An End-to-End Gluing Construction for Surfaces of Constant Mean Curvature

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Abstract

An End-to-End Gluing Construction for Surfaces of Constant Mean Curvature

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In this dissertation we present a method for constructing new surfaces of constant mean curvature in Euclidean space by gluing known surfaces of constant mean curvature together end-to-end, provided the summands satisfy some technical conditions. We also show that there exist surfaces of constant mean curvature which satisfy the conditions for gluing and use this construction to explore the topology of the moduli space of surfaces of constant mean curvature.

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ACKNOWLEDGMENTS

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Chapter 1

INTRODUCTION

The mean curvature of surfaces embedded in $\mathbb{R}^3$ has been studied since the late 1700's (see, for example [Lag60]). Embeddings with mean curvature 1 (referred to as CMC surfaces below) are particularly interesting, as they are critical points for the functional *Area - Volume*. Current research regarding CMC surfaces centers on the following two questions: how can one construct examples of these surfaces, and how well can one describe the set of such surfaces with fixed topology? At the heart of both these questions lies the attempt to understand solutions to the mean curvature equation, which is a nonlinear partial differential equation, on a fixed surface. If one writes the surface as the graph of a function u (which one can always do locally), then the mean curvature equation becomes

$$\frac{1}{2}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 1, \tag{1.1}$$

which is a well-studied quasilinear elliptic PDE in divergence form.

Below we will study properly embedded, noncompact CMC surfaces of finite topological type. The goal of this dissertation is to describe a method to construct many new examples of such CMC surfaces.

The first example of a noncompact, complete, embedded CMC surface is the right circular cylinder of radius $\frac{1}{2}$. One can also consider a string of mutually tangent unit spheres such that the points of contact between consecutive spheres all lie on the same line, which collectively form a singular surface. In 1841 C. Delaunay [Del41] classified all the rotationally symmetric CMC surfaces, including the two examples above. These surfaces are periodic and determined up to rigid motion by their necksize ϵ, which is the minimum radius of a cross-section perpendicular to the axis of symmetry. A necksize of $\frac{1}{2}$ corresponds to a cylinder and as $\epsilon \to 0$ the surfaces tend to the string of unit spheres mentioned above.

Kapouleas constructed the next examples of noncompact, complete, embedded, CMC surfaces in [Kap90] via a gluing construction. Since then, several other new examples of complete, noncompact, embedded CMC surfaces have appeared, including the gluing constructions of Mazzeo and Pacard [MP01] and Mazzeo, Pacard, and Pollack [MPP]. Each of these gluing constructions uses tools from modern analysis, particularly partial differential equations. Kapouleas uses singular perturbation theory while Mazzeo, Pacard, and Pollack solve boundary value problems for equation (1.1) and match Cauchy data. In each construction one must carefully account for the behavior of solutions to equation (1.1) on noncompact domains.

The main result of this dissertation is the construction of new CMC embeddings by gluing two CMC embeddings together end-to-end in a sense described below. We start with two noncompact, proper embeddings $X_1 : \Sigma_1 \to \mathbb{R}^3$ and $X_2 : \Sigma_2 \to R^3$ where Σ_1 (respectively Σ_2) is topologically a closed surface of genus g_1 with k_1 punctures (respectively of genus g_2 with k_2 punctures). The properness condition forces $\operatorname{dist}(X_i(p), (0,0,0)) \to \infty$ as p approaches any of the punctures in Σ_i. By a result of Korevaar, Kusner, and Solomon [KKS89], each end E_j of X_1 or X_2 (image of a small neighborhood of a puncture) is asymptotic to a Delaunay embedding D_j. Pick two ends $E_1 \subset X_1$ and $E_2 \subset X_2$ such that the E_1 and E_2 are asymptotic to congruent Delaunay surfaces. Next align X_1 and X_2 so that E_1 and E_2 are asymptotic to opposite ends of the same Delaunay surface D, so in particular E_1 and E_2 are graphs over opposite ends of the same cylinder. One can then patch X_1 and X_2 together using a cut-off function along this cylinder to obtain an embedding $\bar{X} : \Sigma \to \mathbb{R}^3$. Topologically, Σ is a closed surface of genus $g = g_1 + g_2$ with $k = k_1 + k_2 - 2$ punctures. The mean curvature of of $\bar{X}$ is 1 away from the gluing region and is globally close to 1, and so $\bar{X}$ is an approximate solution to equation (1.1), which we will describe more explicitly in Section 3. We have two parameters in this construction: R, which we can think of as the distance along the ends E_1 and E_2 at which we glue, and ϕ, which specifies a rotation of X_2 about the axis of D. The translation parameter R is discrete, as we can only translate by periods of the Delaunay surface D. To indicate the dependence on these parameters we will denote the embedding as $\bar{X}_{R,\phi}$. However, much of the analysis is independent of one (or both) of these parameters, and so we will often suppress this dependence.

The goal now is to perturb $\bar{X}_{R,\phi}$ using normal perturbations and geometric deformations to obtain an embedding $X_{R,\phi} : \Sigma \to \mathbb{R}^3$ which has mean curvature 1. This is equivalent to solving a nonlinear partial differential equation on Σ in the following way. Given an exponentially decaying function v and a geometric deformation parameter u we obtain from $\bar{X}_{R,\phi}$ another embedding $\bar{X}_{R,\phi}(u,v)$ by applying the geometric deformation associated to u to the normal perturbation of $\bar{X}_{R,\phi}$ by v. We parameterize this space of geometric deformations by translations, rotations, and variations of the necksize parameters and use this parameterization to obtain a norm on the space of geometric deformations. We denote the mean curvature of this new embedding as $H(u,v)$. The equation we wish to solve is

$$1 = H(u,v) = H(0,0) + \mathcal{L}_{\bar{X}_{R,\phi}}(u+v) + \mathcal{Q}_{\bar{X}_{R,\phi}}(u,v).$$

Here we have expanded the mean curvature $H(u,v)$ in a Taylor series about $(0,0)$. $\mathcal{L}_{\bar{X}_{R,\phi}}$ is the linearized mean curvature operator and $\mathcal{Q}_{\bar{X}_{R,\phi}}$ contains all the quadratic and higher order terms. If we write $H(0,0) = 1 - \psi$ then the above equation becomes

$$\mathcal{L}_{\bar{X}_{R,\phi}}(u+v) = \psi - \mathcal{Q}_{\bar{X}_{R,\phi}}(u,v). \tag{1.2}$$

The linearized equation is

$$\mathcal{L}_{\bar{X}_{R,\phi}}(u+v) = \psi. \tag{1.3}$$

In Chapter 4 we show that if X_1 and X_2 satisfy certain conditions one can always find a tempered solution to equation (1.3). The conditions can be summarized as follows:

- we want both X_1 and X_2 to be nondegenerate (their linearized mean curvature operators should have no exponentially decaying solutions) and

- we want X_1 to admit a deformation through CMC surfaces which changes the asymptotic necksize of E_1 to first order.

We also show that the Green's operator we construct is uniformly (in R) bounded in an appropriate norm. In Section 5 we solve equation (1.2) using a contraction mapping. This yields the following theorem.

Theorem 1. *Let X_1 and X_2 be noncompact, proper, CMC embeddings with finite topology which are nondegenerate. Suppose one can chose ends $E_1 \subset X_1$ and $E_2 \subset X_2$ which are asymptotic to congruent Delaunay surfaces and suppose further that X_1 admits a deformation through CMC surfaces which changes the asymptotic necksize of E_1 to first order. Let $\bar{X}_{R,\phi}$ be the approximate solution of Section 3. Then there exists $R_0 > 0$ and $\eta > 0$ such that for $R \geq R_0$ one can find a geometric deformation parameter u with $|u| \leq \eta$ and an exponentially decaying function v such that the embedding $\bar{X}_{R,\phi}(u, v)$ has constant mean curvature equal to one. Moreover, this CMC embedding is nondegenerate.*

Finally, in Section 6 we use this construction to prove the following.

Corollary. *For $k \geq 4$ the moduli space of $\mathcal{M}_k$ of k-ended, genus-zero CMC surfaces has connected components which are not simply connected.*

We also show that every nondegenerate genus-zero three-ended CMC surface satisfies the gluing hypotheses for all ends and that the gluing hypotheses are stable under perturbation. One particular example of this gluing construction is a construction we will call *doubling*. In this case, we take X_2 to be congruent to X_1 and patch the chosen end to a copy of itself.

Chapter 2

NOTATION

Below, we will always use the symbols g and A to denote the first and second fundamental forms (respectively) of an embedding $X : \Sigma \to \mathbb{R}^3$. More explicitly, if we have coordinates (s, θ) on Σ, then we write $g = Eds^2 + 2Fdsd\theta + Gd\theta^2$ and $A = Lds^2 + 2Mdsd\theta + Nd\theta^2$. With these coordinates, we will also orient the surface with the normal $\nu = \frac{\partial_s X \times \partial_\theta X}{\|\partial_s X \times \partial_\theta X\|}$. We will denote the mean curvature by $H = \frac{1}{2} \operatorname{tr}_g A$ and the Gaussian curvature by $K = \det A$. If we have another surface $\bar{\Sigma}$, we will denote its metric as $\bar{g}$, and so on. Also, we will often use subscripts to denote derivatives.

We will always consider noncompact, proper embeddings $X : \Sigma \to \mathbb{R}^3$ of surfaces of finite topology. The function χ will always be a cut-off function, either centered about 0 on a line or radially symmetric with its support a suitably chosen ball. Given a rotationally symmetric surface, we will denote its axis by $\vec{a}$ where $\{\vec{a}, \vec{b}, \vec{c}\}$ form an oriented orthonormal basis for Euclidean three-space.

In general, we will be able to decompose all embedded surfaces we encounter into a compact piece K and some number of ends, each of which is a graph over some half-infinite cylinder. We will parameterize the jth end E_j with coordinates $(t_j, \theta_j) \in (0, \infty) \times \mathbb{S}^1$. Also, we will have a graph over a long, but finite, cylinder in our approximate solution $\bar{X}_{R,\phi}$, which we will usually parameterize with coordinates $(t, \theta) \in (-R, R) \times \mathbb{S}^1$.

Chapter 3

THE APPROXIMATE SOLUTION

The first step is to construct an approximate solution. We start with two complete, embedded, noncompact CMC surfaces $X_i : \Sigma_i \hookrightarrow \mathbb{R}^3$ with finite topology. The ends of the surfaces are the unbounded connected components of $X_i(\Sigma_i) \cap (\mathbb{R}^3 \backslash \mathbb{B}_{r_0})$ where r_0 is taken large enough so that the number of such components remains constant if r_0 increases. Roughly speaking, we can decompose the surface $X_i(\Sigma_i)$ into a union of a compact piece and k_i noncompact ends. By a theorem of Korevaar, Kusner, and Solomon [KKS89] each end of $X_i(\Sigma_i)$ is asymptotic to a Delaunay surface $D = D_\tau$ of Delaunay parameter τ. We will explain this convergence below. We will assume that we can choose ends E_i of $X_i(\Sigma_i)$ which are asymptotic to congruent Delaunay surfaces.

