Omega_\Lambda, 0) \in \mathbb{Z}$

This is an infinite dim'l generalization of the Browder degree.

$\deg(F_p, \Omega_{\Lambda}, 0)$ is the Leray-Schauder degree (Ref.: Nirenberg, Topics in Nonlinear Functional Analysis; Schwarz, Nonlinear Functional Analysis or Possibly Gilbarg and Trudinger)

The Key Property of the degree is:

Homotopy Invariance: If $F_p \neq 0$ on $\partial\Omega_\Lambda$ $\forall\, p \in [1, \frac{n+2}{n-2}]$ the $\deg_{p,\Lambda}$ is independent of p.

Thus the Main Estimate guaranties the hypothesis of the homotopy invariance, giving us the

Corollary: $\deg_{\frac{n+2}{n-2},\Lambda} = \deg_{1,\Lambda} \quad \forall \Lambda \geq \Lambda_0$

How to compute $\deg_{p,\Lambda}$?

If all zeros of F_p are nondegenerate then the Existence theorem says that there are a finite # of zeros in Ω_Λ; $u_1, u_2, \ldots, u_K$.

$$\deg_{p,\Lambda} = \sum_{i=1}^{K} \left(\text{local degree of } F_p \text{ at } u_i \ (=\pm 1)\right)$$

How to compute the local degree? Take the linearized operator:

$$\mathcal{L}_i = I + T_i \quad , \quad \mathcal{L}_i \eta = \frac{d}{dt} F_p(u_i + t\eta)\Big|_{t=0}$$

let η_i = # of negative eigenvalues of $\mathcal{L}_i$

= # of eigenvalues of T_i less than 1

(Note: Nondegenerate means that $\mathcal{L}_i$ is non-singular i.e. invertible so 0 is not an eigenvalue of $\mathcal{L}_i$)

T_i has discrete spectrum tending to zero (which may be an accumulation point) $\therefore$ $\exists$ only finitely many eigenvalues ≤ 1.

$$\boxed{\deg_{p,\Lambda} = \sum_{i=1}^{K} (-1)^{\eta_i}}$$

Compute $\deg_{1,\Lambda}$ $(p=1)$

$F_1(u)$ is linear if we think of $E(u)$ as a constant, but in reality the map is non-linear.

$$F_1(u) = u + E(u)\, L^{-1}(u) = 0$$

or $\quad Lu + E(u)u = 0 \quad$ i.e. u is an eigenfunction of L

Multiply by u and integrate $\Rightarrow \int u^2 = 1$

$\therefore$ $\exists$ exactly one solution, u_1 which is the unique positive eigenfunction with L^2-norm 1.

We linearize about u_1:

$$\mathcal{L}_1 \eta = \frac{d}{dt} F_p(u_1 + t\eta)\Big|_{t=0} = \frac{d}{dt}\left(u_1 + t\eta + E(u_1+t\eta)\, L^{-1}(u_1+t\eta)\right)\Big|_{t=0}$$

$$= \eta + \frac{d}{dt}\left(E(u_1+t\eta)\, L^{-1}(u_1+t\eta)\right)\Big|_{t=0}$$

$$= \eta - 2\left(\int (Lu_1)\,\eta\, dw_{g_0}\right) L^{-1}(u_1) + E(u_1)\, L^{-1}(\eta)$$

(using Liebnitz and integration by parts).

using The equation satisfied by u_1:

$$\mathcal{L}_1 \eta = \eta + 2\, E(u_1) \langle u_1, \eta\rangle_{L^2} \left(-\frac{u_1}{E(u_1)}\right) + E(u_1)\, L^{-1}(\eta)$$

$$= \eta - 2 \langle u_1, \eta\rangle_{L^2}\, u_1 + E(u_1)\, L^{-1}(\eta)$$

Say η is an eigenvector for $\mathcal{L}_1$ with eigenvalue λ; $\mathcal{L}_1 \eta = \lambda \eta$

$$\eta - 2\langle u_1, \eta\rangle_{L^2}\, u_1 + E(u_1)\, L^{-1}(\eta) = \lambda\eta$$

Apply L to the equation, noting $Lu_1 = -E(u_1)\, u_1$

$$L\eta + 2\langle u_1, \eta\rangle_{L^2} E(u_1)\, u_1 + E(u_1)\,\eta = \lambda L\eta$$

$$(1-\lambda) L\eta + 2\langle u_1, \eta\rangle_{L^2} E(u_1)\, u_1 + E(u_1)\eta = 0$$

To establish nondegeneracy (0 not an eigenvalue), set $\lambda = 0$

$$\langle u_1, L\eta + 2\langle u_1, \eta\rangle_{L^2} E(u_1)\, u_1 + E(u_1)\,\eta\rangle_{L^2} = 0$$

$$\Rightarrow -E(u_1)\langle u_1, \eta\rangle_{L^2} + 2E(u_1)\langle u_1, \eta\rangle_{L^2} + E(u_1)\langle u_1, \eta\rangle_{L^2} = 0$$

$$\Rightarrow \qquad 2E(u_1)\langle u_1, \eta\rangle_{L^2} = 0$$

$$\Rightarrow \qquad \langle u_1, \eta\rangle = 0$$

[More generally, without assuming $\lambda = 0$ we get $(\cancel{\alpha}(\lambda - 1) + 2 + 1)\, E(u_1)\langle u_1, \eta\rangle = 0$
$\Rightarrow$ for $\lambda \neq -2$, $\langle u_1, \eta\rangle = 0$].

$\therefore$ For $\lambda \neq -2$ we have $(1-\lambda) L\eta + E(u_1)\,\eta = 0$

So $\lambda = 0 \Rightarrow \eta = u_1$ but this is impossible we ~~took~~ showed $\eta \perp u_1$.

$\therefore \eta = 0$. This checks nondegeneracy

To count # of negative eigenvalues of $\mathcal{L}$.

If $\lambda < 0$ $\lambda \neq -2$, by above

$$L\eta + \frac{E(u_1)}{1-\lambda}\,\eta = 0$$

$\frac{E(u_1)}{1-\lambda} < E(u_1)$ since $\lambda < 0$ $\Rightarrow$ $\eta = 0$, since u_1 is the lowest eigenfunction corresponding to lowest eigenvalue $E(u_1)$.

$\therefore$ Only possible eigenvalue is $\lambda = -2$.

$\lambda = -2 \Rightarrow \quad 3L\eta + 2\langle u_1, \eta\rangle E(u_1)\, u_1 + E(u_1)\,\eta = 0$

Now $\eta = u_1$ is a solution $\Rightarrow$ $\dim \text{Eigenspace}(-2) \geq 1$

If $\dim \text{Eigenspace}(-2) > 1$ we can find an η, $\eta \perp u_1$ which is a solution $\Rightarrow$ $L\eta + \frac{E(u_1)}{3}\eta = 0 \Rightarrow \eta = 0$

$\therefore$ $\dim \text{Eigenspace}(-2) = 1$

$$\therefore \deg_{1,\Lambda} = -1.$$

Geometric case $p = \frac{n+2}{n-2}$:

1st consider how $F(u)$ changes under a conformal change of metric $g_1 = v^{4/n-2} g_0$, $v > 0$.

Claim: $F^{g_1}(u) = v^{-1} F^{g_0}(vu)$ [$\therefore F^{g_1} = (M_v)^{-1} \circ F^{g_0} \circ M_v$ so that solutions are trivially related, where M_v is the operator: $M_v(u) = vu$]

This follows from the conformal invariance of the conformal Laplacian:

$$L_{g_0}(v\varphi) = v^{\frac{n+2}{n-2}} L_{g_1}(\varphi), \quad \forall \varphi \in C^\infty(M)$$

Proof: 1st assume $\varphi > 0$, the general case can be handled by writing $\varphi = \varphi_1 - c$, where $\varphi_1 > 0$, c a positive constant, and making use of the linearity of L.

Write $g = \varphi^{4/n-2} g_1 = (v\varphi)^{4/n-2} g_0$ so

$$R(g) = -c(n)^{-1} L_{g_1}(\varphi)\,\varphi^{-\frac{n+2}{n-2}}$$

$$R(g) = -c(n)^{-1} L_{g_0}(v\varphi)\,(v\varphi)^{-\frac{n+2}{n-2}}$$

Equating the two r.h.s's gives the conformal invariance ■

Proof of claim:

$$F^{g_1}(u) = u + E_{g_1}(u)\, L_{g_1}^{-1}\left(u^{\frac{n+2}{n-2}}\right)$$

Let $f = L^{-1}_{g_1}\left(u^{\frac{n+2}{n-2}}\right)$

$u^{\frac{n+2}{n-2}} = L_{g_1}(f) = v^{-\frac{n+2}{n-2}} L_{g_0}(vf)$ by conformal invariance.

$\therefore\ L_{g_0}(vf) = (uv)^{\frac{n+2}{n-2}} \Rightarrow f = v^{-1} L_{g_0}^{-1}\left((uv)^{\frac{n+2}{n-2}}\right)$

Now $E_{g_1}(u) = c(n)\mathcal{R}(u^{4/n-2} g_1) = c(n)\mathcal{R}((uv)^{4/n-2} g_0) = E_{g_0}(uv)$

$\therefore\ F^{g_1}(u) = u + E_{g_0}(uv)v^{-1} L_{g_0}^{-1}((uv)^{\frac{n+2}{n-2}})$

$= v^{-1} F^{g_0}(uv)$ ■

To Compute $\deg_{\frac{n+2}{n-2}, \Lambda}$ ...... (We assume Non-degeneracy)

say $g \in [g_0]_1$ is a nondegenerate solution

We view g as the backround metric (This does not change the degree)

$\therefore\ u \equiv 1$

Our map is $F^g(u) = u + E_g(u) L_g^{-1}(u^{\frac{n+2}{n-2}})$

1 is a solution ie. $F^g(1) = 0$, We'll linearize about 1

$$\mathcal{L}\eta = \eta + \frac{n+2}{n-2} E_g(1) L_g^{-1}(\eta) - 2\langle L_g(1), \eta\rangle_{L^2} L_g^{-1}(1)$$

$$E_g(1) = c(n)\int_M R(g)\,d\omega_g = c(n)R(g) = \frac{n-2}{4(n-1)} R(g)$$

and $L_g(1) = -c(n)R(g) = -\frac{n-2}{4(n-1)} R(g)$

$\therefore\ L_g^{-1}(1) = -\frac{4(n-1)}{n-2} R(g)^{-1}$

So $\mathcal{L}\eta = \eta + \frac{n+2}{4(n-1)} R(g) L_g^{-1}(\eta) - 2\int_M \eta\,d\omega_g$

What is the spectrum, say $\mathcal{L}\eta = \lambda\eta$

$$L_g(\eta) + \frac{n+2}{4(n-1)} R(g)\eta - 2\int_M \eta\,d\omega_g\, L_g(1) = \lambda L_g(\eta)$$

$$(1-\lambda) L_g(\eta) + \frac{n+2}{4(n-1)} R(g)\eta = -\frac{n-2}{4(n-1)} R(g)\int_M \eta\,d\omega_g$$

Integrate both sides $\left(L_g(\eta) = \Delta_g\eta - \frac{n-2}{4(n-1)} R(g)\eta\right)$

$$\left[-(1-\lambda)\frac{n-2}{4(n-1)} + \frac{n+2}{4(n-1)} + \frac{n-2}{4(n-1)}\right] R(g)\int \eta\,d\omega_g = 0$$

$$[(\lambda-1)(n-2) + n+2 + n-2]\, R(g)\int \eta\,d\omega_g = 0$$

$[(n-2)\lambda + n+2] R(g) \int_M \eta \, d\omega_g = 0$

So for $\lambda \neq -\frac{n+2}{n-2}$ we have $\int \eta \, d\omega_g = 0$

So $(1-\lambda) L_g \eta + \frac{n+2}{4(n-1)} R(g) \eta = 0$

i.e. $\Delta_g \eta + \left[-1 + \frac{\frac{n+2}{n-2}}{1-\lambda}\right] c(n) R(g) \eta = 0$

$$\Delta_g \eta + \left[\frac{\lambda - 4/n-2}{1-\lambda}\right] c(n) R(g) \eta = 0$$

Then $\frac{\lambda - 4/n-2}{1-\lambda} c(n) R(g)$ is an eigenvalue of Δ_g.

let $\mu = \frac{\lambda - 4/n-2}{1-\lambda} c(n) R(g)$, we want to show $\mu \in (0, \frac{R(g)}{n-1})$ and that the # of $\mu \in (0, R(g)/n-1)$ which are eigenvalues of Δ_g is the Morse index $+1$, of the constrained variational problem.

11/8/88

We step back, regroup and start over.

(M, g) g considered as backround metric and a critical point.

$R(g) \equiv$ constant

So $u \equiv 1$ is a solution of $F^g_{(u)} = u + E(n) L^{-1}(u^{\frac{n+2}{n-2}}) = 0$.

Compute the linearization about 1, $u + t\eta$ our variation

<u>2 cases</u>

) $\int_M \eta \, d\omega_g = 0$, then $\frac{d}{dt} E(u+t\eta)\Big|_{t=0} = 2c(n) \int_M R(g)\, u\, \eta \, d\omega_g + 2\int_M \eta \Delta u \, d\omega_g$

$u \equiv 1$

$$= 2c(n) R(g) \int_M \eta \, d\omega_g = 0$$

So $\mathcal{L}\eta = \eta + \frac{n+2}{n-2} c(n) R(g) L^{-1}(\eta)$

We want to study the negative eigenvalues, i.e. $\mathcal{L}\eta = \lambda \eta$

Apply L to both sides

$$L(\eta) + \frac{n+2}{n-2}\, c(n) R(g)\, \eta = \lambda L(\eta)$$

$$\Delta_g \eta + 4/n-2\; c(n) R(g)\, \eta = \lambda(\Delta_g \eta - c(n) R(g)\, \eta)$$

$$\Delta_g \eta + \left(\frac{\lambda + 4/n-2}{1-\lambda}\right) c(n) R(g)\, \eta = 0$$

So $\mu = \left(\frac{\lambda + 4/n-2}{1-\lambda}\right) c(n) R(g)$ is an eigenvalue of $-\Delta_g$

μ is increasing for $\lambda < 0$, $\lambda \neq -4/n-2$.
for $u = -4/n-2$, the constants are eigenfunctions, but $\int_M \eta = 0 \therefore \eta = 0$.
So since $\mathrm{Spec}(-\Delta_g) \geq 0$ we have

$$\mu \in (0, 4/n-2\; c(n) R(g)) = (0, R(g)/n-1)$$

i) $\eta \equiv 1$ is an eigenfunction of $\mathcal{L}$, i.e. $\mathcal{L}(1) = 1 - \frac{n+2}{n-2} < 0$
his give a one dimensional eigenspace (the constants), so any other eigenvector is orthogonal (in $L^2(M)$) to the constants i.e. $\int_M \eta = 0$ and we are in the case (i).

Thus we conclude: The dimension of the negative eigenspaces of $\mathcal{L} = 1 + \{\#$ of eigenvalues of $-\Delta_g$ in $(0, \frac{R(g)}{n-1})$.