The embedded Delaunay surfaces are CMC surfaces which are rotationally symmetric about some axis. They have a profile curve which is the graph of some positive function $\rho_D(t)$. Thus D can be parameterized as $D(t,\theta) = t\vec{a} + \rho_D(t)\omega(\theta)$ where $\{\vec{a}, \vec{b}, \vec{c}\}$ is an oriented orthonormal basis for $\mathbb{R}^3$ and $\omega(\theta) = \cos\theta\vec{b} + \sin\theta\vec{c}$. By examining the equation ρ_D must satisfy, one can show that ρ_D is periodic. The necks of the Delaunay surface are small neighborhoods of the circles $\{t_0\vec{a} + \rho_D(t_0)\omega(\theta)\}$ where ρ_D attains a minimum at t_0 and the necksize is the minimum value of ρ_D. It is convenient to parameterize the Delaunay surfaces with the parameter $\tau = 2\epsilon - \epsilon^2$ and to denote the associated Delaunay surface as D_τ. Appendix B.1 contains a longer explanation of these surfaces. The result of [KKS89] states that there is an $r > 0$ such that the unbounded connected components of $X_i(\Sigma_i) \cap (\mathbb{R}^3 \backslash \mathbb{B}_r)$ can each be written as graphs over a cylinder $(r, \infty) \times \mathbb{S}^1$. Moreover, if we parameterize such an end E_i as

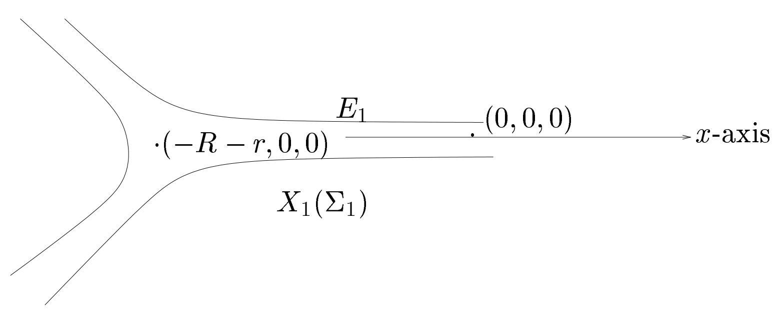
$$(t,\theta) \mapsto t\vec{a} + \rho_{E_i}(t,\theta)\omega(\theta) : (r,\infty) \to \mathbb{R}^3,$$

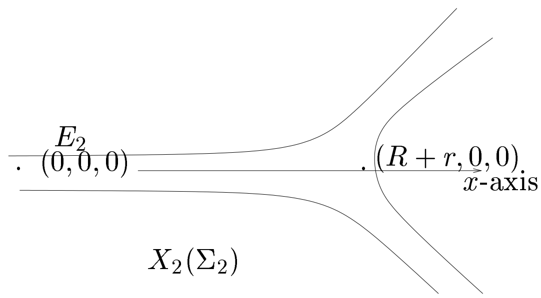
then there exists an embedded Delaunay surface $D(t,\theta) = t\vec{a} + \rho_D(t)\omega(\theta)$ such that the following estimate holds:

$$\|\rho_D(t) - \rho_{E_i}(t,\theta)\|_{2,\alpha,t_0} = O(e^{-\gamma_2(\tau)t_0})$$

for some $0 < \alpha < 1$, all $t_0 \geq r+1$, and some $\gamma_2(\tau) > 0$ which depends on α and the necksize of D. The norm $\|\cdot\|_{2,\alpha,t_0}$ is the standard Hölder norm on $(t_0-1, t_0+1)\times\mathbb{S}^1$. The coefficient $\gamma_2(\tau)$ is called the second indicial root associated to D_τ. The indicial roots arise naturally in the study of Fredholm properties of the linearized mean curvature operator of a Delaunay surface and determine asymptotic expansions of solutions to the homogeneous linearized mean curvature equation. They are positive, depend continuously on τ, and correspond both to poles of an associated operator (see Appendix F) and to exponential growth rates of solutions to certain ODEs (see the next chapter). The indicial roots of a Delaunay surface are covered more thoroughly in [MP01] and [MPPR].

Without loss of generality, we can suppose D has the x axis as its axis of symmetry and that ρ_D has a minimum occurring at $x = 0$. This amounts to a translation and rotation of X_i. Moreover, by another translation of X_i we can take the ball $\mathbb{B}_r$ in the above definition of the ends to be centered at $(\pm(R+r), 0, 0)$ ($-R-r$ for $i = 1$ and $R+r$ for $i = 2$) where R is a large positive parameter (see the figures below). We can write E_1 as a graph over the cylinder $(-R,\infty)\times\mathbb{S}^1$ and E_2 as a graph over the cylinder $(-\infty, R)\times\mathbb{S}^1$. Under this situation $\|\rho_{E_1}(s,\theta) - \rho_D(s-R)\|_{2,\alpha,0} = O(e^{-\gamma_2(\tau)R})$ and $\|\rho_{E_2}(s,\theta) - \rho_D(s+r)\|_{2,\alpha,0} = O(e^{-\gamma_2(\tau)R})$. Notice that the two surfaces X_1 and X_2 are close in the $C^{2,\alpha}$ norm (in fact, the C^1 norm as well) only if is R is an integer multiple of the period of the Delaunay surface D. We also remark that the embedding $X_i : \Sigma_i \to \mathbb{R}^3$ depends on R. We will suppress this dependence, as two such embeddings of Σ_i described above can only differ by a translation along the x axis.





Note that in the region $-R \leq x \leq R$ each of E_1, E_2, and D are graphs over the cylinder $(-R,R)\times\mathbb{S}^1$.

Given a surface parameterized as a graph over a cylinder

$$(t,\theta)\mapsto(t,\rho(t,\theta)\cos\theta,\rho(t,\theta)\sin\theta),$$

one can compute that the mean curvature is given by

$$H=\frac{-\rho^3(\partial_t^2\rho-\partial_\theta^2\rho)+\rho^2(1+(\partial_t\rho)^2)+\rho((\partial_t\rho)(\partial_\theta\rho)(\partial_t\partial_\theta\rho)-\partial_\theta^2\rho)+2(\partial_\theta\rho)^2}{(\rho^2+\rho^2(\partial_t\rho)^2-4(\partial_t\rho)^2(\partial_\theta\rho)^2)\sqrt{\rho^2+(\partial_\theta\rho)^2+\rho^2(\partial_t\rho)^2}}. \tag{3.1}$$

Let $\chi=\chi(t)\geq 0$ be a nonincreasing cutoff function where

$$\chi(t)=\begin{cases}1 & \text{for} \quad t<-1\\ 0 & \text{for} \quad t>1.\end{cases}$$

We construct the approximate solution $\bar{X}:\Sigma\to\mathbb{R}^3$ as follows. The surface Σ is topologically a closed Riemann surface of genus $g=g_1+g_2$ and $k=k_1+k_2-2$ punctures. We will number the ends of $\bar{X}$ as $E_3\ldots E_{k_1+k_2}$, reserving the labeling of E_1 and E_2 for the ends of X_1 and X_2 we truncate in the gluing construction. We can write part of $\bar{X}$ as a graph over the cylinder $(-R,R)\times\mathbb{S}^1$. In the region corresponding to $-R<t<-1$ let $\bar{X}$ be given by

$$(t,\theta)\mapsto(t,\rho_{E_1}(t,\theta)\cos\theta,\rho_{E_1}(t,\theta)\sin\theta).$$

In the region $1<t<R$ let $\bar{X}$ be given by

$$(t,\theta)\mapsto(t,\rho_{E_2}(t,\theta)\cos\theta,\rho_{E_2}(t,\theta)\sin\theta).$$

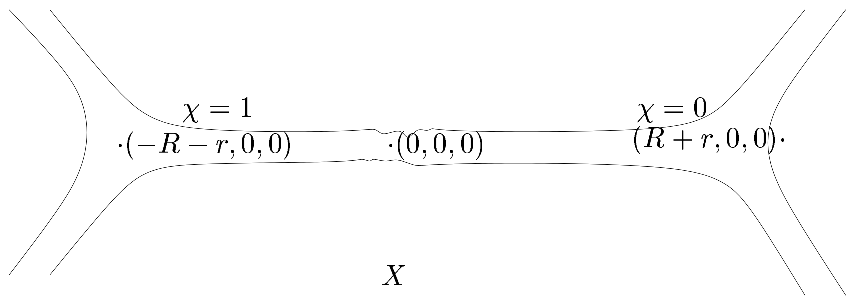
In the region $-1\leq t\leq 1$ parameterize $\bar{X}$ by

$$(t,\theta)\mapsto(t,\rho_{\bar{X}}(t,\theta)\cos\theta,\rho_{\bar{X}}(t,\theta)\sin\theta)$$

where

$$\rho_{\bar{X}}(t,\theta) = \chi(t)\rho_{E_1}(t,\theta) + (1-\chi(t))\rho_{E_2}(t,\theta).$$

This gives a smooth surface with two boundary components written as a graph over a bounded cylinder. Because $\bar{X}$ and X_1 are given as graphs of the same function over the cylinder $(-R,-1)\times\mathbb{S}^1$, we can extend $\bar{X}$ past the boundary component $\{-R\}\times\mathbb{S}^1$ by letting it agree with X_1. We can similarly extend $\bar{X}$ past the boundary component $\{R\}\times\mathbb{S}^1$ to agree with X_2 (see the figure below). Then $\bar{X}$ is a smooth embedding of Σ, and it is CMC in the regions corresponding to $t<-1$ and $t>1$.



In the region $-1 \le t \le 1$ $\bar{X}$ is a graph over the Delaunay surface D of the function $\rho_{\bar{X}}-\rho_D$. The function $\rho_{\bar{X}}$ is a convex combination of ρ_{E_1} and ρ_{E_2} and $\|\rho_{E_i}\|_{C^{2,\alpha}((-1,1)\times\mathbb{S}^1)} = O(e^{-\gamma_2(\tau)R})$. Thus

$$\begin{aligned} H_{\bar{X}} &= H_D + \mathcal{L}_D(\rho_{\bar{X}}-\rho_D) + O((\rho_{\bar{X}}-\rho_D)^2) \\ &= 1 + O(\|\mathcal{L}_D\|\|\rho_{\bar{X}}-\rho_D\|_{C^{2,\alpha}((-1,1)\times\mathbb{S}^1)}) + O(e^{-2\gamma_2(\tau)R}) = 1 + O(e^{-\gamma_2(\tau)R}) \end{aligned}$$

where $\mathcal{L}_D$ is the linearized mean curvature operator about the Delaunay surface D (see the next chapter).We will write this as

$$H_{\bar{X}} = 1-\psi$$

where ψ is supported in the region $-1\le t\le 1$ and $\|\psi\|_{C^{2,\alpha}} = O(e^{-\gamma_2(\tau)R})$. We can adjust this construction by changing the translation parameter R by a multiple of the period of D. In particular, we can make R as large as we please. Thus we can make this error ψ as small as we wish to start the construction.

In addition to the parameter R, we also have an angular parameter ϕ. One can think of this parameter as determining the angle of $X_2(\Sigma_2)$ relative to $X_1(\Sigma_1)$, as measured from some chosen starting position. We can change ϕ by rotating $X_2(\Sigma_2)$ about the axis of D. In constructing the approximate solution $\bar{X}$, we did not use this parameter at all. But later we may reconstruct $\bar{X}$ for a different value of ϕ than the value we initially chose. To indicate the choice of parameters R and ϕ, we will denote this approximate solution as $\bar{X}_{R,\phi}$. As mentioned in the introduction, we will often fix one or both of these parameters, and in these instances we will suppress the subscripts.

In the doubling case, take $\Sigma_1 = \Sigma_2$. Align $X_1(\Sigma_1)$ so that the chosen end E_1 has the x axis as the axis of symmetry for its model Delaunay surface D and so that D has a neck at $x = 0$. Then let $X_2(\Sigma_2)$ be the embedding one obtains by rotating $X_1(\Sigma_1)$ about the z axis by an angle of π.

In addition to this construction, one could also build in "misalignment errors". For example, instead of aligning the surfaces X_1 and X_2 exactly, one could rotate X_2 so that the axis of E_2 differs from the axis of E_1 by an angle α which is $O(e^{-\gamma_2(\tau)R})$. The resulting approximate solution would still have mean curvature 1 outside a small cylindrical region, and globally its mean curvature would still be $1 + O(e^{-\gamma_2(\tau)R})$. Thus one can perform the same analysis as we do below, but starting with this bent approximate solution, obtaining a CMC surface from the bent configuration as well. We can also induce other misalignment errors, such as translating X_2 in a direction perpendicular to the axis of E_2 by a small amount, or starting with an asymptotic necksize of E_2 which is not equal to (but close to) the asymptotic necksize of E_1. We remark that this approximate solution is unbalanced, in that the weight vectors associated to its ends do not sum to zero (see [Kus91] or appendix C.2). However, the geometric parameters we use to deform the surface in the nonlinear part of this construction change these weight vectors, so the contraction mapping we use in the final step produces a balanced surface automatically.

Chapter 4

MAPPING PROPERTIES OF THE LINEARIZED OPERATOR

In Appendix E we show that the linearization of the left hand side of equation (1.1) is $\mathcal{L}_X = \frac{1}{2}(\Delta_X + \|A_X\|^2)$, where Δ_X is the Laplace-Beltrami operator and A_X is the second fundamental form. The operator $\mathcal{L}_X$ is called the Jacobi operator of the embedding X. In this section we will study the mapping properties of $\mathcal{L}_{\bar{X}_{R,\phi}}$ where $\bar{X}_{R,\phi}$ is the approximate solution constructed in the last section. In particular, we are interested in finding tempered solutions u to the equation $\mathcal{L}_{\bar{X}_{R,\phi}} u = \psi$ when ψ decays exponentially along the ends of $\bar{X}_{R,\phi}$.

4.1 The Jacobi operator for a Delaunay surface

First consider the case where $X = D_\tau$ is a Delaunay embedding with necksize ϵ and $\tau = 2\epsilon - \epsilon^2$. In this case, the Jacobi operator is

$$\mathcal{L}_D = \frac{1}{2\tau^2 e^{2\sigma}}(\partial_s^2 + \partial_\theta^2 + \tau^2 \cosh 2\sigma), \tag{4.1}$$

where $\sigma'' + \frac{\tau^2}{2}\sinh 2\sigma = 0$ (see section 4.1 of [MP01]). This parameterization of the Delaunay surface differs from the one given in the previous chapter by the change of coordinates $t = k(s)$ and $\rho(t) = \tau e^{\sigma(s)}$ where $k' = \frac{\tau^2}{2}(e^{2\sigma} + 1)$. We can understand solutions to $\mathcal{L}_D u = 0$ and the spectral properties of $\mathcal{L}_D$ in terms of solutions to the ODEs

$$(\partial_s^2 + \tau^2 \cosh \sigma - j^2) u^{j,\pm} = 0. \tag{4.2}$$

Indeed, if $u^{j,\pm}(s)$ solve equation (4.2) with initial conditions

$$u^{j,+}(0) = 1 \quad \partial_s u^{j,+}(0) = 0 \quad u^{j,-}(0) = 0 \quad \partial_s u^{j,-}(0) = 1, \tag{4.3}$$

and we let

$$\chi_j = \begin{cases} \frac{1}{\sqrt{\pi}} \cos j\theta & j > 0 \\ \frac{1}{\sqrt{2\pi}} & j = 0 \\ \frac{1}{\sqrt{\pi}} \sin j\theta & j < 0, \end{cases}$$

and define

$$u(s,\theta) = \sum_{-\infty}^{\infty} \chi_j(\theta)(a^{j,+}u^{j,+}(s) + a^{j,-}u^{j,-}(s))$$

then $\mathcal{L}_D u = 0$ provided $a^{j,\pm}$ are chosen so that the series converges. We call the Jacobi fields $u = \sum_{|j|\leq 1} \chi_j(a^{j,+}u^{j,+} + a^{j,-}u^{j,-})$ the low eigenmode solutions.

One can identify these low eigenmode Jacobi fields with explicit geometric deformations of D. To demonstrate this phenomenon, we will examine the one-parameter family of Delaunay surfaces obtained by translating a given surface along its axis. We parameterize the Delaunay surface as $D(t,\theta) = (\rho(t)\cos\theta, \rho(t)\sin\theta, t)$. In the (t,θ) coordinates, the normal vector ν is given by $\nu(t,\theta) = \frac{1}{\sqrt{1+\rho_t^2}}(-\cos\theta, -\sin\theta, \rho_t)$. We wish to write a translation $D_\eta(t,\theta) = D(t,\theta) + (0,0,\eta) = D(t',\theta') + u(t',\theta')\nu(t',\theta')$ as a normal variation of $D(t,\theta)$. We are left with three equations

$$\begin{cases} \rho(t)\cos\theta &=& \rho(t')\cos\theta' - \frac{1}{\sqrt{1+\rho_t^2(t')}}u(t',\theta')\cos\theta' \\ \rho(t)\sin\theta &=& \rho(t')\sin\theta' - \frac{1}{\sqrt{1+\rho_t^2(t')}}u(t',\theta')\sin\theta' \\ t+\eta &=& t' + \frac{\rho_t(t')u(t',\theta')}{\sqrt{1+\rho_t^2(t')}}. \end{cases} \tag{4.4}$$

Squaring the first two equations of (4.4) and adding them together we get

$$\rho^2(t) = \rho^2(t') + \frac{u^2(t',\theta')}{1+\rho_t^2(t')} - \frac{2\rho(t')u(t',\theta')}{\sqrt{1+\rho_t^2(t')}}.$$

Notice that from this equation we can take u to be a function of t alone. Multiplying through by $1+\rho_t^2(t')$ and rearranging yields

$$u^2(t') - 2\rho(t')\sqrt{1+\rho_t^2(t')}u(t') + (\rho^2(t') - \rho^2(t))(1+\rho_t^2(t')) = 0.$$

The quadratic formula then implies

$$u(t') = (\rho(t') - \rho(t))\sqrt{1+\rho_t^2(t')}.$$

From the third equation of (4.4),

$$t - t' = \frac{u(t')\rho_t(t')}{\sqrt{1+\rho_t^2(t')}} - \eta.$$

Thus

$$\rho(t) = \rho(t') + (t-t')\rho_t(t') + O(t-t')^2 = \rho(t') + \frac{\rho_t^2(t')u(t')}{\sqrt{1+\rho_t^2(t')}} - \eta\rho_t(t') + O(t-t')^2.$$

Using this expression for $\rho(t)$ yields

$$u(t') = \frac{\eta\rho_t(t')}{\sqrt{1+\rho_t^2(t')}} + O(t-t')^2$$

and thus the Jacobi field which generates this translational deformation of D is the function

$$u = u^{0,+} = \frac{\rho_t}{\sqrt{1+\rho_t^2}} = \sigma_s. \tag{4.5}$$

Notice that $u^{0,+}(s) = \sigma_s(s)$ solves equation (4.2) for $j = 0$:

$$u_{ss} + \tau^2 u \cosh 2\sigma = \sigma_{sss} + \tau^2\sigma_s \cosh 2\sigma = 0.$$

This computation shows that the infinitesimal generator of the one-parameter family of Delaunay surfaces obtained by translating a given surface along its axis is $\chi_0 u^{0,+}$. The infinitesimal generators of the one-parameter families of Delaunay surfaces obtained by translating a given surface perpendicular to its axis are $\chi_{\pm 1}u^{\pm 1,+}$. The infinitesimal generators of the one-parameter families of Delaunay surfaces obtained by rotating the axis of a given surface are $\chi_{\pm 1}u^{\pm 1,-}$. Finally, the infinitesimal generator of the one-parameter family of Delaunay surfaces obtained by varying the necksize of a given surface is $\chi_0 u^{0,-}$.

Notice that all the low eigenmodes Jacobi fields are either bounded and periodic or grow linearly. In fact, these low eigenmodes are the only Jacobi fields which are exponentially bounded (see Lemma 2 below). A precise definition of the indicial roots of $D = D_\tau$ would be that the jth indicial root $\gamma_j(\tau)$ is the coefficients of exponential growth of homogeneous solutions to equation (4.2). From this formulation one sees that $\gamma_1 = \gamma_0 = 0$ and $\gamma_j > 0$ for $|j| \geq 2$. In fact, the potential in equation (4.2) is strictly negative for $|j| \geq 2$. The name arises from the fact that one can recover the indicial roots from the eigenvalues of the matrix which translates the solution by a period of the equation. As equation (4.2) is periodic with period S, there is a matrix T_j such that any homogeneous solution w to equation (4.2) satisfies

$$\begin{pmatrix} w(s+S) \\ \partial_s w(s+S) \end{pmatrix} = T_j \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix}$$

and one can show that the indicial roots are the real parts of $\frac{1}{S}\log\lambda_j$ where λ_j is an eigenvalue of T_j. These growth properties motivate the use of the following function spaces. Recall that given an embedding $X : \Sigma \to \mathbb{R}^3$ with k asymptotically Delaunay ends we have a decomposition $X(\Sigma) = K \cup (\cup_1^k E_j((0,\infty)\times\mathbb{S}^1))$ where

$$E_j(t_j,\theta_j) = t_j\vec{a}_j(\rho_{D_j}(t_j) + \rho_j(t_j,\theta_j))(\cos\theta_j\vec{b}_j + \sin\theta_j\vec{c}_j)$$

and $K = X(\Sigma)\backslash(\cup_1^k E_j)$.

Definition 1. *Given a noncompact proper embedding $X : \Sigma \to \mathbb{R}^3$ as above and a function $u : X(\Sigma) \to \mathbb{R}$ with $u \in C^{k,\alpha}_{loc}(X(\Sigma))$ we define the weighted Hölder space $C^{k,\alpha}_\delta$ to be*

$$\{\|u\|_{C^{k,\alpha}_\delta} = \sup_{t'_j>1}\|e^{-\delta t_j}u(E_j(t_j,\theta_j))\|_{C^{k,\alpha}([t'_j-1,t'_j+1]\times\mathbb{S}^1)} + \|u\|_{C^{k,\alpha}(K)} < \infty\}.$$

Functions in $C^{k,\alpha}_\delta(X)$ can grow at most like $e^{\delta t_j}$ on each end E_j. For the approximate solution we will need to use a more refined weighting function, defined below. Recall that $\bar{X}_{R,\phi}$ has a decomposition $\bar{X}_{R,\phi}(\Sigma) = K \cup (\cup_3^{k_1+k_2} E_j((0,\infty)\times\mathbb{S}^1)) \cup C_{R,\phi}((-R,R)\times\mathbb{S}^1)$ where the ends E_j are parameterized as above,

$$C_{R,\phi}(t,\theta) = t(1,0,0) + (\rho_D(t) + \chi(t)\rho_{E_1}(t,\theta) + (1-\chi(t))\rho_{E_2}(t,\theta))(0,\cos\theta,\sin\theta)$$

where $-R \le t \le R$ and χ is a cut-off function, and $K = \bar{X}_{R,\phi}\backslash(\cup_3^{k_1+k_2} E_j((0,\infty)\times\mathbb{S}^1) \cup C_{R,\phi}((-R,R)\times\mathbb{S}^1))$.

Definition 2. *Let $\bar{X}_{R,\phi}$ be the approximate solution constructed above and let $u : \bar{X}_{R,\phi}(\Sigma) \to \mathbb{R}$ with $u \in C^{k,\alpha}_{loc}(\bar{X}(\Sigma))$. Then define the weighted Hölder space $F^{k,\alpha}_\delta$ to be*

$$\begin{aligned}\{\|u\|_{F^{k,\alpha}_\delta} &= \sup_{|t_0|\le R-1}\|\frac{\cosh^\delta t}{\cosh^\delta R}u(C_{R,\phi}(t,\theta))\|_{C^{k,\alpha}([t_0-1,t_0+1]\times\mathbb{S}^1)} \\ &\quad + \sup_{t'_j>1}\|e^{-\delta t_j}u(E_j(t_j,\theta_j))\|_{C^{k,\alpha}([t'_j-1,t'_j+1]\times\mathbb{S}^1)} + \|u\|_{C^{k,\alpha}(K)} < \infty\}.\end{aligned}$$

Note that the space of functions $F_{\delta}^{k,\alpha}(\bar{X}_{R,\phi})$ is the same as $C_{\delta}^{k,\alpha}(\bar{X}_{R,\phi})$, where the middle cylinder $C_{R,\phi}$ is unweighted, but the norms are different. The effect of this weighting function on the norms of functions in this space will become important later when we want to find a choice of Green's operator for $\mathcal{L}_{\bar{X}_{R,\phi}}$ which is uniformly bounded in R.

Definition 3. *A noncompact, properly embedded surface $X : \Sigma \to \mathbb{R}^3$ with asymptotically Delaunay ends is called nondegenerate if the operator*

$$\mathcal{L}_X : C_{-\delta}^{k+2,\alpha}(X) \to C_{-\delta}^{k,\alpha}(X)$$

is injective for all $\delta > 0$.

Following Proposition 20 of [MP01], we have the following Lemma.