Hence equating $\deg_{1,\Lambda}$, $\deg_{\frac{n+2}{n-2},\Lambda}$: $-1 = -\sum_{b \neq} (-1)^{\eta_i}$ $\eta_i = 1$, eigenvalues of $-\Delta$ on $(0, \frac{R(g)}{n-1}]$

__Variational formulation__ : The relevent # is the Morse Index.
g backround metric, we extremize $c(n) R(u^{4/n-2} g) = E(u)$, subject to the volume 1 constraint $\int_M u^{2n/n-2}\, dw_g = 1$

We consider $g(t) = u(t)^{4/n-2} g$, $\int_M u(t)^{2n/n-2}\, dw_g = 1$

Differentiating the constrait $\Rightarrow \int_M u(t)^{\frac{n+2}{n-2}}\, \dot{u}\, dw_g \overset{t=0}{\Longrightarrow} \int_M \dot{u}\, dw_g = 0$

Differentiating a second time and evaluating at $t=0$ $(u=1)$:

$$\frac{n+2}{n-2}\int_M (\mathring{u})^2\, d\omega_g + \int_M \ddot{u}\, d\omega_g = 0. \qquad ②$$

Compute derivatives of the functional:

$$E(u) = \int_M (|\nabla u|^2 + c(n) R(g) u^2)\, d\omega_g$$

$$\dot{E}(u) = 2\int_M (\langle \nabla u, \nabla \mathring{u} \rangle + c(n) R(g) u \mathring{u})\, d\omega_g$$

$$\ddot{E}(u)\Big|_{t=0} = 2\int_M (|\nabla \mathring{u}|^2 + c(n) R(g)(\mathring{u})^2 + c(n) R_g \ddot{u})\, d\omega_g$$

By ② above: $\quad = 2\int_M \left((|\nabla \mathring{u}|)^2 - \frac{R(g)}{n-1}(\mathring{u})^2\right) d\omega_g = -2\int_M \mathring{u}\, \hat{\mathcal{L}} \mathring{u}\, d\omega_g$

where $\hat{\mathcal{L}}\mathring{u} = \Delta \mathring{u} + \frac{R(g)}{n-1}\mathring{u}$ with constraint $\int_M \mathring{u}\, d\omega_g = 0$

The Morse index is the # of negative eigenvalues of $-\hat{\mathcal{L}}\mathring{u}$.

So $\operatorname{Ind}(g)$ = Morse Index of g = # of eigenvalues of $-\Delta_g$ in $\left(0, \frac{R(g)}{n-1}\right)$.

$\therefore$ The local degree of F at $g = (-1)^{1+\operatorname{Ind}(g)}$

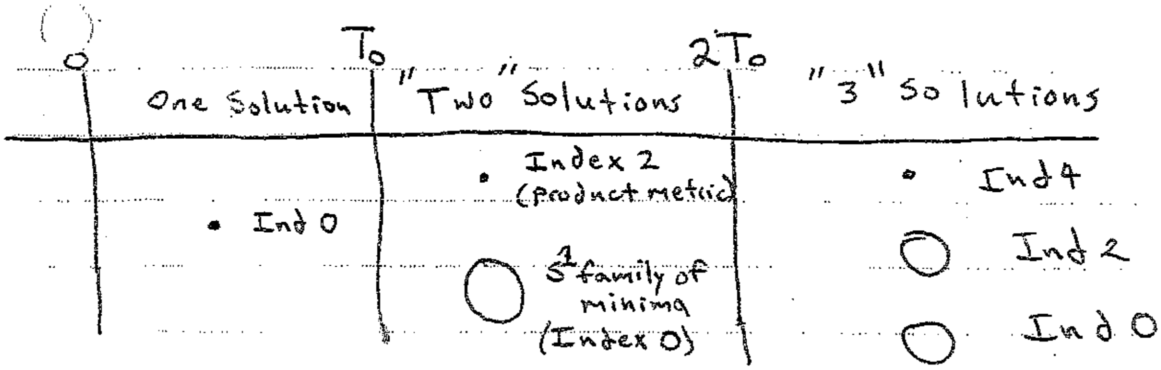
<u>Corollary from last time</u>: $\deg(F^{g_0}, \Omega_\Lambda, 0) = -1$ for $\Lambda > \Lambda$.

If the critical points of F are nondegenerate, $g_1, \dots, g_K$

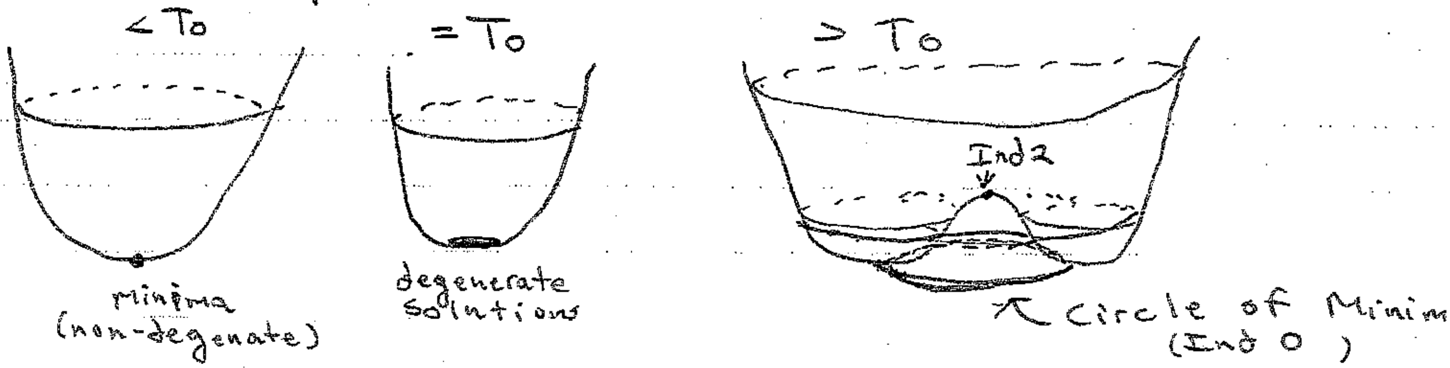
$$\sum_{i=1}^{K} (-1)^{1+\operatorname{Ind}(g_i)} = -1 \implies \sum_{i=1}^{K} (-1)^{\operatorname{Ind}(g_i)} = 1.$$

Back to our concrete example: $M_T = S^1(T) \times S^{n-1}(1)$.

We can give in nice picture (finite dimensional analog) of the bifurcation of solution as T goes past the integer multiples of the critical value T_0.



Think of a proper function on $\mathbb{R}^n$:



Σ nondegenerate critical manifold of Morse index k, the local degree at $\Sigma = (-1)^k \chi(\Sigma)$

<u>Back to the Main Estimate</u>. (To Prove:)

<u>Recall</u>: $p \in [1, \frac{n+2}{n-2}]$, $\exists \Lambda_0 = \Lambda_0(g_0)$ s.t. $F_p^{-1}(0) \subset \Omega_{\Lambda_0}$
$\forall p \in [1, \frac{n+2}{n-2}]$, Provided $(M, g_0) \not\cong (S^n, \text{stand.})$

1^{ST}: <u>Subcritical p case</u>: (Get an estimate independent of Energy b
(Due to Gidas - Spruck)

<u>Model Equations</u> : $(*)_p$: $\Delta u + u^p = 0$ on $\mathbb{R}^n$, $u > 0$, $1 \leq p <$

<u>Theorem</u>: If $p \in [1, \frac{n+2}{n-2})$, then no entire solutions exist.

[Note: When $p = \frac{n+2}{n-2}$, all entire solutions extend to S^n where we Know all solutions explicitely].

For <u>$p = 1$</u>, easy case.

$$\Delta u + u = 0$$

claim: This implies $\lambda_0(\Omega) > 1$. say u_0 is the positive first eigenfunction $-\Delta u_0 = \lambda_0 u_0$, $u_0 > 0$, $\frac{\partial u_0}{\partial \eta} < 0$ since $u_0|_{\partial\Omega} = 0$ where η = unit outward pointing normal.

$$\int_\Omega u_0 \Delta u - u \Delta u_0 = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} u_0 - \frac{\partial u_0}{\partial \eta} u \qquad \forall\, u \in C^2(\Omega)$$

$$\int_\Omega u_0 \Delta u + \lambda_0 u u_0 = \int_{\partial\Omega} -\frac{\partial u_0}{\partial \eta} u \qquad \text{let } u \text{ be our } p=1 \text{ solution.}$$

$$\Rightarrow (\lambda_0 - 1)\int_\Omega u u_0 = -\int_{\partial\Omega} \frac{\partial u_0}{\partial \eta} u > 0 \quad \Rightarrow \lambda_0 - 1 > 0 \quad \lambda_0 > 1$$

But if $\Omega = B_R(0)$ and $R \to \infty$ then $\lambda_0 \searrow 0$ a contradiction
so $\nexists$ entire solutions of $\Delta u + u = 0$, $u > 0$

For $p \in (1, \frac{n+2}{n-2})$, observe that the equations $(*)_p$ have a natural invariance under similarities: $x \mapsto \lambda A x + b$, $\lambda \in \mathbb{R}^+$, $b \in \mathbb{R}^n$, $A \in O(n)$.

check dilation invariance:

Say u is a solution of $(*)_p$, then $u_\lambda(x) = \lambda^{2/p-1} u(\lambda x)$ is also a solution. Since

$$\Delta(u_\lambda) = \lambda^{2/p-1+2} \Delta u(\lambda x)$$
$$u_\lambda^p = \lambda^{2p/p-1} u^p(\lambda x)$$
$$0 \Rightarrow 0$$

We want to exploit the non-invariance of the subcritical equations under the full conformal group of S^n (similarities fix ∞).

We compute the change under an inversion $x \to \frac{x}{|x|^2} = y$

$$R(u^{4/n-2}\delta_{ij}) = -c(n)^{-1} u^{-\frac{n+2}{n-2}} \Delta u = + c(n)^{-1} u^{-\delta}$$

where $\delta = \frac{n+2}{n-2} - p$.

$$u^{4/n-2} \sum (dx^i)^2 = |y|^{-4} u(x)^{4/n-2} \sum (dy^i)^2 = v(y)^{4/n-2} \sum (dy^i)^2$$

where $v(y) = |y|^{2-n} u\left(\frac{y}{|y|^2}\right)$

so that near ∞ (i.e. for $y \gg 0$) $v \simeq |y|^{2-n}\left(a + \frac{b \cdot y}{|y|^2} + O(|y|^{-

So we also have $R(u^{4/n-2}\delta_{ij}) = -c(n)^{-1}\dfrac{\Delta v}{v^{\frac{n+2}{n-2}}}$

or $\Delta v + u^{-\delta} v^{\frac{n+2}{n-2}} = 0$ on $\mathbb{R}^n \setminus \{0\}$

$\therefore$ v satisfies

$$\Delta v + |y|^{-\delta(n-2)} v^p = 0 \quad \text{on } \mathbb{R}^n \setminus \{0\}$$

So for $p \neq \frac{n+2}{n-2}$ ($\delta \neq 0$) the equation is not symmetric about any plane except for planes passing through the origin. An application of the Alexandroff reflection principle can show that $\exists$ a plane of symmetry, by the above this is necessarily through the origin. We actually get symmetry about any plane through the origin $\Rightarrow$ 0 is a regular ~~value~~ point $\Rightarrow$ the solution is symmetric about 2 points, which is a contradiction. $\therefore$ No entire solutions.

Note: We've glossed over a detail in the above and similar in the proof of the upper bound for solutions on $\Omega \subset S^n$. Namely we need:

Prop. $\Delta u + |y|^{-\delta(n-2)} v^p \leq 0$ on $\mathbb{R}^n$ distributionally

and $\displaystyle\int_{B_1(0)} |y|^{-\delta(n-2)} v^p \, dy < \infty$

Exercise break it up, i.e. by parts, $\mathbb{R}^n = A \cup B$, $A = \{v \leq a\}$, $B = \{v$

Proof: For a large number $a>0$, let $v_a(y) = \min\{v(y), a\}$

$\Delta v_a \leq (\Delta v)\chi_{\{v \leq a\}}$ distributionally away from the origin.

$\Rightarrow$ $\Delta v_a \leq -|y|^{-\delta(n-2)} v^p \chi_{\{v \leq a\}}$

Claim: This inequality holds weakly on $\mathbb{R}^n$.

say $\varphi \in C^\infty$ compact support, $\varphi \geq 0$

want to show $\displaystyle\int_{\mathbb{R}^n} (\Delta\varphi) v_a \leq -\int_{\mathbb{R}^n} |y|^{-\delta(n-2)} v^p \chi_{\{v \leq a\}} \varphi$

Let $\xi_\varepsilon(|x|) = \begin{cases} 0 & \text{for } |x| < \varepsilon/2 \\ 1 & \text{for } |x| \geq \varepsilon \end{cases}$, and $|\nabla \xi_\varepsilon| < C/\varepsilon$, $|\nabla\nabla \xi_\varepsilon| \leq C/\varepsilon^2$

then $\varphi \xi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and is supported away from zero, so by the weak inequality away from 0, we have:

$$\int_{\mathbb{R}^n} \Delta(\varphi \xi_\varepsilon) v_a \leq -\int_{\mathbb{R}^n} |y|^{-\delta(n-2)} v^p \chi_{\{v \leq a\}} \varphi \xi_\varepsilon$$

Now $\Delta(\varphi \xi_\varepsilon) = \xi_\varepsilon \Delta\varphi + 2\nabla\varphi \nabla\xi_\varepsilon + \varphi \Delta \xi_\varepsilon$

So

$$\int_{\mathbb{R}^n} \xi_\varepsilon \Delta\varphi \cdot v_a + 2\int_{\mathbb{R}^n} \nabla\varphi \nabla\xi_\varepsilon v_a + \int_{\mathbb{R}^n} \varphi \Delta\xi_\varepsilon \cdot v_a + \int_{\mathbb{R}^n} |y|^{-\delta(n-2)} v^p \chi_{\{v \leq a\}} \varphi \xi_\varepsilon \leq$$

We have v_a is bounded above by and the bounds on derivatives ξ_ε above, this combined with the observation that the two middle integrands are supported on annuli of measure $\sim c\varepsilon$ allows us to let $\varepsilon \to 0$ giving:

$$\int_{\mathbb{R}^n} \Delta\varphi \cdot v_a + |y|^{-\delta(n-2)} v^p \chi_{\{v \leq a\}} \varphi \leq 0 \qquad *$$

Take $\varphi \equiv 1$ near $y = 0$ $\Rightarrow$ $\Delta\varphi \equiv 0$ near $y = 0$

$$\Rightarrow \int_{B_1(0)} |y|^{-\delta(n-2)} \chi_{\{v \leq a\}} v^p \, dy \leq -\int_{\mathbb{R}^n \setminus B_1(0)} \Delta\varphi \cdot v_a + |y|^{-\delta(n-2)} v^p \chi_{\{v \leq a\}} \varphi$$

$$\leq C$$

$\Rightarrow$ letting $a \to \infty$ and using the Monotone convergence theorem.

$$\int_{B_1(0)} |y|^{-\delta(n-2)} v^p \, dy < \infty, \text{ hence } v \in L^1_{loc}(\mathbb{R}^n).$$

Let $a \to \infty$ in $*$, and apply the dominated convergence theorem to conclude that

$$\Delta v + |y|^{-\delta(n-2)} v^p \leq 0 \text{ on } \mathbb{R}^n, \text{ distributionally.}$$

■

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<u>Proposition</u>: There is no solution of $\Delta u + u^p = 0$, $u > 0$ on $\mathbb{R}^n$ for $p \in [1, \frac{n+2}{n-2})$.