Lemma 2. *Let $D(s,\theta) : (0,\infty) \times \mathbb{S}^1 \to \mathbb{R}^3$ be one half of an embedded Delaunay surface with necksize ϵ and with $\tau = 2\epsilon - \epsilon^2$ and for $j > 1$ let $u^j(s)$ be a solution of equation (4.2) which is square integrable, normalized so that $u^j(0) = 1$. Then*

$$|u^j(s)| \leq e^{-s\sqrt{j^2-2+\tau^2}}.$$

Proof: First consider the two point boundary value problem

$$\begin{cases} (-\partial_s^2 - \tau^2 \cosh 2\sigma + j^2)u_{s_0} = 0 \\ u_{s_0}(0) = 1, \qquad u_{s_0}(s_0) = 0 \end{cases}$$

for $s_0 > 0$. Because

$$\tau^2 \leq \tau^2 \cosh 2\sigma \leq 2 - \tau^2$$

(see Proposition 11 of [MP01]), the zero order term of this ODE satisfies

$$-\tau^2 \cosh 2\sigma + j^2 \geq j^2 - 2 + \tau^2 > 0.$$

Thus one can apply the Maximum Principle to solutions of this ODE. Letting

$$\hat{\gamma}_j = \sqrt{j^2 - 2 + \tau^2},$$

we have

$$\begin{aligned}(-\partial_s^2 - \tau^2\cosh 2\sigma + j^2)e^{-\hat\gamma_j s} &= (-\hat\gamma_j^2 - \tau^2\cosh 2\sigma + j^2)e^{-\hat\gamma_j s}\\ &= (-j^2 + 2 - \tau^2 - \tau^2\cosh 2\sigma + j^2)e^{-\gamma_j s}\\ &= (2 - \tau^2 - \tau^2\cosh 2\sigma)e^{-\hat\gamma_j s} \ge 0.\end{aligned}$$

Moreover, $e^{-\hat\gamma_j s}$ bounds u_{s_0} above at the end points 0 and s_0. Therefore, $u_{s_0}(s) \le e^{-\hat\gamma_j s}$ on the entire interval $[0, s_0]$. Also, by the Minimum Principle $-u_{s_0}(s) \ge -e^{-\hat\gamma_j s}$. So the family $\{u_{s_0}\}$, for $s_0 > 0$, is uniformly bounded. By standard ODE theory, the solution u_{s_0} depends continuously on the parameter s_0. By the Arzela-Ascoli theorem, the limit $u^j(s) = \lim_{s_0\to\infty} u_{s_0}$ exists and satisfies the bound $|u^j(s)| \le e^{-\hat\gamma_j s}$. ■

The above lemma gives a lower bound for $\gamma_j(\tau)$ when $j \ge 2$, but it is not sharp. In fact, Proposition 1 in [MPPR] shows that $\lim_{\tau\to 0}\gamma_j(\tau) = j$.

A general solution in $C^{2,\alpha}_{-\delta}((0,\infty)\times\mathbb{S}^1)$ (for any $0 < \delta < \gamma_2(\tau)$) to $\mathcal{L}_D u = 0$ can be written as

$$u(s,\theta) = \sum_{j=2}^{\infty} c_j \chi_j(\theta) u^j(s)$$

where $\chi_j(\theta) = \frac{1}{\sqrt{\pi}}\sin j\theta$ and u^j is the solution in Lemma 2. Then

$$\|u\|^2_{L^2((0,\infty)\times\mathbb{S}^1)} = \sum_{j=2}^{\infty} c_j^2 \|u^j\|^2_{L^2(0,\infty)}.$$

The boundary data for u is

$$u(0,\theta) = \sum_{j=2}^{\infty} c_j\chi_j(\theta)$$

and so the L^2 norm of the boundary data is

$$\|u(0,\theta)\|^2_{L^2} = \sum_{j=2}^{\infty} c_j^2.$$

By Proposition 21 of [MP01] the Poisson operator for the equation $\mathcal{L}_D u = 0$ is a bounded linear map from $L^2(\mathbb{S}^1)$ to $L^2((0,\infty)\times\mathbb{S}^1)$. So there is a positive constant m which depends only on the Delaunay parameter τ such that

$$m^2 \sum_{j=2}^{\infty} c_j^2 \|u_j\|_{L^2(0,\infty)} = \sum_{j=2}^{\infty} c_j^2.$$

In particular, if we normalize $\|u\|_{L^2((0,\infty)\times\mathbb{S}^1)} = 1$ then

$$\sum_{j=2}^{\infty} c_j^2 = m^2$$

where m depends only on the necksize of the Delaunay surface D. Thus $|c_j| \leq m$ for each j and so $|u(s,\theta)| \leq me^{-\gamma_2(\tau)s}$ where m depends only on τ and $\gamma_2(\tau)$ is the second indicial root mentioned above. Summarizing this argument we have the following lemma.

Lemma 3. *Let $u : (0,\infty) \times \mathbb{S}^1 \to \mathbb{R}$ be in the kernel of operator (4.1) for the Delaunay parameter τ and let $\|u\|_{L^2} = 1$. Then there exist positive constants m and $\gamma_2(\tau)$ which depend only on τ such that $|u(s,\theta)| \leq me^{-\gamma_2(\tau)s}$.*

4.2 The Jacobi operator on k-ended CMC surfaces

Now consider a more general k-ended, properly embedded surface with asymptotically Delaunay ends $X : \Sigma \to \mathbb{R}^3$. Over the jth end, one can write the Jacobi operator as

$$\mathcal{L}_X = \mathcal{L}_D + e^{-\gamma_2(\tau)t_j}\mathcal{R}$$

where (t_j, θ_j) are cylindrical coordinates for the end and $\mathcal{R}$ is a second order operator with smooth bounded coefficients. The deformations of the Delaunay surfaces corresponding to the low eigenmode solutions found above yield asymptotic Jacobi fields on $X(\Sigma)$. Below we will make this precise.

As X has asymptotically Delaunay ends, we can write $X(\Sigma) = K \cup (\cup_1^k E_j)$ where K is a compact set and each E_j is a graph of some exponentially decaying function $\rho_j(t_j, \theta_j)$ (for $t_j \geq 0$) over a Delaunay surface D_j. Now pick some cut-off function χ such that

$$\chi(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases}$$

and for $|i| \leq 1$ define

$$w_j^{i,\pm}(t_j, \theta_j) = \chi(t_j) u_j^{i,\pm}(\Phi^{-1}(t_j, \theta_j)).$$

Here Φ is the diffeomorphism between E_j and D_j defined by the graphing function and $u_j^{i,\pm}$ is the $i, \pm$ eigenmode Jacobi field on the Delaunay surface D_j with initial conditions given by (4.3). This defines $w_j^{i,\pm}$ on the end E_j and we extend it by zero on the rest of the surface. We will refer to $w_j^{i,\pm}$ as the asymptotic Jacobi field arising from the ith eigenmode on D_j.

Definition 4. *Let $X : \Sigma \to \mathbb{R}^3$ be a noncompact k-ended genus g proper embedding with asymptotically Delaunay ends. The deficiency space W_X is the span of all the asymptotic Jacobi fields arising from low eigenmode deformations of the underlying Delaunay end; $W_X = \operatorname{span}\{w_j^{i,\pm} \mid 1 \le j \le k; i = -1, 0, 1\}$. The bounded null space B_X is the linear span of all Jacobi fields which do not grow exponentially. In other words,*

$$B_X = \{u \in C^{2,\alpha}_{\delta}(X) \mid \mathcal{L}_X u = 0\}$$

where $0 < \delta < inf_j \gamma_2(\tau_j)$. The spectral theory for $\mathcal{L}_X$ shows that any Jacobi fields which grow exponentially on the end E_j must grow with an exponential rate of at least $\gamma_2(\tau_j)$, so it suffices to find this kernel for one value of δ.

Notice W_X is a vector space of dimension $6k$ with a geometrically natural basis given by $\{w_j^{i,\pm}\}$ where $|i| \le 1$. We will use this basis to induce the Euclidean norm on W_X. These two spaces are related as follows (see the Linear Decomposition Lemmas of [KMP96] and [MPU96]):

Theorem 4. (Kusner, Mazzeo, Pollack) *Let $X : \Sigma \to \mathbb{R}^3$ be a proper embedding of a noncompact surface with finite topology and asymptotically Delaunay ends. For $0 < \delta < inf_j \gamma_2(\tau_j)$, let $u \in C^{k+2,\alpha}_{\delta}(X)$ and $f \in C^{k,\alpha}_{-\delta}(X)$ such that $\mathcal{L}_X u = f$. Then $u = w + \phi$ where $w \in W_X$ and $\phi \in C^{k+2}_{-\delta}(X)$.*

Strictly speaking, this theorem is only stated for CMC surfaces and for weighted Sobolev spaces. However, the proof only requires that $\mathcal{L}_X$ is the linearized mean curvature operator and that the ends of the surface in question are asymptotic to Delaunay surfaces (see Appendix F). Therefore, the conclusion of this and related theorems (in particular, their results regarding the weights for which the Jacobi operator is Fredholm) in [KMP96] and [MPU96] remain true for all the surfaces with which we will work.

Suppose $X : \Sigma \to \mathbb{R}^3$ is a nondegenerate embedding with asymptotically Delaunay ends $E_1 \ldots E_k$, each with Delaunay parameter τ_j. Then for $0 < \delta < \inf_j \gamma_2(\tau_j)$

$$\mathcal{L}_X : C^{2,\alpha}_{-\delta}(X) \to C^{0,\alpha}_{-\delta}(X)$$

is injective. By duality and elliptic regularity,

$$\mathcal{L}_X : C^{2,\alpha}_{\delta}(X) \to C^{0,\alpha}_{\delta}(X)$$

is surjective. So if $f \in C^{0,\alpha}_{-\delta} \subset C^{0,\alpha}_{\delta}$, then there is a function $u \in C^{2,\alpha}_{\delta}$ such that $\mathcal{L}_X u = f$. Then by the Linear Decomposition theorem $u \in W_X \oplus C^{2,\alpha}_{-\delta}$. Thus we see that in the nondegenerate case

$$\mathcal{L}_X : W_X \oplus C^{2,\alpha}_{-\delta} \to C^{0,\alpha}_{-\delta}$$

is surjective with kernel B_X. We will use the Euclidean norm $\|u+v\|_{W_X \oplus C^{2,\alpha}_{-\delta}(X)} = \sqrt{\|u\|^2_{W_X} + \|v\|^2_{C^{2,\alpha}_{-\delta}(X)}}$ on the direct sum. This paragraph summarizes why nondegeneracy is such an important property.

The above argument shows there is a well defined map $\Pi : B_X \to W_X$ given by projection. If $u, v \in B_X$ and $\Pi(u) = \Pi(v) = w \in W_X$ then $\mathcal{L}_X(u-v) = 0$ and $u - v \in C^{k+2,\alpha}_{-\delta}(X)$. If X is also nondegenerate, then $u = v$. So in the nondegenerate case this map $B_X \to W_X$ is injective. In this case, we will often identify B_X with its image in W_X. For the general immersion (which may be degenerate) the element $\Pi(u) = w \in W_X$ determines $u \in B_X$ only up to terms which decay exponentially.

In fact, W_X and B_X carry more structure. To see this, first recall that given two solutions u_1 and u_2 to a linear second order ODE $u'' + pu' + qu = 0$, the Wronskian $Wr(u_1, u_2) = u_1 u_2' - u_2 u_1'$ satisfies the equation $(Wr)' + Wr \cdot p = 0$. Notice that equation (4.2) is a linear second order ODE with no first order terms. So the Wronskian $Wr(u^{i,+}, u^{i,-}) = u^{i,+}(s)\partial_s u^{i,-}(s) - u^{i,-}(s)\partial_s u^{i,+}(s)$ is 1 by the initial conditions (4.3).

Let W_j be the part of W_X arising from the j eigenmodes of the model Delaunay surfaces for the ends of Σ. Write $u, v \in W_0$ as

$$u = \sum_1^k (a_i u_i^{0,+} + b_i u_i^{0,-})$$

and

$$v = \sum_1^k (\alpha_i u_i^{0,+} + \beta_i u_i^{0,-})$$

where $u_i^{0,\pm}$ is the element of W_X arising from the $0,\pm$ eigenmode of the model Delaunay surface (with Delaunay parameter τ_i) for the ith end. As in [MPU96] we define

$$\Omega(u,v) = \lim_{r\to\infty} \int_{\Sigma\cap\mathbb{B}_r(0)} (\mathcal{L}_\Sigma u)v - u(\mathcal{L}_\Sigma v) = \lim_{r\to\infty} \int_{\Sigma\cap\mathbb{B}_r(0)} (\Delta u)v - u(\Delta v)$$

where $\mathbb{B}_r(0)$ is a large ball as in the definition of W. Upon integrating by parts, we find

$$\begin{aligned}\Omega(u,v) &= \lim_{r\to\infty} \int_{\partial(\Sigma\cap\mathbb{B}_r(0))} \frac{\partial u}{\partial\nu}v - u\frac{\partial v}{\partial\nu} \\ &= \lim_{r\to\infty} [\sum_1^k [(a_i\beta_i - b_i\alpha_i) Wr(u_i^{0,+}, u_i^{0,-}) \frac{1}{2\pi} \int_0^{2\pi} d\theta] + O(e^{-\gamma_2(\tau_i)r})] \\ &= \sum_1^k (a_i\beta_i - b_i\alpha_i).\end{aligned}$$

Thus Ω is the standard symplectic structure on $\mathbb{R}^{2k}$. Similarly, W_1 and W_{-1} also carry the standard symplectic structure on $\mathbb{R}^{2k}$ and so W_X carries the standard symplectic structure on $\mathbb{R}^{6k}$. From the definition of Ω, $B_X \subset W_X$ is an isotropic subspace. By a relative index theorem (see [KMP96]), $\dim B_X = 3k = \frac{1}{2}\dim W_X$ and thus $B_X \subset W_X$ is Lagrangian.

Given an end $E = E_j$, let $W_E = \mathrm{span}\{w_j^{i,\pm}; i = -1, 0, 1, j \text{ fixed}\}$. These are asymptotic Jacobi fields which are zero on all ends except E. Functions $u \in B_X$ such that $\Pi(u) \in W_E$ are Jacobi fields of X which decay exponentially on all but one end of $X(\Sigma)$ and grow at most linearly on the remaining end E. As remarked earlier, we can identify B_X with a subspace of W_X in the case that X is nondegenerate. In this case, we will again abuse notation and say that the functions u described above lie in $B_X \cap W_E$. Such a function u corresponds to a deformation of $X(\Sigma)$ which fixes the asymptotics of all ends except E and changes the asymptotics of E. The existence of such a u is limited by the following Lemma.

Lemma 5. *If $u \in B_X$ and $\Pi(u) \in W_E$ for some end E of a noncompact, proper embedding $X : \Sigma \to \mathbb{R}^3$ with asymptotically Delaunay ends, then u can only correspond to an asymptotic translation along the axis of E.*

Proof: Suppose u corresponded to a change in the necksize of the asymptotic Delaunay surface for the end E. Let w be the Jacobi field which arises from translating the embedding X along the axis of E. Such a Jacobi field always exists, as it arises from a global rigid motion of the surface. Then $\Omega(u,w) \neq 0$, which contradicts the fact that B_X is a Lagrangian subspace of W_X. Similarly, one can eliminate the translations off the axis of E (using global rotations) and the rotations of the axis of E (using global translations). $\blacksquare$.

4.3 Nondegeneracy of the Jacobi operator on the approximate solution

In this gluing construction, we want the approximate solution $\bar{X}_{R,\phi}$ to be nondegenerate, at least when the summands X_1 and X_2 are. Unfortunately, we cannot show this is always the case and must make the additional assumption that X_1 allows a deformation through CMC surfaces which changes the asymptotic necksize of E_1 to first order.

Remark 1. *We remark that the existence of a deformation of X_1 through CMC surfaces which changes the asymptotic necksize of E_1 to first order implies that $B_{X_1} \cap W_{E_1} = \{0\}$. To see this, recall that by Lemma 5 any nonzero Jacobi field u in W_{E_1} would have to correspond to an asymptotic translation of E_1 along its axis. Let v be the infinitesimal generator of the deformation of X_1 which changes the asymptotic necksize of E_1. Then $\Omega(u,v) \neq 0$, which contradicts the fact that B_{X_1} is Lagrangian.*

First, let us recall the construction of Chapter 3. We start with two complete CMC embeddings $X_i : \Sigma_i \to \mathbb{R}^3$ of noncompact surfaces. The surfaces Σ_i have genus g_1 and g_2 and have k_1 and k_2 punctures respectively. We choose ends E_i of $X_i(\Sigma_i)$ such that E_1 and E_2 are asymptotic to congruent Delaunay surfaces with Delaunay parameter $\tau_1 = \tau_2 = \tau$. Align the two surfaces so that the ends E_i are asymptotic to opposite ends of a Delaunay embedding D. Now patch the embeddings together at a neck of D using a cut-off function to get an embedding $\bar{X}_{R,\phi} : \Sigma \to \mathbb{R}^3$. Here Σ is topologically a closed surface of genus $g_1 + g_2$ with $k_1 + k_2 - 2$ punctures. We will label the ends of $\bar{X}_{R,\phi}$ as $E_3, \ldots E_{k_1+k_2}$, reserving the labels E_1 and E_2 for the ends we truncate in the gluing construction. The ends of $\bar{X}_{R,\phi}$ are all congruent to ends of either X_1 or X_2. We will label the ends congruent to ends of X_1 as $E_3, \ldots, E_{k_1+1}$. The embedding depends on a discrete parameter R, which one can think of as a distance along the end at which we place the gluing region, and the rotation parameter ϕ, which we can think of as the relative angle between the two summands $X_1(\Sigma_1)$ and $X_2(\Sigma_2)$. $\bar{X}_{R,\phi}$ has mean curvature 1 outside a compact set and the deviation from 1 of the mean curvature of $\bar{X}$ is pointwise $O(e^{-\gamma_2(\tau)R})$.

We need the condition stated above (that X_1 admits a deformation which changes the asymptotic necksize of $E_1$to first order) to guard against the following behavior. If we do not make this assumption, then it is possible for both X_1 and X_2 to admit Jacobi fields which correspond only to translations of E_1 (respectively E_2) along its axis (see Remark 1). If both X_1 and X_2 have Jacobi fields which correspond to asymptotic translations along the axis of the end we are gluing and decay exponentially on all other ends, then we can patch these two Jacobi fields together to construct an approximate Jacobi field u_R with $\|u_R\|_{F^{2,\alpha}_{-\delta}(\bar{X}_{R,\phi})} = 1$. By construction, $\mathcal{L}_{\bar{X}_{R,\phi}}(u_R) = O(e^{-\gamma_2 R})$ pointwise. So $\mathcal{L}_{\bar{X}_{R,\phi}}$ has an exponentially small eigenvalue, which leads one to suspect that $\mathcal{L}_{\bar{X}_{R,\phi}}$ might be degenerate. Moreover, in this case any Green's operator for $\mathcal{L}_{\bar{X}_{R,\phi}}$ cannot be uniformly bounded in R. It is natural to ask whether any embedding X admits a deformation which preserves its mean curvature up to first order and changes the asymptotic necksize of a given end E to first order. If we denote by ξ the variable in W_X which corresponds to changing the necksize of E, then the failure of such a deformation to exist is equivalent to the condition that B_X lies in the hyperplane $\{\xi = 0\}$. Thus we expect that the condition we will assume for the summands is generically satisfied among nondegenerate CMC surfaces. In fact, all known nondegenerate CMC surfaces satisfy this condition for all ends. However, we should remark that we know very few examples of nondegenerate CMC surfaces. See Chapter 6 for examples of surfaces which satisfy the gluing hypotheses for all choices of ends.

Proposition 6. *Suppose both X_1 and X_2 are nondegenerate and that X_1 admits a deformation through CMC surfaces which changes the asymptotic necksize of E_1 to first order. Fix $0 < \delta < \inf_j \gamma_2(\tau_j)$. Then there is an $R_0 > 0$ such that for $R \geq R_0$ one can find a Green's operator $\mathcal{G}_{\bar{X}_{R,\phi}} : F^{0,\alpha}_{-\delta} \to F^{2,\alpha}_{-\delta} \oplus W_{\bar{X}_{R,\phi}}$, uniformly bounded in R, such that $\mathcal{L}_{\bar{X}_{R,\phi}} \circ \mathcal{G}_{\bar{X}_{R,\phi}} = \mathrm{Id}$. In particular, in this case the approximate solution $\bar{X}_{R,\phi}$ is nondegenerate.*

The idea behind the proof of this Proposition was originally communicated to me by F. Pacard.

Remark 2. *The weighting of the middle cylinder $(-R, R) \times \mathbb{S}^1$ is necessary to obtain a uniform bound on the Green's operator. To see this, consider the problem*

$$u'' = f$$

on the segment $(-R,R)$ with $u(\pm R)=0$. The first eigenvalue of this problem is $\frac{\pi^2}{4R^2}$. Indeed, one can check that $u(t)=\cos(\frac{\pi t}{2R})$ is an eigenfunction associated to this eigenvalue. Thus the norm of the inverse of this operator (in an unweighted function space) grows like R^2.

Proof: Suppose X_1 admits a deformation through CMC surfaces which changes the asymptotic necksize of E_1 to first order; notice we are not making the corresponding assumption about X_2. Given a function $f \in F^{0,\alpha}_{-\delta}(\bar{X}_{R,\phi})$ we wish to solve the equation $\mathcal{L}_{\bar{X}_{R,\phi}}(u)=f$. The method employed in this proof is to first truncate f using a cut-off function χ and solve the equations $\mathcal{L}_{X_1}(U_1)=\chi f$ and $\mathcal{L}_{X_2}(U_2)=(1-\chi)f$ with appropriate decay. Then we will glue these two functions together and show that the result is an exponentially small (in R) perturbation of the desired solution $\mathcal{G}_{\bar{X}_{R,\phi}}(f)$.

Let χ be a smooth monotone nonincreasing function such that

$$\chi(t)=\begin{cases} 1 & t\leq -1 \\ 0 & t\geq 1. \end{cases}$$

First let $u_2+v_2 \in W_{X_2}\oplus C^{2,\alpha}_{-\delta}(X_2)$ be a solution to $\mathcal{L}_{X_2}(u_2+v_2)=(1-\chi)f$, which exists because X_2 is nondegenerate. Moreover, (recall we are using the Euclidean norm on the direct sum $W_{X_2}\oplus C^{2,\alpha}_{-\delta}(X_2)$)

$$\sqrt{\|u_2\|^2_{W_{X_2}}+\|v_2\|^2_{C^{2,\alpha}_{-\delta}(X_2)}}\leq c_2\|(1-\chi)f\|_{C^{0,\alpha}_{-\delta}(X_2)}. \tag{4.6}$$

Also, on the end E_2, u_2 has an asymptotic expansion

$$u_2\sim\sum_{i,\pm}\alpha^{i,\pm}u^{i,\pm} \tag{4.7}$$

where $u^{i,\pm}$ are the normalized low-eigenmode Jacobi fields on the model Delaunay surface D for E_2. Notice that $\sqrt{\sum(\alpha^{i,\pm})^2}\leq c_2\|(1-\chi)f\|_{C^{0,\alpha}_{-\delta}(X_2)}$.

Next let $u_1+v_1\in W_{X_1}\oplus C^{2,\alpha}_{-\delta}(X_1)$ be a solution to $\mathcal{L}_{X_1}(u_1+v_1)=\chi f$, which exists because X_1 is nondegenerate. We also have the bound

$$\sqrt{\|u_1\|^2_{W_{X_1}}+\|v_1\|^2_{C^{2,\alpha}_{-\delta}(X_1)}}\leq c_1\|\chi f\|_{C^{0,\alpha}_{-\delta}(X_1)}. \tag{4.8}$$

This time, we can choose u_1 so that one E_1

$$|u_1(t,\theta)|\leq c_1\|\chi f\|_{C^{0,\alpha}_{-\delta}(X_1)}e^{-\gamma_2(\tau)(R+t)} \tag{4.9}$$

for $t \geq -R$. This is because there exist global Jacobi fields on X_1 which are asymptotic on E_1 to any of $u^{i,\pm}$ for $i = 0, \pm 1$, by assumption that X_1 admits a deformation through CMC surfaces which changes the necksize of E_1 to first order. The infinitesimal generator of this deformation yields a global Jacobi field on X_1 asymptotic to $u^{0,-}$. The global Jacobi fields asymptotic to $u^{\pm 1,\pm}$ and to $u^{0,+}$ arise from global translations and rotations of X_1. In short, we can find a global Jacobi field on X_1 with prescribed asymptotics on E_1.