<u>Proof</u>: $\Delta u + u^p = 0$, let $v(x) = |x|^{2-n} u(\frac{x}{|x|^2}) = |x|^{2-n}(a + b\cdot(\frac{x}{|x|^2}) + C$

(Note $a > 0$, and by scaling u we may assume $a \equiv 1$)

$\Delta v + |x|^{-\delta(n-2)} v^p = 0$, v smooth on $\mathbb{R}^n \setminus \{0\}$, $\delta = \frac{n+2}{n-2} - p >$

We've previously shown $\Delta v + |x|^{-\delta(n-2)} v \leq 0$ weakly on $\mathbb{R}^n$.

$\Rightarrow \Delta v \leq 0$ weakly on $\mathbb{R}^n$

i.e. v is superharmonic. We thus have the mean value inequ

$v(x) \geq \fint_{B_1(x)} v \quad \forall x \in \mathbb{R}^n$, in particular $v(0) \geq \fint_{B_1(0)} v > 0$

so $v(0) > 0$.

W want to apply the Alexandroff reflection principle:

Choose any direction, say the x_1-coordinate direction.

let $\Pi_\lambda = \{x : x_1 = \lambda\}$, $v^*(x) = v(2\lambda - x_1, x')$ for $\{x : x_1 \leq \lambda$

To get started we need to show that for λ sufficiently lar

$v^*(x) \leq v(x)$ for $\{x : x_1 \leq \lambda\}$.

We do this first for $\{x : x_1 \leq \lambda, |x| > R\}$ by computing the asymtotics of $v^*(x)$ and comparing them to the given asymtotics of $v(x)$.

$$v^*(x) = |x^*|^{2-n}\left(a + \frac{b\cdot x^*}{|x^*|^2} + O(|x^*|^{-2})\right)$$

Now

(i) $$|x^*|^2 = |x|^2\left(1 + \frac{4\lambda(\lambda - x_1)}{|x|^2}\right)$$

so (ii) $$|x^*|^{2-n} = |x|^{2-n}\left(1 + \frac{4\lambda(\lambda - x_1)}{|x|^2}\right)^{\frac{2-n}{2}}$$

$$= |x|^{2-n}\left(1 + \frac{2(2-n)\lambda(\lambda - x_1)}{|x|^2} + O(x^{-2})\right)$$

(iii) $|x^*|^{-2} = |x|^{-2}\left(1 - \frac{4\lambda(\lambda - x_1)}{|x|^2} + O(|x|^{-2})\right)$

(iv) $b\cdot x^* = b\cdot(2\lambda - x_1, x') = b\cdot x + 2b_1(\lambda - x_1)$

where $b = (b_1, b_2, \dots, b_n)$

So $\frac{b\cdot x^*}{|x^*|^2} = \frac{b\cdot x}{|x|^2} + \frac{2b(\lambda - x_1)}{|x|^2} + O(|x|^{-2})$

Assume $a = 1$, i.e. $v^* = |x^*|^{2-n}\left(1 + \frac{b\cdot x^*}{|x^*|^2} + O(|x^*|^{-2})\right)$

so

$v^*(x) = |x|^{2-n}\left(1 + \frac{b\cdot x}{|x|^2} + \frac{2b_1(\lambda - x_1)}{|x|^2} + \frac{2(2-n)\lambda(\lambda - x_1)}{|x|^2} + O(|x|^{-2})\right)$

These expansions holds for $|x| \geq R >> 0$ and $\{x : x_1 < \lambda\}$

So for $v^*(x) \leq v(x)$ in this region we must have.

$b_1(\lambda - x_1) + (2-n)\lambda(\lambda - x_1) < 0 \iff (\lambda - x_1)(b_1 + (2-n)\lambda) < 0$

Now $\lambda - x_1 > 0$ so we want $b_1 + \lambda(2-n) < 0$

or $\lambda > -b_1/2-n$.

Now for $|x| \leq R$, by the mean-value inequality we have $\inf_{B_R(0)} v > 0$, but by choosing λ sufficiently large, we see using the expansion above that we can make $v^*(x) = v(x^*)$ arbitrarily small (since given any $N >> 0$ by choosing λ large enough we can guarantee that $x^* \in \{x : |x| >$ $\forall x \in B_R(0)$, and $v(x) \longrightarrow 0$ as $x \longrightarrow \infty$). So in partic. we can ensure that $\sup_{x\in B_R(0)} v^*(x) < \inf_{x\in B_R(0)} v(x)$

$\therefore \exists\, \lambda$ - such that $v^*(x) \leq v(x)$ for $\{x : x_1 \leq \lambda\}$

The set of $\lambda > 0$?λ where $v^*_\lambda(x) \leq v(x)$ or $v^*_\lambda(x) \equiv v(x)$ is non-empty (by the above) and open (the condition is open, i.e. the inequality is strict and the solution v decays at ∞).

To show the set is closed: We take a limit of planes getting $v^* \le v$ at some point. $(x: x_1 < \lambda)$

Now $\Delta(v - v^*) = |x^*|^{-\delta(n-2)}(v^*)^p - |x|^{-\delta(n-2)}(v)^p$

and $|x^*| > |x|$ so $|x^*|^{-\delta(n-2)} < |x|^{-\delta(n-2)}$

$\therefore \Delta(v - v^*) \le |x^*|^{-\delta(n-2)}[(v^*)^p - (v)^p]$

$\Rightarrow \Delta(v - v^*) \le 0$ since $p \ge 1$ and $v^* \le v$

Now by the mean value inequality

$$(v - v^*)(x) \ge \fint_{B_1(x)} (v - v^*) > 0 \cdot \text{ strictly}$$

$\therefore$ The Maximum principle is applicable

i.e. either $v^* < v$ strictly or $v^* \equiv v$

$\therefore$ we must get some plane of symmetry (i.e. $\exists \lambda$ s.t. $v^* \equiv v$) else by letting $\lambda \to -\infty$ we conclude that $v \equiv 0$.

The plane of symmetry must be Π_0 $(\lambda = 0)$ since the equation is not invariant under the inversion.

Our choice of direction was completely arbitrary, so w conclude $v(x) \equiv v(|x|)$ i.e. v is rotationally symmetric. This implies our original $u = u(|x|)$, is rotationally symmetric.

Finally we recall, that we could have inverted abou some other origin, concluding that u is rotational symmetric about two points $\Rightarrow$ u is constant.

$\Rightarrow u \equiv 0$.

Q.E.D ■

<u>Proposition</u> : (M, g_0) M compact, For $p \in [1, p_0]$, $p_0 < \frac{n+2}{n-2}$
$\exists \Lambda_0 = \Lambda(p_0)$ such that $F_p^{-1}(0) \subset \Omega_{\Lambda_0}$

$(*)_p$: $Lu + E(u) u^p = 0$, $u > 0$; $\Rightarrow \int u^{p+1} = 1$

The following arguement "blows up" as $p \nearrow \frac{n+2}{n-2}$, linear estimates can give the result for $p=1$ and for p slightly greater than 1.

<u>Proof</u> :

<u>Case(i)</u> : $p \in [1, p_1]$ where $p_1 < n/n-2$.

Think of the equation as a perturbation of a linear equation :
$Lu + E(u) u = 0$ (regarding $E(u)$ as a constant.)

Now $Lu = \Delta u - c(n) R(g_0) u$

so integrating $(*)_p \Rightarrow c(n) \int_M R(g_0) u = E(u) \int_M u^p$

so $E(u) \int_M u^p \le c_1 \int_M u \le c_2 \left(\int_M u^p \right)^{1/p}$, $c_1 = c(n) \cdot \max_M R(g_0)$
$c_2 = c_1 \cdot (\mathrm{Vol}(M))^{1/q}$

$\therefore$ $E(u) \left(\int_M u^p \right)^{\frac{p-1}{p}} \le C$ i.e. $E(u) \| u^{p-1} \|_{L^{p/p-1}} \le C$.

Once we have a solution u, we can think of $V = E(u) u^{p-1}$, a given function. This gives the linear equation $Lu + Vu = 0$

so $E(u) \| u^{p-1} \|_{L^{p/p-1}} = \| V \|_{L^{p/p-1}} \le C$

Now There is a Harnack inequality for u with constant dependi[ng]
on $\| V \|_{L^q(M)}$, $q > n/2$; $p/p-1 > n/2 \Leftrightarrow p < n/n-2$

(See Gilbarg-Trudinger §8.6 and §8.5)

$\Rightarrow$ u has upper and lower bounds : $\Lambda^{-1} \le u \le \Lambda$ for some Λ.

$\Rightarrow$ $E(u)$ is bounded ($E(u) \le C \| u^{p-1} \|^{-1}_{L^{p/p-1}}$), so V is bound[ed]

$\therefore$ applying linear P.D.E. estimates to $Lu + Vu = 0$

we get $\| u \|_{C^{2,\alpha}(M)} \le C$, ~~iterating once give the result~~

Note : A similar argument gives a regularity therem for we solutions. This is not true for $p \geq n/n-2$. In general for thi nonlinear equations estimates on solutions do not convert into regularity theorems, this is also the case for the Harmonic map problem.

Case (ii) : Fix $p_1 \in (1, n/n-2)$ and we need only consider $p \in [p_1, p_0]$

Replace $(*)_p$ by $(*)_{p}'$ $Lu + u^p = 0$ by rescaling u

Assume the estimate is not true, i.e. suppose $\exists \{u_i\}$ solutions of $(*)_{p_i}$, $p_i \in [p_1, p_0]$ where $\max_M \bar{u}_i \longrightarrow \infty$, $\bar{u}_i = u_i(x_i)$

(or else, if u_i bounded we can apply the linear theory as above) get a suitable estimate).

$\{x_i\} \in M$ are the maximum points of u_i.

For each i choose normal coordinates $y^1, \dots, y^n$ centered at x_i (we surpress the dependence on i).

so $(g_0)_{ij} = \delta_{ij} + O(|y|^2)$.

Take a dilational scaling $v_i(y) = (\bar{u}_i)^{-1} u_i\left(\frac{y}{\bar{u}_i^{\frac{p-1}{2}}}\right)$ (here w use the fact that p is away from 1). $v_i(0) = 1$

By passing to a subsequence, we may assume $p_i \to p$.

we have $L_{g_i} v_i + v_i^{p_i} = 0$, $g_i \Longrightarrow \delta$.

As in our blow up argument on p. 36-37, we may use Harnack inequality and linear theory to get

$v_i \Rightarrow v$, C^∞ convergence on compact subsets of $\mathbb{R}^n$. $\therefore$ $\Delta v + v^p = 0$ on $\mathbb{R}^n$

which is a contradiction with the previous propositio (p. 52). ∎

<u>Proposition</u>: (M, g_0) M compact, For $p \in [1, p_0]$, $p_0 < \frac{n+2}{n-2}$
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<u>roof</u>:

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$c_2 = c_1 \cdot (\mathrm{Vol}(M))^{1/q}$

$\therefore E(u)\left(\int_M u^p\right)^{\frac{p-1}{p}} \le C$ i.e. $E(u)\|u^{p-1}\|_{L^{p/p-1}} \le C$.

Once we have a solution u, we can think of $V = E(u)u^{p-1}$ as a given function. This gives the linear equation $Lu + Vu = 0$

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Now There is a Harnack inequality for u with constant depending on $\|V\|_{L^q(M)}$, $q > n/2$; $p/p-1 > n/2 \Leftrightarrow p < n/n-2$

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$\therefore$ applying linear P.D.E. estimates to $Lu + Vu = 0$

we get $\|u\|_{C^{3,\alpha}(M)} \le C$, ~~iterating an can give the result~~.

Note: A similar argument gives a regularity theorem for weak solutions. This is not true for $p \geq n/n-2$. In general for this nonlinear equations estimates on solutions do not convert into regularity theorems, this is also the case for the Harmonic map problem.

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Replace $(*)_p$ by $(*)_p'$ $Lu + u^p = 0$ by rescaling u

Assume the estimate is not true, i.e. suppose $\exists \{u_i\}$ solutions of $(*)_{p_i}$, $p_i \in [p_1, p_0]$ where $\max_M \bar{u}_i \longrightarrow \infty$, $\bar{u}_i = u_i(x_i)$

(or else, if u_i bounded we can apply the linear theory as above to get a suitable estimate).

$\{x_i\} \in M$ are the maximum points of u_i.

For each i choose normal coordinates $y^1, \ldots, y^n$ centered at x_i.

(We surpress the dependence on i).

so $(g_0)_{ij} = \delta_{ij} + O(|y|^2)$.

Take a dilational scaling $v_i(y) = (\bar{u}_i)^{-1} u_i\left(\frac{y}{\bar{u}_i^{\frac{p-1}{2}}}\right)$ (here we use the fact that p is away from 1). $v_i(0) = 1$

By passing to a subsequence, we may assume $p_i \to p$.

we have $L_{g_i} v_i + v_i^{p_i} = 0$, $g_i \Longrightarrow \delta$.

As in our blow up argument on p. 36-37, we may use a Harnack inequality and linear theory to get

$v_i \Rightarrow v$, C^∞ convergence on compact subsets of $\mathbb{R}^n$. $\therefore$ $\Delta v + v^p = 0$ on $\mathbb{R}^n$

which is a contradiction with the previous proposition (p. 52).

Geometric part: Allow the exponents to approach $\frac{n+2}{n-2}$.

In the previous argument we get a spherical region (bubble) if the solutions blow up, since the exponent always remained subcritical we were able to get a contradiction since no entire solutions of the subcritical equation exist on $\mathbb{R}^n$. In the general case, i.e. for the critical exponent, this sort of bubbling is allowed, so to rule out solutions blowing we must use much more delicate arguments. (Recall that in the conformally flat case this behavior contradicts the convexity of balls in the regular region, thus giving a relatively easy estimate).

Proposition: Given $\varepsilon > 0$, $R >> 0$, $\exists\ C = C(\varepsilon, R)$ such that if u is a solution of $(*)_p : Lu + \bar{c}\, u^p = 0$ (where c is choosen sothat $\bar{v}(x) = (1+|x|^2)^{-\frac{n-2}{2}}$ is an exact solution of $\Delta \bar{v} + c\, \bar{v}^{\frac{n+2}{n-2}} = 0$), for any $p \in [p_1, \frac{n+2}{n-2}]$, with $\max_M u > C$; then $\exists \{x_1, \dots, x_N\} \subset M$, with $N \geq 1$, such that:

— (i) Each x_i is a local max of u and $\{B_{r_i}(x_i)\}_{i=1}^N$ (geodesic balls in the backround metric g_0) with $r_i = R\, u(x_i)^{-\frac{p-1}{2}}$ is a disjoint collection.

— (ii) $|\frac{n+2}{n-2} - p| < \varepsilon$ and if y coordinates are choosen such that $z = \frac{y}{u(x_i)^{\frac{p-1}{2}}}$ is a normal coordinate system centered at x_i, then $\| u(0)^{-1} u\left(\frac{y}{u(0)^{\frac{p-1}{2}}}\right) - \bar{v}(y) \|_{C^2(B_{2R}(0))} < \varepsilon$ where $B_{2R}(0)$ is in the y-coordinates

(i.e. All the $B_{r_i}(x_i) = B_i$ are "spherical regions".)

— (iii) If $x \in M \setminus \bigcup_i^N B_{r_i}(x_i)$ then
$$u(x) \le c\,[\operatorname{dist}(x, \cup B_{r_i}(x_i))]^{-\frac{2}{p-1}}.$$

Before we proceed with the proof of this proposition, we give a Lemma, from which the proposition shall follow easily.

<u>Lemma</u>: $K \subset M$, compact, u a solution of $(*)_p$ on $\overline{M \setminus K}$.
Given $\varepsilon, R > 0$, $\exists\, C = C(\varepsilon, R)$ such that if $\max_{M \setminus K} d(x)^{\frac{2}{p-1}} u(x) \ge C$, where $d(x) = \operatorname{dist}(x, K)$ (if $K = \{\phi\}$, then let $d(x) \equiv 1$ so that this say that u is large), then $\frac{n+2}{n-2} - p < \varepsilon$ (or else we could employ the earlier estimates to get a contradiction) and $\exists\, x_0 \in M \setminus K$, such that x_0 is a local max. of u and $\| u(0)^{-1} u\left(\frac{y}{u(0)^{\frac{p-1}{2}}}\right) - \bar{v}(y) \|_{C^2(B_{2R}(0))} < \varepsilon$ and $B_r(x_0) \subset M \setminus K$ where $r = R\,u(0)^{-\frac{p-1}{2}}$ and y, $B_{2R}(0)$ are as in the prop.

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<u>Proof</u>: Suppose no such C exists.
Then $\exists\ \{u_i\}$, p_i and K_i such that $\max_{M \setminus K_i} d_i(x)^{\frac{2}{p_i-1}} u_i(x) \ge i$ and for each i no such point x_0 exists.

Let $f_i(x) = d_i(x)^{\frac{2}{p_i-1}} u_i(x)$ and let x_i be a maximum point of $f_i(x)$ in $\overline{M \setminus K_i}$.

Let z be normal coordinates relative to the background metric g_0 and centered at x_i

$$y = u_i(x_i)^{\frac{p-1}{2}} z$$

$$w_i(y) = u_i(x_i)^{-1}\, u_i\left(\frac{y}{u_i(x_i)^{\frac{p-1}{2}}}\right)$$

$$w_i(0) = 1$$

The ball of radius $\frac{d(x_i)}{2}$ about x_i in x-coordinates corresponds to a ball of radius $R_i = \frac{d_i}{2} u_i(0)^{\frac{p_i-1}{2}} > \frac{i^{\frac{p_i-1}{2}}}{2}$, about 0 in the y-coordinates.

In this ball the ratio of the max value to the min. value of $d_i(x)$ is bounded by a constant.

So since x_i is a maximum point of f_i in $M \setminus K$ we conclude: $\sup_{B_{R_i}} w_i < C$ for C independent of i.

As in our previous blow up argument (p. 37) we use the Harnack principle and linear P.D.E. theory to show $\exists\ \hat{v}$ s.t.

$$w_i \longrightarrow \hat{v} \quad C^\infty\text{- convergence on compact subsets of } \mathbb{R}^n$$

and since $p_i \to \frac{n+2}{n-2}$, $\hat{v}$ satisfies

$$\Delta \hat{v} + c\hat{v}^{\frac{n+2}{n-2}} = 0.$$

(We fix one R_i and now work in the ball of radius $2R_i$)

Now the maximum point $\hat{y}$ of $\hat{v}$ occurs within a fixed distance of the origin in B_{R_i} (independent of i). The corresponding distance in the x_i coordinates is like $u_i(x_i)^{\frac{1-p_i}{2}}$ i.e. is very small. So since we are working in a ~~fixed~~ large fixed ball in y space, the shifting of the origin to $\hat{y}$, does not effect the argument.

$\therefore$ $\hat{v} \equiv v$ which is the unique solution whose maximum value is 1 at the origin. This clearly contradicts our assumption since $\|w_i - v\|_{C^2(B_{2R_i}(0))} \longrightarrow 0$ as $i \to \infty$

fixed. ∎

<u>Proof of Proposition</u>: Take x_1 to be maximum point of u.

Applying the lemma with $K = \{\emptyset\}$ we get $B_{r_1}(x_1)$ satisfying properties (i) and (ii).

Now let $K = \overline{B_{r_1}(x_1)}$, if $d(x)^{\frac{2}{p-1}} u(x) \leq C$, then property (iii) is satisfied and we're done.

Otherwise, we apply the lemma again: $\exists\, x_2 \in M \setminus B_{r_1}(x_1)$ such that $\{x_1, x_2\}$ satisfy (i) and (ii) with $B_{r_1}(x_1)$ and $B_{r_2}(x_2)$.

Repeating this argument, we are certain that this stops after a finite number of steps since M is compact and $r_i \geq R/(\sup_M u)^{\frac{p-1}{2}}$. This proves the proposition. ∎

<u>Corollary</u>: $u(x) \leq C_1 \left[\operatorname{dist}(x, \{x_1, \dots, x_N\})\right]^{-\frac{2}{p-1}}$, $\forall x \in M$

~~from~~ $\cancel{\bigcup_{i=1}^{N} B_{r_i}(x_i)}$

<u>Proof</u>:

(I) If $\operatorname{dist}(x, x_i) > 2r_i$

Then
$$\operatorname{dist}(x, \cup B_{r_i}(x_i)) = \operatorname{dist}(x, B_{r_i}(x_i)) = d(x, x_i) - r_i$$
$$\geq \tfrac{1}{2} d(x, x_i)$$
$$\geq \tfrac{1}{2} d(x, \{x_1, \dots, x_N\})$$

So (iii) $\Longrightarrow$ $u(x) \leq 2^{2/p-1} C \left[\operatorname{dist}(x, \{x_1, \dots, x_N\}\right]^{-2/p-1}$

(II) If $x \in B_{2r_i}(x_i)$ i.e. x is in a spherical region.

$$r_i \geq \tfrac{1}{2} d(x, x_i) \geq \tfrac{1}{2} d(x, \{x_1, \dots, x_N\})$$

Now $v(y) < 1$

So (ii) $\Longrightarrow$
$$u(x) \leq (1+\varepsilon)\, u(x_i)$$
$$\leq (1+\varepsilon) R^{2/p-1} r_i^{-2/p-1}$$
$$\leq C_2 \left[\operatorname{dist}(x, \{x_1, \dots, x_N\}\right]^{-2/p-1}$$

where $C_1 = \max\left\{ 2^{2/p-1} C,\ (1+\varepsilon) R^{2/p-1} \right\}$ ∎

Back to

Main Estimate: $\exists \Lambda_0 = \Lambda_0(g_0)$ such that $F_p^{-1}(0) \subset \Omega_{\Lambda_0}$ for all $p \in [1, \frac{n+2}{n-2}$
O.K. for $p \in [1, p_0]$ for any $p_0 < \frac{n+2}{n-2}$.)

Proof: Suppose not; $\exists u_i, p_i$ $\max_M u_i \to \infty$, $p_i \to \frac{n+2}{n-2}$

Proof proceed is three steps

Step 1: Reduction to isolated point blow up with dimensional bound away from the point.

Fix ε small, R large.

By the proposition $\exists$ points $\{x_{1,i}, \dots, x_{N(i),i}\}$ satisfying conclusions (i), (ii) and (iii).

Note: If we had an apriori energy bound, we could estimate, the # of balls, since each ball has energy uniformly bounded below.

Let $\delta_i = \min\{\operatorname{dist}(x_{\alpha,i}, x_{\beta,i}) : \alpha \neq \beta, 1 \le \alpha, \beta \le N(i)\}$
relabel the points so that $\delta_i = \operatorname{dist}(x_{1,i}, x_{2,i})$.

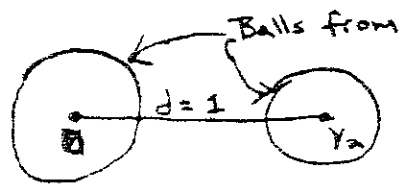
From the proposition we have balls of radius $r_{1,i} = R u_i(x_{1,i})^{-\frac{p-}{2}}$

Let us rescale the coordinates so that the minimal distance between points becomes 1.

i.e. let z be a normal coordinate system centered at $x_{1,i}$ and let $y = \delta_i^{-1} z$

so $x_{1,i} \rightsquigarrow 0$ and $x_{2,i} \rightsquigarrow y_{2,i}$: $\operatorname{dist}(0, y_2) = 1$

What happens when $\delta_i \to 0$? i.e. we want to show that the points around which we have spherical region can not accumulate (as $i \to \infty$).



Note that at this new scale all the other points are spread out (dist (y_i, y_j) for i or $j \neq 1$ or 2 is greater than or equal to 1).

The conformal metric $g_{(i)} = u_i^{4/n-2} g$ is then given by

$g_{ij} = \delta_i^2 \sum_{i,j} g_{(i)\,ij} (\delta_i y) dy^i dy^j = \delta_i^2 \hat{g}_{(i)}$

where $\hat{g}_{(i)ij} = \delta_{ij} + O(\delta_i^2 |y|^2) \longrightarrow \delta_{ij}$ as $\delta_i \to 0$ for y in a bounded set.

The solution in y-coordinates is given by $v_i(y) = \delta_i^{\frac{p_i-1}{2}} u_i(\delta_i y)$

By passing to a subsequence, $\exists$ points y_j such that $y_{j,i} \longrightarrow y_j$ as $i \to \infty$. In particular $\exists\, y_2$, $y_{2,i} \to y_2$.

We want to show that either 0 or y_2 is a point of blow up of v, where v is the limit of v_i, i.e. $v_i \to v$ uniformly on compact subsets of $\mathbb{R}^n \setminus \{y_1, \dots, y_{n_0}\}$, where v satisfies $\Delta v + v^{\frac{n+2}{n-2}} = 0$.

Now from the corollary we have for each i

$v_i(y) \le c_1 [\mathrm{dist}(y, \{y_{1,i}, y_{2,i}, \dots, y_{N(i),i}\})]^{-2/p_i - 1}$.

This gives a bound on $\sup_{\Omega} v$, where Ω is a compact subset of $\mathbb{R}^n \setminus \{y_1^0, \dots, y_{N_0}\}$. By Harnacks inequality this gives a positive lower bound on $\inf_{\Omega} v$.

$\therefore\; v > 0$

Now we can show, using a similar argument to the one on p 50-51, that

v is a global weak supersolution on S^n.

i.e. $\int_{S^n} (v \Delta \varphi + c v^{\frac{n+2}{n-2}} \varphi) d\sigma \le 0 \quad \forall \varphi \in C_+^\infty(S^n)$.

Then two posibilities arise if neither 0 nor y_2 are point of blow up for v both of which may be ruled out by our understanding of solutions on S^n.

1) If none of the points y_j are points of blow up then v is a regular solution on S^n. From the proposition we get two <u>disjoint</u> balls $B_{r_0}(0)$ and $B_{r_2}(y_2)$ in which the solution v, (suitably normalized) is C^2-close to the standard spherical solution, on a <u>large</u> ball. This implies that the metric corresponding to v is close to the spherical metric on a large ball in S^n, i.e. $B_{r_0}(0)$ and $B_{r_2}(y_2)$ are <u>disjoint</u> <u>concave</u> balls in S^n - a contradiction

(2) If the singular set of v is nonempty then, <u>no</u> concave balls are allowed in the domain where v is regular. Thus by the above one of 0 and y_2 need be a point of blow up for v.

This completes Step 1.

So Two possibilities exist:

i) $\delta_i \not\to 0 \Rightarrow N(i) = N$ for sufficiently large i
$\{u_i\}$ has <u>isolated points of blow-up</u> at $\{x_{1,i}, \dots, x_{N,i}\}$ and $u_i \to u$ uniformly in C^∞, on compact subsets of $M \setminus \{x_1, \dots, x_N\}$, where x_j is the limit of $x_{j,i}$, for sufficiently large i.

ii) $\exists \{g_{(i)}\}$ $g_{(i)} \to \delta$, the Euclidean metric, on compact subsets of $\mathbb{R}^n$, with corresponding solutions v_i, such that 0 is an <u>isolated point of blow-up</u> for v_i, $\forall i$. And $v_i \to v$ uniformly in C^∞, on compact subsets of $\mathbb{R}^n \setminus \{y_1 = 0, y_2, \dots\}$ and $\operatorname{dist}\{y_\alpha, y_\beta\} \geq 1$ for $\alpha \neq \beta$.

<u>Def'n</u>: $\bar{x} \in M$ is an <u>isolated point of blow-up</u> for a sequence $\{u_i\}$ corresponding to metrics $g_i \to g_0$ near $\bar{x}$, if $\exists \{x_i\}$, local maxima of u_i such that, $x_i \to \bar{x}$, and

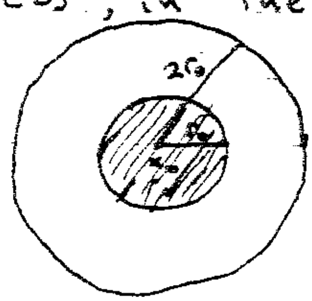
(i) $u_i(x) \leq C[\operatorname{dist}(x, x_i)]^{-\frac{2}{p_i - 1}}$ $\forall x \in B_{\bar{r}}(x_i)$, $\bar{r} > 0$.

(ii) $\lim\limits_{i \to \infty} (\max\limits_{B_r(\bar{x})} u_i) = \infty$ $\forall r > 0$

<u>Step 2</u>: Reduction to simple point blow up.

let z be normal coordinates about the point x_0

From the proposition we get a "spherical region" of the solution $u(z)$, in the ball of radius $r_0 = R\, u(0)^{-\frac{p-1}{2}}$



Say $u(z) \leq C|z|^{-\frac{2}{p-1}}$ for $|z| \leq 2r_0$

Suppose $0 < r \leq r_0$ and let $S_r = \{z : |z| = r\}$

<u>Claim</u>: $\max\limits_{S_r} u \leq C \min\limits_{S_r} u$

To see this, 1^{st} normalize so that $r = 1$, (denote by $'$ the sets in these coords.) Then the solution is bounded in a neighborhood of S_1', so we may cover S_1' by balls in which we can apply Harnacks inequality, combining these estimates gives.

$$\max_{S_1'} u' \le C \min_{S_1'} u' \Rightarrow \max_{S_r} u \le C \min_{S_r} u$$

This suggests that we look at spherical averages

$$\bar{u}(r) = \fint_{S_r} u(z) \, d\Sigma_{g_0}(z)$$

$$\min_{S_r} u \le \bar{u}(r) \le \max_{S_r} u \, .$$

Let $w(z) = |z|^{2/p-1} u(z)$, $\bar{w}(r) = r^{2/p-1} \bar{u}(r)$

<u>claim</u>: $\bar{w}(r) \le C$

what is the behavior of $\bar{w}(r)$. ?

<u>Definition</u>: $\bar{x}$ is a <u>simple point</u> of blow up, if $\bar{x}$ is an isolated point of blow up and $\exists \, \bar{r}$ so that $\bar{w}_i(r)$ has only one critical point for $r \in (0, \bar{r})$. ($\bar{r}$ is indepedent of i).

1^{st} note: $w(z) = |z|^{2/p-1} u(z)$ is scale invariant.

i.e. Define $u_1(y) = u(0)^{-1} u\left(\frac{y}{u(0)^{\frac{p-1}{2}}}\right)$ where $y = u(0)^{\frac{p-1}{2}} z$

so $u_1(y) \sim \bar{v}(y)$ for $|y| < 2R$

where $\bar{v}(y) = (1 + |y|^2)^{-\frac{n-2}{2}}$

So $|y|^{2/p-1} u_1(y) = w(z)$

$$|y|^{2/p-1}\,\bar{v}(y) = |y|^{2/p-1}\,(1+|y|^2)^{-\frac{n-2}{2}}$$

$$p \le \frac{n+2}{n-2} \Rightarrow 2/p-1 \ge \frac{n-2}{2}$$

for $|y|$ large

$$|y|^{2/p-1}\,\bar{v}(y) \sim |y|^{2/p-1-(n-2)}$$

so that for p close to $\frac{n+2}{n-2}$ $\quad n-2 \ge 2/p-1$

and thus $|y|^{2/p-1}\,\bar{v}(y) \longrightarrow 0$.