Now let Φ be a global Jacobi field on X_1 such that on X_1, $\Phi \sim -\sum \alpha^{i,\pm} u^{i,\pm}$. This is possible because E_1 and E_2 are asymptotic to the same Delaunay surface. Let η_1 and η_2 be cut-off functions so that

$$\eta_1(t) = \begin{cases} 1 & t \leq R-2 \\ 0 & t \geq R-1 \end{cases}$$

and

$$\eta_2(t) = \begin{cases} 1 & t \geq -R+2 \\ 0 & t \leq -R+1. \end{cases}$$

Define the operator $\tilde{\mathcal{G}} : F^{0,\alpha}_{-\delta}(\bar{X}_R) \to W_{\bar{X}_R} \oplus F^{2,\alpha}_{-\delta}(\bar{X}_R)$ by

$$\tilde{\mathcal{G}}(f) = \eta_1(u_1 + v_1 + \Phi) + \eta_2(u_2 + v_2).$$

Notice now that in the region corresponding to $-R \leq t \leq R-2$, we have the estimate that

$$|u_2(t,\theta) + \Phi(t,\theta)| = O(\left(\frac{\cosh t}{\cosh R}\right)^{\delta} \|f\|_{F^{0,\alpha}_{-\delta}(\bar{X}_R)}) \tag{4.10}$$

because we have chosen Φ precisely to cancel out the parts of u_2 which do not decay. We wish to prove the following two estimates:

- $\|\tilde{\mathcal{G}}(f)\|_{W_{\bar{X}_R} \oplus F^{2,\alpha}_{-\delta}(\bar{X}_R)} \leq c\|f\|_{F^{0,\alpha}_{-\delta}(\bar{X}_R)}$
- $\|\mathcal{L}_{\bar{X}_R} \circ \tilde{\mathcal{G}}(f) - f\|_{F^{0,\alpha}_{-\delta}(\bar{X}_R)} \leq \tilde{c}\|f\|_{F^{0,\alpha}_{-\delta}(\bar{X}_R)} e^{-\tilde{\delta}R}$ for some $\tilde{\delta} > 0$.

The second estimate shows that we can write the composition $\mathcal{L}_{\bar{X}_R} \circ \tilde{\mathcal{G}}$ as $\mathrm{Id} + \mathcal{R}_{\mathrm{R}}$, where the operator norm of $\mathcal{R}_R$ is $O(e^{-\tilde{\delta}R})$, and is thus invertible with uniformly bounded inverse once R is sufficiently large, which will complete the proof of the proposition.

First we estimate $\|\tilde{\mathcal{G}}(f)\|_{W_{\bar{X}_R}\oplus F^{2,\alpha}_{-\delta}(\bar{X}_R)}$. In the region C_R which is parameterized by $(t,\theta)\in[-R,R]\times\mathbb{S}^1$, we have the bound

$$|\tilde{\mathcal{G}}(f)(t,\theta)| \le |u_1(t,\theta)+v_1(t,\theta)| + |u_2(t,\theta)+v_2(t,\theta)+\Phi(t,\theta)| = O(\left(\frac{\cosh t}{\cosh R}\right)^{\delta}\|f\|_{F^{0,\alpha}_{-\delta}})$$

by combining the estimates (4.6), (4.8), (4.9) and (4.10). The desired bound on $\|\tilde{\mathcal{G}}(f)\|_{W_{\bar{X}_R}\oplus F^{2,\alpha}_{-\delta}(\bar{X}_R)}$ in $X_1(\Sigma_1)\backslash E_1$ follows from (4.8), while the similarly desired bound in $X_2(\Sigma_2)\backslash E_2$ follows from (4.6), which completes the proof of the desired estimate.

Finally we wish to estimate $\|\mathcal{L}_{\bar{X}_R}(\tilde{\mathcal{G}}(f)) - f\|_{F^{0,\alpha}_{-\delta}(\bar{X}_R)}\|$. Notice that by construction $\mathcal{L}_{\bar{X}_R}(\tilde{\mathcal{G}}(f)) \neq f$ only in the regions $R-2\le t\le R-1$ and $-R+1\le t\le -R+2$. In the region $R-2\le R-1$,

$$\begin{aligned}
|\mathcal{L}_{\bar{X}_R}(\tilde{\mathcal{G}}(f))(t,\theta) - f(t,\theta)| &= |\mathcal{L}_{\bar{X}_R}(\eta_1(u_1+v_1+\Phi))(t,\theta)| \\
&\le \|\mathcal{L}_{\bar{X}_R}\|(|\eta_1 u_1(t,\theta)| + |\eta_1 v_1(t,\theta)|) + \|\mathcal{L}_{X_1} - \mathcal{L}_{X_2}\||\Phi(t,\theta)| \\
&= \|\mathcal{L}_{\bar{X}_R}\|O(\|f\|_{F^{0,\alpha}_{-\delta}} e^{-2\delta R}) + |\Phi(t,\theta)|O(e^{-\tilde{\delta}R}) \\
&= \|f\|_{F^{0,\alpha}_{-\delta}}(O(e^{-2\delta R}) + O(Re^{-\tilde{\delta}R}))
\end{aligned}$$

where $\tilde{\delta}$ is any positive number less than $\gamma_2(\tau)$. One can obtain the desired estimate in the region $-R+1\le t\le -R+2$ similarly. ∎

Chapter 5

NONLINEAR ANALYSIS AND SOLVING THE GLUING PROBLEM

In previous sections of this dissertation we have constructed an approximate solution $\bar{X}_{R,\phi}$ with mean curvature $1-\psi$ (where ψ is compactly supported and pointwise small) and shown that we can solve the linearized mean curvature equation (1.3) provided the summands X_1 and X_2 satisfy some conditions. It remains to solve the nonlinear equation (1.2), which we will restate here:

$$\mathcal{L}_{\bar{X}_{R,\phi}}(u+v) = \psi - \mathcal{Q}_{\bar{X}_{R,\phi}}.$$

Recall that given $(u,v) \in W_{\bar{X}_{R,\phi}} \oplus F^{2,\alpha}_{-\delta}(\bar{X}_{R,\phi})$ we obtained this equation by measuring the mean curvature of embeddings $\bar{X}_{R,\phi}(u,v)$ which is the deformation associated to u of the normal variation of $\bar{X}_{R,\phi}$ by v. If we can solve equation (1.2) then the embedding $\bar{X}_{R,\phi}(u,v)$ has mean curvature 1. In Section 5.1 we first make these geometric deformations precise and in Section 5.2 we solve equation (1.2) by using a contraction mapping.

5.1 Deformations of the Approximate Solution

In this subsection, we will deform the approximate solution $\bar{X}_{R,\phi}$ using elements in its deficiency space $W_{\bar{X}_{R,\phi}}$. For the remainder of this section, we will suppress the subscripts R and ϕ for the approximate solution, as we will be working with a fixed distance R and a fixed angle ϕ. Roughly speaking, each element $u \in W_{\bar{X}}$ corresponds to some combination of rotations, translations, and deformations of necksizes applied to the model Delaunay surfaces for the ends of $\bar{X}$. We then deform $\bar{X}$ to obtain a deformed approximate solution $\bar{X}(u)$ in two steps. We first change the necksizes on the model Delaunay surfaces as prescribed by u and use these new Delaunay surfaces as asymptotic models for the ends of $\bar{X}(u)$. Then we apply the rotations and translations to these new Delaunay surfaces as prescribed u, which rotates and translates the ends of $\bar{X}(u)$. Below we will make these deformations precise.

First recall the construction of the deficiency space W_X for a CMC embedding $X : \Sigma \to \mathbb{R}^3$. As this construction relies only on the asymptotic structure of an immersion, we can carry it out for our approximate solution $\bar{X} : \Sigma \to \mathbb{R}^3$. We constructed the approximate solution by patching together two CMC embeddings X_1 and X_2. In this construction, we translate the embedding X_1 so that the asymptotic estimate of [KKS89] holds outside a ball of radius r centered at $(-R-r, 0, 0)$. Let χ be a smooth monotone function such that

$$\chi(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1. \end{cases}$$

As per [KKS89], we can write the connected components of X_1 as ends $E_1, \ldots, E_{k_1}$ where each E_j is given as a graph over a cylinder as follows:

$$(t_j, \theta_j) \mapsto t_j \vec{a_j} + (\rho_{D_j}(t_j) + \rho_j(t_j, \theta_j))\omega_j(\theta_j).$$

Here D_j is the model Delaunay surface for the end E_j with profile curve ρ_{D_j} and axis $\vec{a}_j$, $\{\vec{a}_j, \vec{b}_j, \vec{c}_j\}$ is an oriented orthonormal basis, $\omega_j = \cos\theta \vec{b}_j + \sin\theta \vec{c}_j$, and

$$\|\rho_j\|_{C^{2,\alpha}((t_j-1, t_j+1)\times\mathbb{S}^1)} = O(e^{-\gamma t_j})$$

for $t_j > 1$. Let $\tilde{X}$ be the embedding which agrees with $\bar{X}$ except on the ends, which we replace with the graph over the cylinder given by

$$(t_j, \theta_j) \mapsto t_j \vec{a}_j + (\rho_{D_j}(t_j) + (1-\chi(t_j))\rho_j(t_j, \theta_j))\omega_j(\theta_j).$$

Let Φ be the diffeomorphism which sends $t_j\vec{a}_j + (\rho_{D_j}(t_j) + \rho_j(t_j, \theta_j))\omega_j(\theta_j)$ to $t_j\vec{a}_j + (\rho_{D_j}(t_j) + (1-\chi(t_j))\rho_j(t_j, \theta_j))\omega_j(\theta_j)$. We define the part of $W_{\tilde{X}}$ arising from the summand X_1 as the linear span of

$$w_j^{i,\pm}(s, \theta) = \chi(t_j) u_j^{i,\pm}(\Phi^{-1}(t_j, \theta_j))$$

where $u_j^{i,\pm}$ was the $i, \pm$ eigenmode on the Delaunay surface D_j, for $2 \leq j \leq k$ and $-1 \leq i \leq 1$. This function is only defined on the end E_j, but we can extend it to be zero on the rest of the surface. Recall we are using the end E_1 for gluing and want to preserve its asymptotic structure, so we do not deform that end. The asymptotic Jacobi fields from this end therefore do not contribute to the deficiency space $W_{\tilde{X}}$. The construction of the deficiency space depends on the choice of cut-off function χ, but changing χ only changes the functions in $W_{\bar{X}}$ on a compact set. We define the part of the deficiency space arising from the summand X_2 similarly.

Now we are ready to define the deformations $\bar{X}(u)$ of the approximate solution $\bar{X}$. Recall that the low eigenmodes $u_j^{i,\pm}$ arise from explicit deformations which one can apply to the Delaunay surface D_j. The $u_j^{i,+}$'s arise from translations, the $u_j^{\pm 1,-}$'s arise from rotating the axis of D_j, and the $u_j^{0,-}$'s arise from changing the necksize of D_j. Let $u = \sum \alpha_j^{i,\pm} w_j^{i,\pm}$ where $\alpha_j^{i,\pm}$ are small coefficients and $w_j^{i,\pm}$ are as above. Let T_j^u be translation by $\alpha_j^{0,+}$ in the $\vec{a_j}$ direction, $\alpha_j^{1,+}$ in the $\vec{b_j}$ direction, and $\alpha_j^{-1,+}$ in the $\vec{c_j}$ direction. Let R_j^u be the rotation which rotates the axis $\vec{a_j}$ through the angle $\alpha_j^{1,-}$ towards the $\vec{b_j}$ axis and through the angle $\alpha_j^{-1,-}$ toward the $\vec{c_j}$ axis. Now apply these rigid motions to D_j and change the necksize to $\epsilon_j + \alpha_j^{0,-}$. This results in a new Delaunay surface D_j^u, with a new orthonormal frame $\{\vec{a}_j^u, \vec{b}_j^u, \vec{c}_j^u\} = R_j^u(\{\vec{a}_j, \vec{b}_j, \vec{c}_j\}) + T_j^u$ and $\omega_j^u = \vec{b}_j^u \cos\theta + \vec{c}_j^u \sin\theta$. If $\alpha_j^{0,-} = 0$ then D_j and D_j^u are congruent. We now define the deformed surface $\bar{X}(u)$ by replacing the end E_j with

$$(t_j, \theta_j) \mapsto t_j \chi(t_j)\vec{a}_j + (t_j - \alpha_j^{0,+})(1 - \chi(t_j))\vec{a}_j^u + \chi(t_j)(\rho_{D_j}(t_j) + \rho_j(t_j, \theta_j))\omega_j(\theta_j) +$$

$$(1 - \chi(t_j))(\rho_{D_j^u}(t_j - \alpha_j^{0,+}) + \rho_j(t_j - \alpha_j^{0,+}, \theta_j))\omega_j^u(\theta_j).$$

In the transition region between $\bar{X}$ and $\bar{X}(u)$, both surfaces can be written as graphs over a cylinder. So in this region we can use equation (3.1), which gives the mean curvature of both surfaces in term of their graphing functions. Recall that we already have a perturbation term ψ such that $H_{\bar{X}} = 1 - \psi$. Using equation (3.1), we can write a Taylor expansion of the mean curvature of $\bar{X}(u)$ in terms of the difference of the graphing functions to conclude the Lemma immediately below. We will call the additional perturbation term $\tilde{\psi}(u)$.