So we claim that if $\bar{x}$ is a simple point of blow up for $\{u_i\}$ then $u_i \Rightarrow$ uniformly on compact subsets then $u_i \Rightarrow 0$ uniformly on compact subsets of $B_{\bar{r}}(0) \setminus \{0\}$.

The point is that for a simple point of blow up $\bar{W}_i(r)$ has the profile:

so that $\bar{w}_i(r)$ has the same profile as the spherical solution up to Ru^{-2/p_i-1} and is decreasing outside (since only one critical point up to $\bar{r}$).

as $i \to \infty$ the spherical profile squeezes in and $\bar{W}_i(r)$ is forced to decrease to zero, hence $u_i \to 0$ on compact subsets of $B_{\bar{r}}(0) \setminus \{0\}$.

<u>This brings us to the Key Reduction</u> 11/22/88

We need to understand the possible blow up in greater detail.

<u>Theorem</u>: <u>Two Possibilities exist</u>, either:

1) $\exists \{x_1, \dots, x_N\} \subset M$ such that each x_α is a point of simple blow up for $\{u_i\}$ (actually, we pass to a subsequence).

2) $\exists \{g^{(i)}\}$, $g^{(i)} \Rightarrow \delta$ uniformly on compact subsets of $\mathbb{R}^n$, and $g^{(i)} = v_i^{4/n-2} g_0$, 0 is a point of simple blow up for $\{v_i\}$

$v_i \Rightarrow 0$ on $\mathbb{R}^n \setminus \{\text{discrete set} \equiv D\}$

Note: In case ②, if $\bar{y} \notin D$ let $\underline{w}_i = v_i / v_i(\bar{y})$

v_i satisfy $\Delta_{g_i} v_i + v_i^{p_i} = 0$

so $\underline{w}_i$ satisfy $\Delta_{g_i} \underline{w}_i + v_i(\bar{y})^{p_i - 1} \underline{w}_i^{p_i} = 0$

Now away from D, v_i are locally bounded and therefore satisfy a Harnack inequality $\Rightarrow$ $\underline{w}_i$ also satisfy a Harnack inequality.

So $\underline{w}_i^{p_i}$ is bounded and $v_i(\bar{y})^{p_i-1} \to 0$ uniformly away from D. so $\Delta \underline{w}_i \longrightarrow \Delta h = 0$.

h is a harmonic function singular at 0 and at, at least one, other point of $\mathbb{R}^n \cup \{\infty\}$

(This is equivalent to say that the metric $h^{4/n-2} \cdot \delta$ is not flat)

h is regular at ∞ if h is regular outside a compact set of $\mathbb{R}^n$ and $|x|^{2-n} h(\frac{x}{|x|})$ has a removable singularity at $x = 0$ i.e. $h(y) = O(|y|^{2-n})$ for y large).

Proof of reduction to simple point blow up.

(We've already done the reduction to isolated point blow up).

Case (i). $\delta_i \geq \delta > 0$ (δ_i is the minimum distance of the points from the proposition). $\Rightarrow$ we get $\{x_1, \ldots, x_{N_0}\}$ isolated points of blow up.

Suppose x_1 is not a simple point of blow up.

Let z be normal coordinates (in the backround metric) centered at x_1

recall $\quad w_i(z) = |z|^{2/p_i - 1} u_i(z)$

Then $\operatorname{Max} |\omega_i(z)| \leq C$, so we looked at spherical averages, $\overline{\omega}_i(r) = \fint_{S_r} \omega_i$, a bounded function of one variable

Assume x_1 is not a point of simple blow up.
i.e. $\exists\ \sigma_i$ such that $\sigma_i \to 0$ and σ_i is the second critical point of $\overline{\omega}_i$.
we have $\omega_i(\sigma_i) \to 0$.
Set $y = z/\sigma_i$, $v_i(y) = \sigma_i^{2/p_i - 1} u_i(\sigma_i y)$
and let $\overline{v}_i$ denote the spherical average of v_i
Then $\overline{\omega}_i(\sigma_i) = |y|^{\frac{p_i-1}{2}} v_i(|y|)$ has a 2^{nd} critical point at $|y| = 1$.
Let $w_i = v_i / v_i(\bar{y})$, $w_i \Rightarrow h$ on $\mathbb{R}^n \setminus \{0\}$
where h is harmonic.
Note: h is radial by Liouville's theorem
$\Rightarrow h(x) = a + b|x|^{2-n}$, $a, b \geq 0$
<u>claim</u> $a, b > 0$ (i.e. both 0, and ∞ are singular points)
since $\overline{\omega}_i(1) = 1 \Rightarrow h(1) = 1 \Rightarrow a + b = 1$
We know $p_i \to \frac{n+2}{n-2}$ and we know that the function $|y|^{\frac{n-2}{2}}(a + b|y|^{2-n})$ has a critical point at $y = 1$ (it is the limit of the spherical averages)
$= a|y|^{\frac{n-2}{2}} + b|y|^{\frac{2-n}{2}}$
Take a derivative $\Rightarrow \left(\frac{n-2}{2}\right) a + \left(\frac{2-n}{2}\right) b = 0 \Rightarrow a = b$.
so $a = b = 1/2$

$\therefore h(y) = \frac{1}{2}\left(1 + |y|^{2-n}\right)$.

This places us into case (ii) of the reduction.

A geometric picture of the above is gotton by thinking of the spherical portions of the solution either shrinking down or we get down to a "neck region" (corresponding to the critical point). The neck region has two infinities and has zero scaler curvature, it is conformally equivalent to $S^n \setminus \{0, \infty\}$.

This completes case (i) of the reduction.

Case (ii): $\delta_i \to 0$

By rescaling arguments we were able to get ourselves into the case $g^{(i)} \Rightarrow \delta$ on $\mathbb{R}^n$, with $\{v_i\}$ and 0 is an isolated point of blow up for $\{v_i\}$.

$v_i \Rightarrow v$ on $\mathbb{R}^n \setminus D$ (D is a discrete set)

We may assume 0 is a simple pt of blow up.

Now look at $\underline{w_i} \Rightarrow h$ on $\mathbb{R}^n \setminus D$.

claim: 0 is a singlar point of h.

If not then $|y|^{\frac{n-2}{2}} h(|y|)$ is increasing for $|y|$ small but this contradicts the hypothesis of simple blow up.

So by using the concave regions, we can claim that 0 and y_2 are simple points of blow up and hence h is singular at 0 and y_2. ∎

We want to show that neither possibility of the previous reduction can occur.

We will use a variational argument involving the Pohozaev identity, which we ~~...~~ now derive.

$\Omega \subset (M,g)$, X a vector field on Ω, ν outward pointing unit normal

Define: $\mathcal{D}X = \frac{1}{2}(X_{i;j} + X_{j;i} - \frac{2}{n}\,\mathrm{div}_g X\, g_{ij}) = \frac{1}{2}(\mathcal{L}_X g - \frac{\mathrm{Tr}_g(\mathcal{L}_X g)}{n} g)$

(recall: $\mathcal{L}_X g = X_{i;j} + X_{j;i}$)

$\mathcal{D}$: Vector fields $\longrightarrow$ Symmetric $(0,2)$ tensors

$\mathcal{D}X = 0 \iff$ flow generated by X consists of conformal transformations

Def'n: X is a conformal Killing vector field if $\mathcal{D}X = 0$.

Ex $\mathbb{R}^n$, $r\frac{\partial}{\partial r} = \frac{1}{2}\nabla r^2 = \sum_i x^i \frac{\partial}{\partial x^i}$, is conformal Killing.

$F_t(x) = e^t x$

Let T_g = Trace free Ricci (g). Analyse $\int_\Omega \langle T_g, \mathcal{D}X\rangle_g \, d\omega_g$

1st Observe

$$\int_\Omega \langle T_g, \mathcal{D}X\rangle_g \, d\omega_g = \int_\Omega \sum_{i,j} T^{ij} X_{i;j} \, d\omega_g$$

integrate by parts noting: $T(X,\nu) = T^{ij} X_i \nu_j$.

$$= \int_{\partial\Omega} T(X,\nu)\, d\Sigma_g - \int_\Omega (\sum T^{ij}{}_{;j}) X_i \, d\omega_g$$

Recall: Twice contracted 2nd Bianchi identity: $\sum_j R^{ij}{}_{;j} = \frac{1}{2}\sum_j R_{;j} g^{ij}$

So $\sum_j T^{ij}{}_{;j} = \sum_j R^{ij}{}_{;j} - \frac{1}{n} R_{;j} g^{ij} = \frac{n-2}{2n} \sum R_{;j}\, g^{ij}$

Where $R^{ij} = \mathrm{Ricci}^{ij}(g)$ and R = Scaler Curvature (g)

This gives us:

$$\frac{n-2}{2n}\int_{\Omega} X(R)\,d\omega_g + \int_{\Omega} \langle T, \mathcal{D}X\rangle_g\,d\omega_g = \int_{\partial\Omega} T(X,\nu)\,d\Sigma_g$$

Applications:

(i) $\Omega \subset \mathbb{R}^n$, $X = r\frac{\partial}{\partial r} = \sum x^i \frac{\partial}{\partial x^i}$, satisfies $\mathcal{D}X = 0$

$u > 0$, $u = 0$ on $\partial\Omega$ $\quad g = u^{4/n-2}\delta$

$\nu = u^{-2/n-2}\nu_0$, where ν, ν_0 are unit outward normals to Ω w.r.t. g and δ respectively

$\Delta u + u^p = 0 \iff R(u^{4/n-2}\delta) = R(g) = -c(n)^{-1}u^{p-\frac{n+2}{n-2}}$

let $\delta = \frac{n+2}{n-2} - p$

$$\frac{n-2}{2n}(-c(n)^{-1})\int_{\Omega}\sum x^i \frac{\partial u^{-\delta}}{\partial x^i}\cdot u^{\frac{2n}{n-2}}\,dx = \frac{n-2}{2n}\int_{\Omega} X(R)\,d\omega_g$$

$$= \int_{\partial\Omega} T(X,\nu)\,d\Sigma_g$$

Now recall that if $g = u^{4/n-2}g_0$ then

$$T_{ij} = T_{ij\,0} + (n-2)u^{2/n-2}\left[(u^{-2/n-2})_{;ij} - \frac{1}{n}\Delta(u^{-2/n-2})g_{ij}\right]$$

for us $g_0 = \delta$ so $T_{ij\,0} \equiv 0$, and $d\Sigma = u^{\frac{2(n-1)}{n-2}}d\Sigma_0$

so $$\int_{\partial\Omega} T(X,\nu)\,d\Sigma = (n-2)\int_{\partial\Omega}\left((u^{-2/n-2})_{;ij} - \frac{1}{n}\Delta(u^{-2/n-2})\delta_{ij}\right)\nu_{0i}X_j u^{\frac{2(n-1)}{n-2}}d\Sigma$$

(Note since this is in $(\mathbb{R}^n, \delta)$ the derivative $(\ )_{;ij}$ are just $(\ )_{ij}$ and we may raise and lower indices without regard).

So:

$$-\left(\frac{n-2}{2n}\right)c(n)^{-1}\int_{\Omega}\sum x^i\frac{\partial u^{-\delta}}{\partial x^i}\cdot u^{\frac{2n}{n-2}}\,dx = (n-2)\int_{\partial\Omega}\left[(u^{-2/n-2})_{ij} - \frac{1}{n}\Delta(u^{-\frac{2}{n-2}})\delta_{ij}\right]\nu_{0i}X_j \cdot u^{\frac{2(n-1)}{n-2}}d\Sigma_0$$

1^{ST} deal with interior term:

rewrite as $-c(n)^{-1}\frac{(n-2)}{2n}\frac{\delta}{p+1}\int_\Omega \sum x^i \frac{\partial u^{p+1}}{\partial x^i}\, dx$

Integrate by parts to

get $\qquad \bar{c}(n,p)\delta\int_\Omega u^{p+1}\, dx \qquad$ ($u \equiv 0$ on $\partial\Omega$ so no boundary term)

2^{nd} back to boundary term:

1^{st} note: $(u^{-2/n-2})_{ij} = \frac{-2}{n-2}(u^{-\frac{n}{n-2}} u_i)_j = \frac{2n}{(n-2)^2} u^{-\frac{2(n-1)}{n-2}} u_j u_i - \frac{2}{n-2} u^{-n/n-2} u_{ij}$

Now because $d\Sigma = u^{\frac{2(n-1)}{n-2}} d\Sigma_0$ and $u \equiv 0$ on $\partial\Omega$, term like the second derivative of u term does not contribute to the integral. So we may rewrite the integrand of the boundary term as

$$\left[\frac{2n}{(n-2)^2} u_i u_j - \frac{2}{(n-2)^2}|\nabla u|^2 \delta_{ij}\right] \nu_i X_j$$

Since $u \equiv 0$ on $\partial\Omega$, tangential derivatives are 0

i.e. $\nabla u = |\nabla u|\nu_0 \Rightarrow u_i = |\nabla u|\nu_{0i}$

So the integrand is

$$\left[\frac{2n}{(n-2)^2}|\nabla u|^2 \nu_{0i}\nu_{0j} - \frac{2}{(n-2)^2}|\nabla u|^2 \delta_{ij}\right] \nu_{0i} X_j$$

$$= \frac{2|\nabla u|^2}{(n-2)^2}\left[n\delta_{ij} - \delta_{ij}\right]\nu_{0i} X_j = \frac{2(n-1)}{(n-2)^2}|\nabla u|^2 \sum_{i=1}^n \nu_{0i} X_i$$

so letting $\bar{\bar{c}}(n) = \frac{2(n-1)}{n-2} > 0$ we get

$$\bar{c}(n,p)\delta\int_\Omega u^{p-1}\, dx = \bar{\bar{c}}(n)\int_{\partial\Omega} |\nabla u|^2 \nu_0 \cdot X \, d\Sigma_0$$

If Ω is starshaped w.r.t. some point $x \in \Omega$ then $\nu_0 \cdot X > 0$. So if $p \geq \frac{n+2}{n-2}$ then $\delta \leq 0$, so that this identity cannot hold for starshaped.

his is: Theorem: (Pohozaev) $\nexists u$ solving $\Delta u + u^p = 0$ on a starshaped domain in $\mathbb{R}^n$ if $p \geq \frac{n+2}{n-2}$.

Note: We can show that for the Dirichlet problem on an arbitrary smooth domain there exist uniform energy estimates near the boundary, this is done by an application of the Alexandroff reflection principle. So, if a solution has singularities (or if a sequence of solutions to the subcritical equation blow up as $i \to \infty$, $p_i \to \frac{n+2}{n-2}$) as must be the case for starshaped domains, they must occur in the interior.

(ii) Kazdan-Warner identity

on (S^n, g_0) $\exists$ lots of X's with $\mathcal{D}X = 0$

e.g. $X = \nabla(\ell(x))$ where $\ell(x) = \sum_{i=1}^{n+1} a_i x^i$ satisfy $\mathcal{D}X = 0$

Take $g = u^{4/n-2} g_0$, $u > 0$; the equation, $\mathcal{D}X$ is clearl conformally invariant.

$$\int_{S^n} X(R)\, d\omega_g = 0 \iff \int_{S^n} \langle \nabla_{g_0} \ell, \nabla_{g_0} R \rangle_{g_0} u^{\frac{2n}{n-2}}\, d\omega_{g_0} = 0$$

Theorem: If $\overline{R} = a + b\ell(x)$ then $\nexists g \in [g_0]$ with $R(g) = \overline{R}$.