Lemma 7. *There exists $\eta > 0$ such that for $u = \sum \alpha_j^{\pm,i} w_j^{\pm,i} \in W_{\bar{X}}$ with $|\alpha| \leq \eta$, the mean curvature of the deformed embedding $\bar{X}(u)$ is given by $1 - \psi - \tilde{\psi}(u)$, where $\|\tilde{\psi}(u)\|_{C^{0,\alpha}((t_j-1,t_j+1)\times\mathbb{S}^1)} = O(e^{-\gamma_2 t_j})$ for $t_j > 1$. Moreover, $\tilde{\psi}(u) = O(|\alpha|)$, but not $o(|\alpha|)$, pointwise.*

In the case that none of the asymptotic necksizes change (i.e. $\alpha_i^{0,-} = 0$ for all i) the error term $\tilde{\psi}$ will be compactly supported. In general, because the ends of $\bar{X}$ and $\bar{X}(u)$ are written as graphs of the same function over different Delaunay surfaces, this error will not be compactly supported. The estimate that $\|\tilde{\psi}(u)\|_{C^{0,\alpha}((t_j-1,t_j+1)\times\mathbb{S}^1)} = O(e^{-\gamma_2 t_j})$ follows from the fact that an end of $\bar{X}(u)$ is still written as the graph of ρ_j over some Delaunay surface D_j^u and ρ_j satisfies the estimate $\|\rho_j\|_{C^{2,\alpha}((t_j-1,t_j+1)\times\mathbb{S}^1)} = O(e^{-\gamma_2 t_j})$.

5.2 Solving the Nonlinear Equation

Recall that our goal is to solve the nonlinear equation

$$\mathcal{L}_{\bar{X}_{R,\phi}}(u+v) = \psi - \mathcal{Q}_{\bar{X}_{R,\phi}}$$

where $(u,v) \in W_{\bar{X}_{R,\phi}} \oplus F^{2,\alpha}_{-\delta}(\bar{X}_{R,\phi})$ and ψ is the initial perturbation from 1 in the mean curvature of the approximate solution $\bar{X}_{R,\phi}$. Given $u \in W_{\bar{X}_{R,\phi}}$ and $v \in F^{2,\alpha}_{-\delta}(\bar{X}_{R,\phi})$ we defined a new embedding $\bar{X}_{R,\phi}(u,v)$ first taking the normal perturbation of $\bar{X}_{R,\phi}$ by v and then adjust it with the geometric deformation determined by u, as in the last section. We perform these operations in this order so that v is always in the fixed function space $F^{2,\alpha}_{-\delta}(\bar{X}_{R,\phi})$. The term $\mathcal{Q}_{\bar{X}_{R,\phi}}(u,v)$ is defined to be the quadratic and higher order terms in the Taylor series for $H(u,v)$ developed about $(0,0)$. To solve equation (1.2), we will examine the operator

$$\mathcal{K}_{\bar{X}_{R,\phi}}(u,v) = \mathcal{G}_{\bar{X}_{R,\phi}}(\psi - \mathcal{Q}_{\bar{X}_{R,\phi}}(u,v)).$$

Notice that this is a well-defined operator, because $\mathcal{Q}_{\bar{X}_{R,\phi}}(u,v)$ does lie in the domain of $\mathcal{G}_{\bar{X}_{R,\phi}}$ by the estimate in Lemma 7.

Proposition 8. *There is an $\eta > 0$ and an $R_0 > 0$ such that for $R \geq R_0$ the mapping $\mathcal{K}$ is a contraction on the ball of radius η in $W_{\bar{X}_{R,\phi}} \oplus F^{2,\alpha}_{-\delta}(\bar{X}_{R,\phi})$.*

Proof: First we estimate $\|\mathcal{K}(u,v)\|_{W_{\bar{X}}\oplus F^{2,\alpha}_{-\delta}(\bar{X})}$. By the uniform bound on $\mathcal{G}_{\bar{X}}$,

$$\|\mathcal{K}(u,v)\| \leq c(\|\psi\|_{F^{0,\alpha}_{-\delta}(\bar{X})} + \|(u,v)\|^2_{W_{\bar{X}}\oplus F^{2,\alpha}_{-\delta}(\bar{X})}).$$

Recall that we started with $|\psi(t,\theta)| = O(e^{-\gamma_2(\tau)R})$, so $\|\psi\|_{F^{0,\alpha}_{-\delta}(\bar{X})} = O(e^{-(\gamma_2(\tau)-\delta)R})$, which we can take to be $o(\eta^2)$. Thus if $\|(u,v)\| = O(\eta)$, then $\|\mathcal{K}(u,v)\| = O(\eta^2)$, from which is follows that $\mathcal{K}$ maps a ball of radius η to itself for sufficiently small η. Moreover,

$$\begin{aligned}
\|\mathcal{K}(u_1,v_1)-\mathcal{K}(u_2,v_2)\| &\leq c\|\mathcal{Q}_{\bar{X}}(u_1,v_1)-\mathcal{Q}_{\bar{X}}(u_2,v_2)\|_{F^{0,\alpha}_{-\delta}(\bar{X})} \\
&\leq \tilde{c}\|(u_1,v_1)-(u_2,v_2)\|^2_{W_{\bar{X}}\oplus F^{2,\alpha}_{-\delta}(\bar{X})} \\
&\leq \tilde{c}\max_{i=1,2}(\|(u_i,v_i)\|_{W_{\bar{X}}\oplus F^{2,\alpha}_{-\delta}(\bar{X})})\|(u_1,v_1)-(u_2,v_2)\|_{W_{\bar{X}}\oplus F^{2,\alpha}_{-\delta}(\bar{X})} \\
&\leq \frac{1}{2}\|(u_1,v_1)-(u_2,v_2)\|_{W_{\bar{X}}\oplus F^{2,\alpha}_{-\delta}(\bar{X})}
\end{aligned}$$

for $\eta \leq \frac{1}{2\tilde{c}}$. The second inequality follows from the fact that

$$\mathcal{Q}_{\bar{X}_{R,\phi}}(0,0)=0 \quad \nabla\mathcal{Q}_{\bar{X}_{R,\phi}}(0,0)=0.$$

∎

The existence part of Theorem 1 follows immediately from the proposition above and the Contraction Mapping Principle.

5.3 Nondegeneracy of the Solution

It remains to see that the solution $X_{R,\phi}$ is nondegenerate. Below we will suppress the dependence on the relative angle ϕ, as none of the analysis depends on it. This proof of nondegeneracy is very similar to the proof of nondegeneracy in [MPP].

It will be convenient to define the following decomposition of X_R. Let $X_{1,R}=\{p=(p_1,p_2,p_3)\in X_R \mid p_1\leq -R\}$, $X_{2,R}=\{p=(p_1,p_2,p_3)\in X_R \mid p_1\geq R\}$ and $X_{3,R}=\{p=(p_1,p_2,p_3)\in X_R \mid -R\leq p_1\leq R\}$. Notice that $X_{1,R}$ and $X_{1,R'}$ differ only by a translation, so we can (and will) identify these surfaces. We will further decompose $X_{1,R}$ and $X_{2,R}$ as $X_{1,R}=K_1\cup(\cup_{j=3}^{k_1+1}E_j)$ and $X_{2,R}=K_2\cup(\cup_{j=k_1+2}^{k_1+k_2}E_j)$ where K_1 and K_2 are fixed compact sets in X_1 and X_2 (respectively) and E_j are the ends of X_1 and X_2.

Suppose this were not the case. Then there would exist a sequence of $R_i\to\infty$ and nontrivial Jacobi fields u_i of X_{R_i}. We may normalize u_i so that $\|u_i\|_{F^{0,0}_{-\delta}(X_{R_i})}=1$. This means $\sup_{p\in X_{R_i}}\rho_i(p)|u_i(p)|=1$ where the supremum is realized at p_i. Here ρ_i is the weighting function

$$\rho_i(p) = \begin{cases} e^{\delta t_j} & (t_j, \theta_j) \in (0,\infty) \times \mathbb{S}^1 \sim E_j \\ \frac{\cosh^\delta t}{\cosh^\delta R_i} & (t,\theta) \in (-R_i, R_i) \times \mathbb{S}^1 \sim X_{3,R_i} \\ 1 & \text{else} \end{cases}$$

where $\delta > 0$ is strictly less than $\inf_j \gamma_2(\tau)$, including $j = 1, 2$ (i.e. including the Delaunay parameter for the ends we are gluing). We will have to consider different cases which correspond to the different possible places where $\rho_i(p)|u_i(p)|$ can achieve its maximum. Notice we always have $|u_i(p)| \le \rho_i(p)^{-1}$ with equality at p_i.

The structure of this proof is the following. We have normalized u_i so that $\rho_i|u_i|$ attains its maximum at some point p_i, and at this point p_i we know the value of $|u_i|$. The weighting function is chosen to try to force the point p_i to occur in the compact sets K_1 and K_2. If p_i occurs elsewhere we can obtain an easy contradiction from the nondegeneracy of the original summands X_1 and X_2 and the model Delaunay surface D for the ends we are gluing together. When p_i occurs in K_1 we will obtain a Jacobi field on X_1 which decays exponentially on all ends but E_1, which contradicts the assumption that X_1 admits a deformation which changes the necksize of E_1 to first order. We obtain a similar Jacobi field on X_2 is p_i occurs in K_2. In this case we will use a transmission argument (the two ends we are gluing together must transmit asymptotic information about Jacobi fields to each other) to obtain a similar Jacobi field on X_1.

First consider the case where $p_i = (t_i, \theta_i) \in X_{3,R_i}$ with $|t_i|$ bounded and let

$$w_i(t,\theta) = (\cosh^{-\delta} R_i) u_i(t,\theta)$$

on $(-R, R) \times \mathbb{S}^1$. The function w_i still solves $\mathcal{L}_{X_{R_i}} w_i = 0$ on $(-R, R) \times \mathbb{S}^1$ and satisfies the bound $|w_i(t,\theta)| \le \cosh^{-\delta} t$, with equality at (t_i, θ_i). Thus a subsequence exists such that w_i converges to a Jacobi field $\bar{w}$ for the model Delaunay surface D (with $(t_i,\theta_i) \to (\bar{t}, \bar{\theta})$) where $|\bar{w}(t,\theta)| \le \cosh^{-\delta} t$ with equality at $(\bar{t}, \bar{\theta})$. In particular, $\bar{w}$ is not identically zero. However, this contradicts the fact that Delaunay Jacobi fields cannot decay on both ends.

Next consider the case where $p_i = (t_i, \theta_i) \in X_{3,R_i}$ with none of $|t_i|$, $|t_i + R_i|$, and $|t_i - R_i|$ bounded. We will take the case where $t_i < 0$, as the case where $t_i > 0$ is similar. Now define

w_i by

$$w_i(t,\theta) = \frac{\cosh^\delta t_i}{\cosh^\delta R_i} u(t+t_i,\theta),$$

defined on $(-R_i, |t_i|) \times \mathbb{S}^1$. We have rescaled this function so that $|w_i(0,\theta_i)| = 1$. Moreover,

$$\begin{aligned} |w_i(t,\theta)| &\leq \frac{\cosh^\delta t_i}{\cosh^\delta (t+t_i)} \\ &= \frac{(e^{t_i}+e^{-t_i})^\delta}{(e^{t+t_i}+e^{t-t_i})^\delta} \\ &\leq \frac{(e^{t_i}+e^{-t_i})^\delta}{(2e^{-t-t_i})^\delta} \\ &= 2^{-\delta} e^{\delta t}(1+e^{2t_i})\delta \\ &\leq 2^{1-\delta} e^{\delta t}. \end{aligned}$$

Choose a subsequence such that $\theta_i \to \bar{\theta}$ and $w_i \to \bar{w}$. Then we obtain a Jacobi field $\bar{w}$ on the Delaunay surface D such that $|\bar{w}(0,\bar{\theta})| = 1$ and $\bar{w}$ decays on one end and grows exponentially at a rate of less than $\gamma_2(\tau)$ on the other. Such a Jacobi field cannot exist on D.