(More generally if $\langle \nabla \overline{R}, \nabla \ell(x) \rangle \neq 0$ on S^n).

iii) We claim that the Obata identity (Obata's uniqueness theorem) is a third instance of this.

(M, g_0), $g = u^{4/n-2} g_0$. $R(g) \equiv$ constant.

Choose: $$X = \nabla_g u^{2/n-2} = u^{-4/n-2} \nabla_{g_0} u^{2/n-2}$$
$$= -u^{4/n-2} \nabla_{g_0}(u^{-2/n-2})$$

So $X(R) = 0$ and $\partial M = 0 \Rightarrow \int_M \langle T, \mathcal{D}_g X \rangle_g \, d\omega_g = 0$

$\mathcal{D}_g X = \mathrm{Hess}_g (u^{2/n-2})$ and $T_0 = T + (n-2) u^{-2/n-2} \mathrm{Hess}_g (u^{2/n-2})$

So $\mathrm{Hess}_g (u^{2/n-2}) = \dfrac{(T_0 - T)\, u^{2/n-2}}{n-2}$

$\therefore \int_M u^{2/n-2} \langle T - T_0, T \rangle_g \, d\omega_g = 0$

So if $T_0 \equiv 0$ then $T \equiv 0$, i.e. g_0 Einstein

$\Rightarrow g$ is Einstein

Applications (M, g_0)

1) The locally conformally flat case, $R(g_0) > 0$.

φ a positive function, $L\varphi = \Delta\varphi - c(n) R(g_0) \varphi$.

L is a negative definate operator hence is invertible.

$\exists$ a fundamental solution G_p with pole at p.

i.e. $L G_p = -c(n) \delta_p$, $G_p > 0$

G_p is unique up to a multiplicative constant.

We can normalize so that

$$G_p(x) = |x|^{2-n} + \text{l.o. terms}$$

then $g = G_p^{4/n-2} g_0$ give a Complete, asymtotically Euclidean metric on $M \setminus \{p\}$ such that $R(g) \equiv 0$

---

g_0 locally conformally flat $\Rightarrow \exists$ a local conformal Killing vector field near p.

i.e. we can choose coordinates near p such that

$g_0 = \lambda^2(x) \sum (dx^i)^2$ (conformally flat coordinates.

Let $X = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}$, let Σ be a hypersurface surrounding p (i.e. Σ bounds a region Ω containing p).

(Σ, ν), ν is the unit outward normal w.r.t. g.

Define: $E(p) = -c(n) \int_\Sigma T_g(X, \nu)\, d\Sigma$

Claim: $E(p)$ is independent of Σ

what we're required is that $\Sigma = \partial\Omega$, $p \in \Omega$, $\Omega \subset$ domain of X.

and $X = \sum x^i \frac{\partial}{\partial x^i} +$ l.o. terms

Proof: Applying the Pohozaev identity in an "Annulus" i.e.

$\Omega' \subset \Omega$ $\Sigma' = \partial\Omega'$, $\Sigma = \partial\Omega$ $p \in \Omega'$ let $A = \Omega \setminus \Omega'$

$R \equiv 0$ so $X(R) = 0$ $\mathcal{D}X = 0$

$$\Rightarrow \quad 0 = \int_{\partial A} T(X, \nu)\, d\Sigma_g$$

$$\Rightarrow \quad 0 = \int_\Sigma T(X, \nu)\, d\Sigma + \int_{\Sigma'} T(X, \nu)\, d\Sigma'$$

$$\Rightarrow \quad \int_\Sigma T(X, \nu)\, d\Sigma = \int_{\Sigma'} T(X, \nu')\, d\Sigma'$$ where ν' is the unit outward normal to Σ'.

$\therefore$ E is well defined and gives a function.

$E : M \to \mathbb{R}$ ∎

Analogous Construction

(M, g) g l.c.f. $R(g) \equiv$ constant > 0.

Take any embedded hypersurface $\Sigma^{n-1}, \nu \hookrightarrow M$, with ν a globally defined normal.

Assume $\pi_1(\Sigma) \hookrightarrow \pi_1(M)$ is trivial ($\Rightarrow \Sigma$ lifts compactly in $\widetilde{M}$).

Recall (Schoen-Yau) $\widetilde{M} \sim \Omega \subset S^n$

So we choose any X, a conformal vector field on S^n.

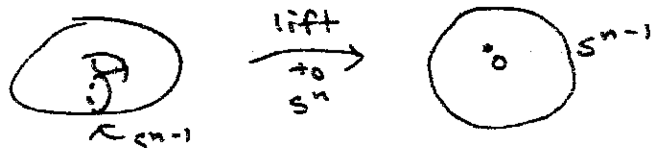
and lift Σ to $\tilde{M}$.

Define $L_\Sigma(X) = \int_\Sigma T(X,\nu)\, d\Sigma_g$.

If $\mathfrak{g}$ is the lie Algebra of $\mathrm{Conf}(S^n)$ then $L_\Sigma \in \mathfrak{g}^*$ (the dual of $\mathfrak{g}$), $L_\Sigma : \mathfrak{g} \to \mathbb{R}$.

L_Σ is a homotopy, but not homological invariant, $L_\Sigma = L_{\Sigma'}$ if Σ' is homotopic to Σ.

Example: Consider $S^1 \times S^{n-1}$, Σ = an S^{n-1}



By an explicit calculation we can show L_Σ is nontrival. Can also give an example where the universal cover is $S^n \setminus \{$a cantor set$\}$, and Σ is zero in homology but L_Σ is not zero.

Back to the Main Estimate. (M, g_0)

Assume we are in the locally conformally flat case.

Assume $\{u_i\}$ s.t. $\max_M u_i \to \infty$ as $i \to \infty$.

Assume Case (i) $\exists \{x_1, \ldots, x_{N_0}\} \in M$ such that $u_i \Rightarrow 0$ on $M \setminus \{x_1, \ldots, x_{N_0}\}$ and each $x_1, \ldots, x_{N_0}$ are simple points of blow up.

let $x^1, \ldots, x^n$ be conformally flat coordinates about x_1, Σ be the boundary of a small ball of radius σ centered at x_1

u_i satisfy $\quad Lu_i + u_i^{p_i} = 0 \quad$ with $g_i = u_i^{4/n-2} g_0$

Since g_0 is l.c.f. $\exists X$ s.t. $\mathcal{D}X = 0$ ($\sum x^i \frac{\partial}{\partial x^i}$ for example). Scaler curvature of g_i is not quite constant since p_i is subcritical.

Applying the Pohozaev identity:

$$\frac{n-2}{2n} \int_{B_\sigma} X(R_i)\, d\omega_{g_i} = \int_{\Sigma_\sigma} T_i(X, \nu_i)\, d\Sigma_i$$

write $g_0 = \lambda^{4/n-2}(x) \sum (dx^i)^2$, note $R_i = +c(n)^{-1} u_i^{-\delta_i}$

where $\delta_i = \frac{n+2}{n-2} - p_i \longrightarrow 0$ as $i \to \infty$

we can write $g_i = (\lambda u_i)^{4/n-2} \sum (dx^i)^2$

So we can write the L.H.S. out as follows (using $X = \sum x^i \frac{\partial}{\partial x}$,

$$-c(n)^{-1} \frac{(n-2)}{2n} \int_{B_\sigma} \sum_{i=1}^{n} x^i \frac{\partial u_i^{-\delta_i}}{\partial x^i} (\lambda u_i)^{\frac{2n}{n-2}} dx \ .$$

which can be rewritten as

$$-4\, \frac{c(n)^{-1}(n-2)}{2n} \frac{\delta_i}{1+p_i} \int_{B_\sigma} \left(\sum_{i=1}^{n} x^i \frac{\partial u_i^{p_i+1}}{\partial x^i} \right) \lambda^{2n/n-2} dx \, .$$

We can now do a Euclidean integration by parts to get.

$$c(n)^{-1} \frac{(n-2)}{2n} \frac{\delta_i}{1+p_i} \left[\int_{B_\sigma} n\, u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} dx + \int_{B_\sigma} u_i^{p_i+1} \left(\sum_{i=1}^{n} x^i \frac{\partial \lambda^{\frac{2n}{n-2}}}{\partial x^i} \right) dx \right.$$

$$\left. - \sigma \int_{\Sigma_\sigma} u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} d\Sigma_\sigma \right] = \text{R.H.S.}$$

now since $r\,\partial/\partial r = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$, we can choose σ sufficiently small so that $\int_{B_\sigma} \left[n\, u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} + u_i^{p_i+1} \left(\sum_{i=1}^{n} x^i \frac{\partial \lambda^{\frac{2n}{n-2}}}{\partial x^i} \right) \right] dx > 0$

$$\text{So} \quad -\int_{\Sigma_\sigma} T_i(X, \nu_i)\, d\Sigma_{\sigma_i} \le \frac{c(n)^{-1}(n-2)}{2n} \frac{\delta_i \sigma}{1+p_i} \int_{\Sigma_\sigma} u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} d\Sigma_{\sigma_i}$$

$$\le \bar{c}(n)\, \delta_i \sigma \int u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} d\Sigma_{\sigma_i}$$

We would like to let $i \to \infty$, then both sides tend to zero, and we get no information.

recall: $\nu_i = u_i^{-2/n-2}\nu_0$ and $d\Sigma_i = u_i^{\frac{2(n-1)}{n-2}} d\Sigma_0$

So we have $-\int_{\Sigma_\sigma} u_i^2 T_i(X, \nu_0)\, d\Sigma_0$ on the L.H.S.

Take $y \in B_\sigma$, let $\varepsilon_i = u_i(y)$, so that $\varepsilon_i \searrow 0$ as $i \to \infty$

recall: $H = \lim_{i\to\infty} \frac{u_i}{u_i(y)}$ where H is singular at $x_1, \ldots, x_{N_0}$

and $LH = 0$ on $M \setminus \{x_1, \ldots, x_{N_0}\}$

let $\bar{g} = H^{4/n-2} g_0$

we have

$$-\varepsilon_i^{-2}\int_{\Sigma_\sigma} u_i^2 T_i(X,\nu_0)\, d\Sigma_0 \le c(n)\,\delta_i\,\sigma\,\varepsilon_i^{-2}\int_{\Sigma_\sigma} u_i^2\, u^{p_i+1+\frac{2}{n-2}}\,\lambda^{\frac{2n}{n-2}}\, d\Sigma$$

$$\downarrow i \to \infty \qquad\qquad\qquad\qquad \downarrow i\to\infty$$

$$\therefore\ -\int_{\Sigma_\sigma} T_{\bar g}(X, \nu_{\bar g})\, d\Sigma_{\bar g} \le 0 \qquad\qquad\qquad 0$$

We want to relate this to the Energy function, $E(\rho)$

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Since H is singular at $x_1, \ldots, x_{N_0}$, $H = \sum_{\alpha=1}^{N_0} a_\alpha G_{x_\alpha}$, $a_\alpha > 0$

$\alpha = 1, \ldots, N_0$

As before let $x^1, \ldots, x^n$ be conformally flat coordinates centered at x_α, so that $g_0 = \lambda^{4/n-2}\sum (dx^i)^2$.

$\therefore\ \bar g = (\lambda H)^{4/n-2}\sum (dx^i)^2$ has zero scaler curvature

$\Rightarrow \lambda H$ is a Euclidean harmonic function.

i.e. $\Delta(\lambda H) = 0$.

By dilating our coordinates, we can assume $\lambda(0) = 1$

$\therefore\ \lambda H(x) = a_\alpha |x|^{2-n} + A_\alpha + h(x)$

where $\Delta h = 0$, $h(0) = 0$

We can use this to calculate the boundary term.

recall: $T_{\bar{g}} = (n-2)\,(\lambda H)^{2/n-2}\ \mathrm{Hess}_{Tr.Fr.}[(\lambda H)^{-2/n-2}]$

<u>Compute expansions</u>:

$$(\lambda H)^{-2/n-2} = a_\alpha^{-2/n-2}|x|^2\left(1+\frac{A_\alpha}{a_\alpha}|x|^{n-2}+O(|x|^{n-1})\right)^{-2/n-2}$$
$$= a_\alpha^{-2/n-2}|x|^2 - \frac{2}{n-2}\frac{A_\alpha}{a_\alpha^{n/n-2}}|x|^n + O(|x|^{n+1})$$

To compute the Trace free Hessian, 1^{st} note

$(|x|^2)_{x^ix^j} = 2\delta_{ij} \Rightarrow 1^{st}$ term has 0 - trace free Hessian.

$(|x|^n)_{x^ix^j} = n(r^{n-1}r_i)_j = n(n-1)r^{n-2}r_ir_j + n\,r^{n-1}r_{ij}$

Now $\nu_{\bar{g}} = \frac{\vec{x}}{\sigma}(\lambda H)^{-2/n-2}$

So $T_{\bar{g}}(\vec{x},\nu_{\bar{g}}) = (n-2)\,\mathrm{Hess}_{Tr.Fr.}[(\lambda H)^{-2/n-2}]\,\frac{(\vec{x},\vec{x})}{\sigma}$

(The powers of λH cancel).

One checks $\sum r_{ij}x^ix^j = 0$, so there is zero contribution form the $n\,r^{n-1}r_{ij}$ part of $(|x|^n)_{x^ix^j}$.

So taking the trace free part we get $n(n-1)r^n$ from the $(|x|^n)_{x^ix^j}$ part. Now also have $\frac{1}{n}\Delta|x|^n = \frac{2n(n-1)}{n}|x|^{n-2}$

So taking trace free part we get $-2(n-1)r^n$.

$\therefore$ Contribution from the $|x|^n$ portion of the expansion of $\mathrm{Hess}_{Tr.Fr.}[(\lambda H)^{-2/n-2}]$ is $(n-1)(n-2)\,r^n$.

$$\therefore\ T_{\bar{g}}(X,\nu_{\bar{g}}) = -2(n-1)(n-2)\frac{A_\alpha}{a_\alpha^{n/n-2}}\sigma^{n-1} + O(\sigma^n) \text{ on } \partial B_\sigma$$

Volume form will be given by

$$d\Sigma_{\bar{g}} = (\lambda H)^{\frac{2(n-1)}{n-2}}\sigma^{n-1}\,d\Sigma_{S^{n-1}(\sigma)}$$

Now $(\lambda H)^{\frac{2(n-1)}{n-2}} = a_\alpha^{\frac{2(n-1)}{n-2}} \sigma^{-2(n-1)}$ + l.o. terms on Σ_σ.

We can now evaluate the boundary integral

$$+\int_{\Sigma_\sigma} T_{\bar{g}}(X, \nu_{\bar{g}})\, d\Sigma_{\bar{g}} = -2(n-1)(n-2)\left(A_\sigma a_\sigma + O(\sigma)\right).$$

Letting $\sigma \searrow 0$, this shows that the boundary term is independent of σ.

So by our previous result $\left(-\int_{\Sigma_\sigma} T_{\bar{g}}(X, \nu_{\bar{g}})\, d\Sigma_{\bar{g}} \leq 0\right)$, we have shown

$$A_\alpha \leq 0 \quad \text{for} \quad \alpha = 1, 2, \ldots, N_0.$$

Now recall: $H = \sum_{\alpha=1}^{N_0} a_\alpha G x_\alpha \geq a_{\alpha_0} G x_{\alpha_0}$.

Expand out $G x_{\alpha_0}(x)$.

$$G x_{\alpha_0}(x) = |x|^{2-n} + c\, E(x_{\alpha_0}) + O(|x|).$$

$$\text{So} \quad a_{\alpha_0} G x_{\alpha_0}(x) = a_{\alpha_0}\left(|x|^{2-n} + c\, E(x_{\alpha_0}) + O(|x|)\right)$$

$$\leq a_{\alpha_0} |x|^{2-n} + A_{\alpha_0} + O(|x|)$$

(the expansion for $H(x)$)

$$\therefore\ c\, a_{\alpha_0} E(x_{\alpha_0}) \leq A_{\alpha_0} \leq 0$$

Now if $N_0 \geq 2$, the by the strong Maximum principle, we have $H = \sum_{\alpha=1}^{N_0} a_\alpha G x_\alpha > a_{\alpha_0} G x_{\alpha_0}$ (strict inequality)

<u>We Conclude</u>: $E(x_\alpha) \leq 0$, $\alpha = 1, \ldots, N_0$

and $E(x_\alpha) < 0$ if $N_0 \geq 2$

So we could now invoke the Positive Energy theorem to conclude that this could not occur. This takes care of case (i).

Now Assume we are in

<u>Case (ii)</u>: $\exists \{g^{(i)}\}$, $g^{(i)} \Rightarrow \delta$ uniformly in C^k on compacta in $\mathbb{R}^n$. $g^{(i)} = v_i^{4/n-2} g_0$, 0 is a simple point of blow up for $\{v_i\}$ and $v_i \Rightarrow 0$ on $\mathbb{R}^n \setminus D$

We would like to use the above argument to show case (ii) cannot occur.

Recall: for $\bar{y} \notin D$, $\frac{v_i}{v_i(\bar{y})} \to h$, Euclidean harmonic function with pole a $y=0$ and additional poles somewhere else.

$h(y) = a|y|^{2-n} + A + O(|y|)$

Now $A + O(|y|) > 0$ near $y=0$, since its a global positive harmonic function with isolated poles, thus $h(y) > a|y|^{2-n}$.

Applying the previous argument we conclude

$A \le 0$, which is a contradiction.

$\therefore$ Case (ii) can not occur.

Note: This argument, as it stands, only works in the conformally flat case.

<u>Back to the Energy function.</u>

In the above we needed the existence of a conformal Killing vector field to define $E(p)$.

let (M, g_0) be an arbitrary compact manifold $p \in M$, take normal coordinates $x^1, \ldots, x^n$

i.e. $g_0 = \sum g_{ij}\, dx^i dx^j$ where $g_{ij}(x) = \delta_{ij} + O(|x|^2)$.

Then $X = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$ is asymtotically a conformal Killing vector field. How far off is it?

Compute $\mathcal{D}X$:

First recall that for a vector field Y with components Y^ℓ in local coordinates (i.e. $Y = \sum Y^\ell \frac{\partial}{\partial x^\ell}$) we have

$$\nabla_{\frac{\partial}{\partial x^j}} Y = Y^\ell_{;j} \frac{\partial}{\partial x^j} \quad \text{where } Y^\ell_{;j} = \frac{\partial Y^\ell}{\partial x^j} + Y^k \Gamma^\ell_{kj}.$$

So computing out $X_{i;j}$ we get

$$X_{i;j} = g_{ij} + g_{i\ell} x^k \Gamma^\ell_{kj}$$

$$\therefore (\mathcal{D}X)_{ij} = g_{ij} + g_{i\ell} x^k \Gamma^\ell_{kj} - \text{trace term.}$$

and the trace term is given by

$$\frac{1}{n} g_{ij} \operatorname{div}_g X = g_{ij} + \frac{g_{ij} x^k \Gamma^\ell_{kj}}{n} \qquad (\operatorname{div}_g X = X^j_{;j})$$

$$\text{So } (\mathcal{D}X)_{ij} = g_{i\ell} x^k \Gamma^\ell_{kj} - \frac{1}{n} g_{ij} x^k \Gamma^j_{kj}$$

$$\therefore \mathcal{D}X = O(|x|^2).$$

If G_p = Greens function with pole at $p \in M$ then the metric $\bar{g} = G_p^{4/n-2} g_0$ is scaler flat and complete, but not necessarily locally conformally flat.

We can only define a limiting Energy and we can expect it to be independent of σ.

One would hope the following limit exists.

$$E(p) = -c(n) \lim_{\sigma \to 0} \int_{\Sigma_\sigma} T_{\bar{g}}(X, \nu_{\bar{g}})\, d\Sigma_{\bar{g}}.$$

The difference between two such boundary integrals is an interior term (from the Pohozaev identity):

$$\int_{B_{\sigma_2} \setminus B_{\sigma_1}} \langle T_{\bar{g}}, \mathcal{D}_{\bar{g}} X \rangle_{\bar{g}}\, d\omega_{\bar{g}}.$$

B_σ = Ball radius σ
$\Sigma_\sigma = \partial B_\sigma$
$\sigma_2 > \sigma_1 > 0$

So for $E(p)$ to be well defined we need

$$\int_{B_\sigma} |\langle T_{\bar g}, \mathcal{D}_{\bar g} X \rangle_{\bar g}| \, d\omega_{\bar g} < \infty, \quad \forall\ \sigma \text{ sufficiently small.}$$

We compute out the relevant expansions. 1

$\mathcal{D}$ is a conformally invariant operator i.e. $\mathcal{D}_{\bar g} X = G_p^{4/n-2} \mathcal{D}_g X$,
thus $\mathcal{D}_{\bar g} X = G_p^{4/n-2}\, O(|x|^2)$

$T_{\bar g} = T_{g_0} + (n-2)\, G_p^{2/n-2} \operatorname{Hess}(G_p^{-2/n-2})$, where T_{g_0} may be taken to be bounded.

One can show the following expansions (using conformal normal coordinates, for example).

$$G(x) = |x|^{2-n} + O(1) \qquad n=3$$
$$G(x) = |x|^{2-n} + O(\log|x|) \qquad n=4$$
$$G(x) = |x|^{2-n} + O(|x|^{4-n}) \qquad n\geq 5$$

so

$$G^{2/n-2}(x) = |x|^{-2} + O(|x|^{n-4}) \qquad n=3$$
$$G^{2/n-2}(x) = |x|^{-2} + O(|x|^{n-4}\log|x|) \qquad n=4$$
$$G^{2/n-2}(x) = |x|^{-2} + O(1) \qquad n\geq 5$$

and

$$G^{-2/n-2}(x) = |x|^2 + O(|x|^n) \qquad n=3$$
$$G^{-2/n-2}(x) = |x|^2 + O(|x|^n \log|x|) \qquad n=4$$
$$G^{-2/n-2}(x) = |x|^2 + O(|x|^4) \qquad n\geq 5$$

therefore

$$\operatorname{Hess}_{T.F}(G^{-2/n-2}) = O(|x|^{n-2}) \qquad n=3$$
$$\operatorname{Hess}_{T.F}(G^{-2/n-2}) = O(|x|^{n-2}\log|x|) \qquad n=4$$
$$\operatorname{Hess}_{T.F}(G^{-2/n-2}) = O(|x|^2) \qquad n\geq 5$$

thus $T_{\bar g} = O(|x|^{n-4}) = O(|x|^{-1})$ $n=3$

$T_{\bar g} = O(|x|^{n-4}\log|x|) = O(\log|x|)$ $n=4$

$T_{\bar g} = O(1)$ $n \geq 5$.

Now $d\omega_{\bar g} = \cancel{d\omega_g}\, G^{\frac{2n}{n-2}}\, d\omega_g$ ③ so using ①, ② and ③ we have

$$\int_{B_\sigma} |\langle T_{\bar g}, \mathcal{D}_{\bar g} X\rangle_{\bar g}|\, d\omega_{\bar g} = \int_{B_\sigma} G_p^2 |\langle T_{\bar g}, \mathcal{D}_{g_0} X\rangle_{g_0}|\, d\omega_{g_0}$$

$\sim \int_{B_\sigma} |x|^{6-2n} \|T_{\bar g}\|_{g_0}\, d\omega_g$ by using the expansions of G_p and $\mathcal{D}_{g_0} X$.

if $n=3$ integrand $\sim |x|^{-1}$, integrable.

if $n=4$ integrand $\sim \frac{\log|x|}{|x|^2}$, integrable.

if $n=5$ integrand $\sim \frac{\text{bounded}}{|x|^4}$, integrable.

so the integral is finite if $\underline{n<6}$, thus the Energy

$E(p) = -c(n) \lim_{\sigma\to 0} \int_{\Sigma_\sigma} T_{\bar g}(X, \nu_{\bar g})\, d\Sigma_{\bar g}$ is well defined as a function $E: M \to \mathbb{R}$.

For $n \geq 6$ what condition do we need to define the Energy?

The Energy is always well defined if conformally flat manifold. For arbitrary (M,g) we will need "sufficient conformal flatness".

For $p \in M$, we want $W_g = O(|x|^m)$, $\Rightarrow g = \delta + O(|x|^{2+m})$

This gives a gain on our expansion of $\mathcal{D}X$, namely we'd find

$\mathcal{D}X = O(|x|^{2+m})$.

and $G(x) = |x|^{2-n} + O(|x|^{m+4-n})$ if $n > m+4$, so we'll expect $T_g = O(|x|^m)$ then the integral is like $\int_{B_\sigma} |x|^{6-2n+2m} dw_g$.

This is finite if $2n-6-2m < n$ ie $m > \frac{n-6}{2}$.

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Statement of General theorem: (M, g_0) arbitrary compact manifold.

Theorem:

(1) Case (ii) can not occur.

(2) If case (i) occurs; $x_1, \dots, x_{N_0}$ satisfy

(i) $W_{g_0} = O(|x|^m)$ for $m > \frac{n-6}{2}$ near each x_α

($\therefore$ we can define the Energy).

(ii) $E(x_\alpha) \leq 0$, and $E(x_\alpha) < 0$ if $N_0 \geq 2$, for each

$\alpha = 1, \dots, N_0$.

[Then the Positive Energy Theorem states that if (M, g_0) with $R > 0$ and $p \in M$, $W_{g_0}(p) = O(|x|^m]$, $m > \frac{n-6}{2}$ then $E(p) \geq 0$ and $E(p) = 0$ iff (M, g_0) is conformally equivalent to $(S^n$, round$)$].

We need to Modify proof in the locally conformally flat case, we first need to say alot more about detailed estimates for simple point blow up.

As before we can write everything in a fixed coordinates system and let the metric vary.

We first state a proposition that will be required in the proof of the theorem.

Proposition: Suppose 0 is a simple point of blow up for $\{u_i\}$ with respect to $g_i \Rightarrow g_0$.
(We're working in normal coordinates for each of the g_i).
u_i satisfy $Lu_i + cu_i^{p_i} = 0$, let $\delta_i = \frac{n+2}{n-2} - p_i$, $\varepsilon_i = u_i(0)^{-1}$.
Then $u_i^{\delta_i} = 1 + O(\varepsilon_i^{\alpha})$ for some $\alpha > 0$ (so $R(u_i^{4/n-2} g_i) = 1 + O(\varepsilon_i^{\alpha})$)

and $\exists\, c$ s.t.

$$c^{-1} u_i(0)\left(1 + u_i(0)^{4/n-2}|x|^2\right)^{-\frac{n-2}{2}} \le u_i(x) \le c\, u_i(0)\left(1 + u_i(0)^{4/n-2}|x|^2\right)^{-\frac{n-2}{2}}$$

for $|x| \le \rho_0$ (fixed ball) and also $v_i = u_i(0)\, u_i \to G_0$, the Greens function of L with pole at 0.

We assume this for now, and give the proof of the theorem for $N = 3, 4, 5$.
recall that $G^{-2/n-2}(x) \sim |x|^2$, $r\frac{\partial}{\partial r} = \frac{1}{2}\nabla|x|^2$ and $v_i^{-2/n-2} \to G^{-2/n-2}$,
so this suggests we choose $X = \nabla_{g_i} v_i^{-2/n-2} = \nabla_g v^{-2/n-2}$
(we'll work with a fixed g_i for now).
so that $X = u(0)^{-2/n-2}\nabla_g(u^{-2/n-2})$
This is the vector field used in the Obata argument (see pg 22 and pg
let $\bar{g} = v^{4/n-2} g$

$$\Rightarrow X = v^{4/n-2}\nabla_{\bar{g}} v^{-2/n-2} = -\nabla_{\bar{g}} v^{2/n-2}$$

$$\frac{n-2}{2n}\int_{B_\sigma} X(R_{\bar{g}})\, d\omega_{\bar{g}} + \int_{B_\sigma} \langle T_{\bar{g}}, \mathcal{D}_{\bar{g}} X\rangle_{\bar{g}}\, d\omega_{\bar{g}} = \int_{\partial B_\sigma} T_{\bar{g}}(X, \nu_{\bar{g}})\, d\Sigma_{\bar{g}}$$

1st Term: $R_{\bar{g}} = c(n)\cdot\frac{-Lv}{v^{\frac{n+2}{n-2}}} = c(n)\, v^{-\delta}$

So $\bar{X}(R_{\bar{g}}) = c(n) \langle \nabla_g v^{-2/n-2}, \nabla_g v^{-\delta} \rangle_g$

now $\nabla_g v^{-\delta} = \left(\frac{n-2}{2}\right) \delta\, v^{2/n-2-\delta} \nabla_g v^{-2/n-2}$

So $\bar{X}(R_{\bar{g}}) = c(n) \frac{(n-2)}{2} \delta\, v^{2/n-2-\delta} \|\nabla_g v^{-2/n-2}\|_g^2 \geq 0$

<u>2nd term</u>: $\int_{B_\sigma} \langle T_{\bar{g}}, v^{4/n-2} \mathcal{D}_g(X) \rangle_{\bar{g}}\, d\omega_{\bar{g}}$

$$= \int_{B_\sigma} \langle T_{\bar{g}}, v^{4/n-2} \operatorname{Hess}_g (v^{-2/n-2}) \rangle_{\bar{g}}\, d\omega_{\bar{g}}$$

Only the trace free part of the Hessian will contribute to the integral, since $T_{\bar{g}}$ is trace free and the trace free and pure trace components are orthogonal, so the identity give us:

$$\frac{1}{n-2} \int_{B_\sigma} \langle T_{\bar{g}}, T_{\bar{g}} - T_g \rangle_{\bar{g}}\, v^{2/n-2}\, d\omega_{\bar{g}} \leq \int_{\partial B_\sigma} T_{\bar{g}}(X, \nu_{\bar{g}})\, d\Sigma_{\bar{g}}$$

Now $\langle T_{\bar{g}}, T_{\bar{g}} - T_g \rangle_{\bar{g}} = \|T_{\bar{g}}\|_{\bar{g}}^2 - \langle T_{\bar{g}}, T_g \rangle_{\bar{g}}$

and by S $\quad \langle T_{\bar{g}}, T_g \rangle_{\bar{g}} \leq \frac{1}{2}\left(\|T_{\bar{g}}\|_{\bar{g}}^2 + \|T_g\|_{\bar{g}}^2\right)$

So $-\frac{1}{2}\|T_g\|_{\bar{g}}^2 \leq \langle T_{\bar{g}}, T_{\bar{g}} - T_g \rangle_{\bar{g}}$

$$\therefore -\frac{1}{2(n-2)} \int_{B_\sigma} \|T_g\|_{\bar{g}}^2\, v^{2/n-2}\, d\omega_{\bar{g}} \leq \int_{\partial B_\sigma} T_{\bar{g}}(X, \nu_{\bar{g}})\, d\Sigma_{\bar{g}}$$

Now $\|T_g\|_{\bar{g}}^2\, v^{2/n-2}\, d\omega_{\bar{g}} = \|T_g\|_g^2\, v^{-8/n-2}\, v^{2/n-2}\, v^{2n/n-2}\, d\omega_g$

where $\|T_g\|_g^2$ is bounded. So we get

$$-c \int_{B_\sigma} v^{\frac{2n-6}{n-2}}\, d\omega_g \leq \int_{\partial B_\sigma} T_{\bar{g}}(X, \nu_{\bar{g}})\, d\Sigma_{\bar{g}}$$

Now by the Proposition

$$v(x) = u(0)u(x) \leq c\, u(0)^2 \left(1 + u(0)^{4/n-2} |x|^2\right)^{-\frac{n-2}{2}}$$
$$= c\left(u(0)^{-4/n-2} + |x|^2\right)^{-\frac{n-2}{2}}$$
$$\leq c' |x|^{2-n}$$

$\therefore$ we get

$$-c\int_{B_\sigma} |x|^{6-2n}\, d\omega_g \leq \int_{\partial B_\sigma} T_{\bar{g}}(X, v_{\bar{g}})\, d\Sigma_{\bar{g}}$$

Now for $n = 3, 4, 5$

$$-c\int_{B_\sigma} |x|^{6-2n}\, d\omega_g = -c'\,\sigma^{6-n}$$

$$\bar{g}_i = v_i^{4/n-2} g_i \Rightarrow G^{4/n-2} g_0 = \hat{g} \quad \text{as } i \Rightarrow \infty$$

So keeping σ-fixed and letting $i \to \infty$, we get

$$-c\,\sigma^{6-n} \leq \int_{\partial B_\sigma} T_{\hat{g}}(X_0, v_{\hat{g}})\, d\Sigma_{\hat{g}}$$

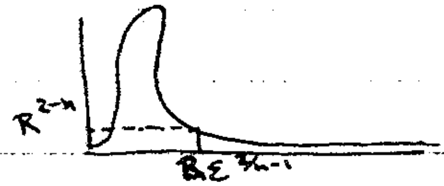
where $X_0 = \nabla_{g_0} G^{-2/n-2} = c\, r\frac{\partial}{\partial r} + O(\sigma^2)$

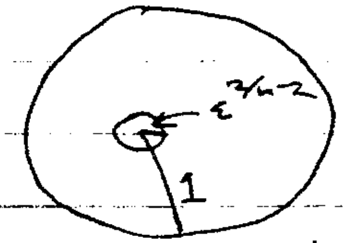
so letting $\sigma \to 0$ we get

$$0 \leq -c\, E(x_\alpha) \quad \blacksquare$$

This proves the The theorem modulo the proof of the proposition.

Note: One can get this arguement to work up to dim 8 by choosing g such that $\|T_g\|_g^2 \leq C|x|^2$.





<u>Proof of Proposition</u>.

Heuristic's:

Simple point of blow up gives good control in a ball of radius $\varepsilon^{2/n-2}$, we want good control in a ball of radius 1

i.e. In the critical limit, at a simple point of blow up $|x|^{\frac{n-2}{2}}\bar{u}(x)$ is decreasing for $|x| \geq c\,\varepsilon^{2/n-2}$, and is very small for $|x| \geq R\,\varepsilon^{2/n-2}$

i.e. $u(x) < \varepsilon_0 |x|^{\frac{2-n}{2}}$ for $|x| \geq R\varepsilon^{2/n-2}$ for ε_0 small

One would hope: $u(x) \leq c\, u(0)^{-1} |x|^{2-n} = c\,\varepsilon |x|^{2-n}$

for $|x| \geq \varepsilon^{2/n-2}$.

Note: the function agree on the inner ball

i.e. $\varepsilon |x|^{2-n} = |x|^{\frac{2-n}{2}}$ for $|x| = \varepsilon^{2/n-2}$

but outside the inner ball, the second bound is better.

<u>Proof</u>:

we have $u(x) \leq \varepsilon_0 |x|^{-2/p-1}$ for $|x| \geq c\,\varepsilon^{2/n-2}$

$Lu + cu^p = 0 \Rightarrow Lu \leq 0$.

Let H be the Greens function for L with $H \equiv 0$ on ∂B_1 with pole at $x = 0$ (i.e. the Dirichlet Green's function)

$H = |x|^{2-n}$ + lower order terms

and $c|x|^{2-n} \leq H(x)$ on $B_{1/2}$

For c_0 small, look at $c_0 \varepsilon H$

$c_0\, \varepsilon\, H(x) < u(x)$ for $|x| = \varepsilon^{2/n-2}$

Maximum principle $\Rightarrow c_0 \varepsilon |x|^{2-n} \leq u(x)$ on $\varepsilon^{2/n-2} \leq |x| \leq 1/2$

This is the lower bound we're after, the upper bound is more subtle.

Look at $\mathcal{L}\varphi = L\varphi + u^{p-1}\varphi$, so $\mathcal{L}u = 0 \Rightarrow \lambda_0(-\mathcal{L})>0$

$u^{p-1} \leq \varepsilon_0^{p-1}|x|^{-2}$, look at $|x|^{-K}$, for $K<(n-2)$.

We want a supersolution, and use a global maximum principle

To technically simplify, we may assume $R_g \equiv 0$ in B_1.

So $L\varphi = \Delta\varphi$.

$\Delta |x|^{-K} = -c[(n-2)-K]|x|^{-2-K}$ in Euclidean space.

By working in local coordinates one can show that on a manifold

$\Delta |x|^{-K} \leq -c(n-2-K)|x|^{-2-K}$

So $\mathcal{L}(|x|^{-K}) \leq [-c(n-2-K)+\varepsilon_0^{p-1}]|x|^{-2-K}$

choose K so that $-c(n-2-K)+\varepsilon_0^{p-1}<0$.

choose $\lambda(K)$ so that $\varepsilon^{\lambda}\varepsilon^{-K(\frac{p-1}{2})} = \varepsilon^{-1}$

look at the function $\varepsilon^{\lambda(K)}|x|^{-k}$.

Let $M = \max_{\partial B_1} u$, $m = \min_{\partial B_1} u$, this (above) gives us agreement on the inner boundary.

look at $\cancel{\mathcal{L}(M+\varepsilon|x|^{-k})} =$

$$\mathcal{L}(M+\varepsilon^{\lambda}|x|^{-K}) = Mu^{p-1} \leq M\varepsilon_0^{p-1}|x|^{2}$$

$M \leq cm$ by the Harnack inequality.

So supersolution $v = M + c\varepsilon^{\lambda}|x|^{-K}$

$$v \geq u \text{ on } \partial(B_1 \setminus B_{c\varepsilon^{\frac{p-1}{2}}}$$

Therefore by the Maximum principle.

$$u(x) \leq M + \varepsilon^{\lambda}|x|^{-K}$$

Since we are at a point of simple blow, we have the monotone decreasing property.

i.e. $|x|^{2/p-1}\bar{u}(x) = \bar{w}(x)$ is monotone decreasing.

Now consider the limiting case $p = \frac{n+2}{n-2}$.

Let $\theta \in (0,1)$ Then

$$\theta^{\frac{n-2}{2}}(M + c_K \varepsilon^\lambda \theta^{-K}) \geq \theta^{\frac{n-2}{2}} \bar{u}(0) \geq \bar{u}(1) \geq m \geq \tfrac{1}{c} M$$

Fix θ small ~~eag~~ enough ~~to~~ so that $\theta^{\frac{n-2}{2}} M < \frac{1}{2c} M$

then $M \leq c_K' \varepsilon^\lambda$

so that $u(x) \leq c'' \varepsilon^\lambda |x|^K$ for $\varepsilon^{2/n-2} \leq |x| \leq 1$.

———o———

Aside: $u^\delta = 1 + O(\varepsilon^\alpha)$

take log $\Rightarrow$ $\delta \log u \sim O(\varepsilon^\alpha)$

at 0, $\log u \sim \log \varepsilon^1$

so that $\delta = \frac{n+2}{n-2} - p = O(\varepsilon^{\alpha'})$

———o———

$X = \sum x^i \frac{\partial}{\partial x^i}$, Analyse the positive term in the argument.

$$\delta \int_{B_1} u^{p+1} + \int_{B_1} \langle T, \mathcal{D}X \rangle_g \, d\omega_g = \text{Boundary term} = O(\varepsilon^2)$$

By using the upper bound derived just above, we can show that $\int_{B_1} \langle T, \mathcal{D}X \rangle_g \, d\omega_g$ is small.

i.e. $\int_{B_1} \langle T, \mathcal{D}X \rangle_g \, d\omega_g \sim O(\varepsilon^\beta)$, $\beta > 0$

and $\int_{B_1} u^{p+1}$ is not small, it's appoximately the Energy.

$$c_0 < \int_{B_1} u^{p+1} \sim E$$

so $\delta = O(u(0)^{-\beta})$

$\therefore$ we have $c^{-1} \varepsilon |x|^{2-n} \leq u(x) \leq c\, \varepsilon^\lambda |x|^{-K}$ for $|x| \geq \varepsilon^{2/n-2}$

Taking δ^{th} Powers of both sides gives the result (δ

$$(c^{-1} \varepsilon |x|^{2-n})^\delta \leq (u(x))^\delta \leq (c \varepsilon^\lambda |x|^{-K})^\delta$$

Using this we sketch the 1^{st} case of the higher dimension result.: $n = 6, 7$.

1^{st}: Lemma: Given g_0 smooth metric near $x=0$

$\exists \quad Q(x) = \sum_{i,j} a_{ij} x^i x^j$ such that $\tilde{g}_0 = e^{2Q(x)} g_0$

satisfies $Ric(\tilde{g}_0) = 0$ at $x=0$

[Note: This is the first step in the constuction of conformal normal coordinates, see the paper on Yamabe Problem by Lee - Parker in the Bulletin.]

Proof of Lemma: $g_{ij}(x) = \delta_{ij} + O(|x|^2)$

$Q(x) = \sum a_{ij} x^i x^j$, $\hat{g} = e^{2Q} g$

$\tilde{R}_{ij} = R_{ij} - (n-2) Q_{ij} + (n-2) Q_i Q_j + (\Delta Q - (n-2)|\nabla Q|^2) g_{ij}$

so at 0 $\quad \tilde{R}_{ij} = R_{ij} - 2(n-2) a_{ij} + 2 (\sum_{k=1}^{n} a_{kk}) \delta_{ij}$

This gives a system of linear equations for the a_{ij}'s.

We want $\tilde{R}_{ij} = 0$

~~Tao~~ Take the trace.

$$0 = \sum R_{ii} - 2(n-2) \sum a_{ii} + 2n \sum a_{kk}$$

$$= \sum R_{ii} + 4 \sum a_{ii}$$

so $\quad \sum a_{ii} = R/4 \quad$ determines a_{ii}.

Plug this into the first equation to reduce the system

Solve the reduced system. ■

Proof $n = 6, 7$:

x_α simple point of blow up for $\{u_i\}$

$v_i = u_i(0) u_i \quad , \quad g_i = v_i^{4/n-2} g_0$

$g_i \Rightarrow g_0$ an Asymtotically Euclidean metric, since
$v_i \longrightarrow G_{x_a}$

From last time $\int_{B_\sigma} v_i^{2/n-2} \langle T_{g_i}, T_{g_i} - T_{g_0} \rangle \, d\omega_{g_i} \leq$ Boundry term.

and $\langle T_{g_i}, T_{g_i} - T_{g_0} \rangle_{g_i} = \|T_{g_i}\|_{g_i}^2 - \langle T_{g_i}, T_{g_0} \rangle_{g_i}$
and $\langle T_{g_i}, T_{g_0} \rangle_{g_i} \leq \frac{1}{2} \|T_{g_i}\|_{g_i}^2 + \frac{1}{2} \|T_{g_0}\|_{g_i}^2$

$\|T_{g_0}\|_g \lessapprox O(|x|^2 v_i^{-8/n-2})$

$$\int_{B_\sigma} v_i^{2/n-2} \|T_{g_i}\|^2 \, d\omega_{g_i} \leq \text{Boundary} + \int_{B_\sigma} |x|^2 v_i^{\frac{2(n-3)}{n-2}} \, d\omega_{g_0}$$
$$\leq \text{Boundary} + c \underbrace{\int_{B_\sigma} |x|^{8-2n} \, dx}_{\text{converges} \to O(\sigma^{8-n})} \quad \text{if } n < 8$$

look at $\varepsilon_i \searrow 0 \qquad g = G_{x_a}^{4/n-2} g_0$

$$\int_{B_\sigma} |x|^2 \|T_g\|_g^2 \, d\omega_g \leq \text{Boundary} + O(\sigma^{8-n})$$
$G = |x|^{2-n}(1 + O(|x|^2))$

Invert coordinates $y = \frac{x}{|x|^2}$, $g_{ij} = \delta_{ij} + O(|x|^{-2})$ for y large

$$\int_{\mathbb{R}^n \setminus B_{\sigma^{-1}}} |y|^2 \|T_g\|_g^2 \, dy \leq \text{Boundary} + O(\sigma^{8-n})$$

g is Scaler flat, so $T_g = Ric(g)$.

This gives decay conditions on $Ric(g)$, we can use this and the fact that g is asymtotically Euclidean to get decay conditions on the Weyl tensor.

1^{ST} we choose Harmonic coordinates (see Bartnik).

i.e. $\exists$ a Harmonic coordinate system $y^1, \ldots, y^n$

$g_{ij} = \delta_{ij} + O(|y|^{-2})$

i.e. $\Delta_g y^i = 0$ near ∞ for $i = 1, \ldots, n$

In Harmonic coordinates $R_{ij} = -\frac{1}{2} \Delta_g g_{ij} + O(|\nabla g|^2)$,

Now $O(|\nabla g|^2) = O(|y|^{-6})$

so $\|Ric\|^2 = \frac{1}{4} \|\Delta_g g\|^2 + O(|y|^{-6})$.

We now use an inequality for weighted Sobolev spaces

$$c^{-1} \int_{\mathbb{R}^n \setminus B_{\sigma^{-1}}} |y|^{-2} \|g - \delta\|^2 \, dy \le \int_{\mathbb{R}^n \setminus B_{\sigma^{-1}}} |y|^2 \|\Delta_g g\|^2 \, dy \le \text{Bdry} + O(\sigma^{8-n})$$

On the other hand $g_{ij} - \delta_{ij} = O(|y|^{-m})$ near ∞, $m \in \mathbb{Z}_+$

So on the left hand side, the integrand is like $\sim |y|^{-2-2m}$ and the integral is finite.

$\therefore$ $2 + 2m > n \Rightarrow m > \frac{n-2}{2}$

i.e. $m \ge 3$ for $n \ge 6$.

So we get $\|Riem_g\|_g = O(|y|^{-2-m})$.

Now Weyl is Conformally invariant

i.e. $W_g{}^i{}_{jkl} = W_{g_0}{}^i{}_{jkl}$.

So $O(|x|^{2+m}) \sim \|W_g\|_g = G^{-4/n-2} \|W_{g_0}\|_{g_0} \sim O(|x|^4) \|W_{g_0}\|_{g_0}$

$\therefore m \ge 3 \Rightarrow W_{g_0} = 0$ at $x = 0$

So $\quad 0 \leq$ Boundary term $+ O(\sigma^{8-n})$

So letting $\sigma \to 0$

we get $0 \leq -E(x_\alpha)$ is defined since

$W_{g_0} = 0$ at x_α

$\therefore E(x_\alpha) \leq 0$

This takes care of case(i)

Case (ii) is similar, here we are in a conformally flat situation, still using $n < 8$, we still have an $O(\sigma^{8-n})$ term.

To be Continued Winter Quarter 1989.