Next consider the case where $p_i = (t_{j,i}, \theta_{j,i}) \in E_j$ and $t_{j,i} \to \infty$. We take a slideback sequence

$$w_i(t_j,\theta_j) = \rho_i(t_{j,i},\theta_{j,i}) u_i(t+t_{j,i},\theta)$$

and argue as in the previous case.

Next consider the case where $p_i \in \bar{K}_1$ where $\bar{K}_1$ is a fixed compact set in X_1 containing K_1. Notice ρ_i is uniformly bounded away from 0 on $\bar{K}_1$. Take a subsequence such that $p_i \to \bar{p}$ and restrict u_i to $X_{1,R_i} \cup \{(t,\theta) \in X_{3,R_i} \mid t < -1\}$. Then on this surface u_i converges uniformly on compact sets to a Jacobi field $\bar{u}$ on X_1. If we parameterize the end E_1 of X_1 by $(s,\theta) = (t+R_i,\theta)$, then the bound $|u_i| \leq \rho_i^{-1}$ can be written as

$$\begin{aligned} |u_i(s,\theta)| &\leq \frac{\cosh^\delta R_i}{\cosh^\delta (s-R_i)} \\ &= \frac{(e^{R_i}+e^{-R_i})^\delta}{(e^{s-R_i}+e^{-s+R_i})^\delta} \\ &\leq 2^{-\delta}(1+e^{-2R_i})^\delta e^{\delta s}. \end{aligned}$$

Thus $|\bar{u}(s,\theta)| \leq ce^{\delta s}$ for some constant c on the end $E_1 \sim \{(s,\theta) \in (0,\infty) \times \mathbb{S}^1\}$. Because $\delta < \gamma_2$, $\bar{u}$ can only grow linearly on E_1, and so $\bar{u} \in B_{X_1} \cap W_{E_1}$. Also, $|\bar{u}(\bar{p})| =$

$\lim_{i\to\infty} \rho_i(p_i)^{-1} > 0$. However, $B_{X_1} \cap W_{E_1} = \{0\}$ because X_1 admits a deformation which changes the asymptotic necksize of E_1 (see Lemma 5 and Remark 1).

Finally consider the case where $p_i \in \bar{K}_2$ where $\bar{K}_2$ is a fixed compact set in X_2 containing K_2. For this case, we will need to define the following function space on the cylinder $[-R, R] \times \mathbb{S}^1$ and prove a preliminary lemma.

Definition 5. *Let $K^{k,\alpha}_{\delta}([-R,R]\times\mathbb{S}^1)$ be the set of functions $u \in C^{k,\alpha}_{loc}([-R,R]\times\mathbb{S}^1)$ for which the norm*

$$\|u\|_{K^{k,\alpha}_{\delta}} = \sup_{|t_0|\leq R-1} \| \left(\frac{\cosh t}{\cosh R}\right)^{\delta} u\|_{C^{k,\alpha}([t_0-1,t_0+1]\times\mathbb{S}^1)}$$

is finite.

Lemma 9. *For each Delaunay parameter τ there is an $R_0 > 0$ such that for $R \geq R_0$ and for any $0 < \delta < \gamma_2(\tau)$ there exists an operator $\hat{\mathcal{G}} : K^{0,\alpha}_{-\delta}([-R,R]\times\mathbb{S}^1) \to K^{2,\alpha}_{-\delta}([-R,R]\times\mathbb{S}^1)$ which is uniformly bounded and such that $u = \hat{\mathcal{G}}(f)$ solves the problem*

$$\begin{cases} \mathcal{L}_D u = f \quad \text{on} \quad [-R,R]\times\mathbb{S}^1 \\ u(\pm R,\cdot) \quad \in \quad \operatorname{span}\{\cos\theta, \sin\theta, 1\}. \end{cases}$$

Proof: Let χ be a cut-off function such that

$$\chi = \begin{cases} 1 & t < -1 \\ 0 & t > 1. \end{cases}$$

First let U_1 be a solution to

$$\begin{cases} \mathcal{L}_D(U_1) = \chi f \quad \text{for} \quad t \geq -R \\ U_1(-R,\cdot) \quad \in \quad \operatorname{span}\{\cos\theta, \sin\theta, 1\}, \end{cases}$$

which exists because the Delaunay surface D is nondegenerate. Moreover, we have the bound

$$|U_1(t,\theta)| \leq c\|\chi f\|_{C^{0,\alpha}_{-\delta}((-R,\infty)\times\mathbb{S}^1)} e^{-\gamma_2(\tau)(R+t)}.$$

Similarly, let U_2 solve $\mathcal{L}_D(U_2) = (1-\chi)f$ on $(-\infty, R)\times\mathbb{S}^1$ with $U_2(R,\cdot) \in \operatorname{span}\{\cos\theta, \sin\theta, 1\}$ and

$$|U_2(t,\theta)| \leq c\|(1-\chi)f\|_{C^{0,\alpha}_{-\delta}((-\infty,R)\times\mathbb{S}^1)} e^{-\gamma_2(\tau)(R-t)}.$$

Let η_1 and η_2 be cut-off functions such that

$$\eta_1 = \begin{cases} 1 & t < R-2 \\ 0 & t > R-1 \end{cases}$$

and

$$\eta_2 = \begin{cases} 1 & t > -R+2 \\ 0 & t < -R+1. \end{cases}$$

Define the operator $\tilde{\mathcal{G}} : K^{0,\alpha}_{-\delta}([-R,R]\times\mathbb{S}^1) \to K^{2,\alpha}_{-\delta}([-R,R]\times\mathbb{S}^1)$ by

$$\tilde{\mathcal{G}}(f) = \eta_1 U_1 + \eta_2 U_2.$$

The lemma the follows from the bounds

- $\|\tilde{G}(f)\|_{K^{2,\alpha}_{-\delta}([-R,R]\times\mathbb{S}^1)} \leq c\|f\|_{K^{0,\alpha}_{-\delta}([-R,R]\times\mathbb{S}^1)}$
- $\|\mathcal{L}_D(\tilde{\mathcal{G}}(f)) - f\|_{K^{0,\alpha}_{-\delta}([-R,R]\times\mathbb{S}^1)} \leq \tilde{c}\|f\|_{K^{0,\alpha}_{-\delta}([-R,R]\times\mathbb{S}^1)} e^{-\tilde{\gamma}R}$ for some positive δ

once one takes R to be sufficiently large by a perturbation argument as in the proof of proposition 6. The bound on $\hat{\mathcal{G}}(f)$ follows from the bounds on

$$|U_i(t,\theta)| \leq \begin{cases} c\|\chi f\|_{C^{0,\alpha}_{-\delta}((-R,\infty)\times\mathbb{S}^1)} & i=1 \\ c\|(1-\chi)f\|_{C^{0,\alpha}_{-\delta}((-\infty,R)\times\mathbb{S}^1)} & i=2. \end{cases}$$

The quantity $\mathcal{L}_D(\eta_1 U_1 + \eta_2 U_2) - f$ is nonzero only in the regions corresponding to $R-2 \leq t \leq R-1$ and $-R+1 \leq t \leq -R+2$. For $R-2 \leq t \leq R-1$,

$$|\mathcal{L}_D \circ \hat{\mathcal{G}}(f) - f| = |\mathcal{L}_D(\eta_1 U_1)| \leq \hat{c}\|\mathcal{L}_D\||U_1| \leq c\|f\|_{K^{0,\alpha}_{-\delta}} e^{-2\delta R}.$$

One can obtain the same bound for $-R+1 \leq t \leq -R+2$ by a similar argument. ■

Now we return to the proof that the solution surface is nondegenerate. Recall that our last remaining case is when $p_i \in \bar{K}_2$ as described above. Take subsequence so that $p_i \to \bar{p}$ and fix an R_0 to be determined later. As in the previous case, u_i converges uniformly on compact sets to a Jacobi field $\bar{u}$ on X_2 when restricted to $X_{2,R_i} \cup \{(t,\theta) \in X_{3,R_i} \mid t > 1\}$. This Jacobi field $\bar{u}$ decays exponentially on all ends of X_2 except E_2. Therefore, by Lemma

5, $\bar{u}$ must correspond to an asymptotic translation of E_2 along its axis. So there is some $\alpha > 0$ such that

$$\bar{u}(t,\theta) = \alpha u^{0,+}(R-t,\theta) + O(e^{-\tilde{\delta}t})$$

for $t > 0$ and some $\tilde{\delta} < \gamma_2(\tau)$. We may as well rescale so that $\alpha = 1$. In any compact set $K \subset \{(t,\theta) \in X_{3,R_i} \mid t > 1\}$ we have

$$u_i(t,\theta) = u^{0,+}(t,\theta) + O(e^{-\tilde{\delta}(R-t)} + \|u_i - \bar{u}\|_{C^0(K)}).$$

Thus (because of the uniform convergence of u_i) there is some i_0 which depends on R_0 such that for $i > i_0$

$$\|u_i - u^{0,+}\|_{C^2((R_i-R_0-1,R_i-R_0+1)\times\mathbb{S}^1)} = O(e^{-\tilde{\delta}R_0}).$$

Similarly, we can restrict u_i to $X_{1,R_i} \cup \{(t,\theta) \in X_{3,R_i} \mid t < -1\}$ and obtain another convergent subsequence on X_1. By the argument in the previous case, this limit must be zero, and so for $i > i_0$,

$$\|u_i\|_{C^2((-R_i+R_0-1,-R_i+R_0+1)\times\mathbb{S}^1)} = O(e^{-\tilde{\delta}R_0}).$$

Now let $\Phi^{0,-}$ be the Jacobi field on the model Delaunay surface D which corresponds to changing the necksize of D. The main point is that we can use the function $\Phi^{0,-}$ to transmit the eigenmode information of u_i from ne end of the cylinder X_{3,R_i} to the other. We can do this because because the cylinder is $C^{2,\alpha}$ close to the Delaunay surface D, u_i is close to the Jacobi field $\Phi^{0,+}$ for D and these two Jacobi fields are dual variables in the symplectic structure of the deficiency space.

We wish to evaluate

$$\int_{[-R_i+R_0,R_i-R_0]\times\mathbb{S}^1} \mathcal{L}_D(u_i)\Phi^{0,-} - \mathcal{L}_D(\Phi^{0,-})u_i.$$

Upon integrating by parts, we find that this integral is

$$\begin{aligned}
\int_{\{R_i-R_0\}\times\mathbb{S}^1}(\frac{\partial u_i}{\partial t}\Phi^{0,-} - \frac{\partial \Phi^{0,-}}{\partial t}u_i) \quad - &\quad \int_{\{-R_i+R_0\}\times\mathbb{S}^1}(\frac{\partial u_i}{\partial t}\Phi^{0,-} - \frac{\partial \Phi^{0,-}}{\partial t}u_i) \qquad (5.1)\\
= &\quad \int_{\{R_i-R_0\}\times\mathbb{S}^1}(\frac{\partial u^{0,+}}{\partial t}\Phi^{0,-} - \frac{\partial \Phi^{0,-}}{\partial t}u^{0,+}) + O(e^{-\tilde{\delta}R_0})\\
&\quad -\int_{\{-R_i+R_0\}\times\mathbb{S}^1} O(e^{-\tilde{\delta}R_0})\\
= &\quad 1 + O(R_0 e^{-\tilde{\delta}R_0}),
\end{aligned}$$

where we have used in the last estimate the facts that $u_i(-R_i + R_0, \cdot) = O(e^{-\tilde{\delta} R_0})$ for $i > i_0$ and that $u_i(R_i - R_0, \cdot) = u^{0,+}(R_0, \cdot) + O(e^{-\tilde{\delta} R_0})$ for $i > i_0$. On the other hand, we can estimate this integral directly. First note that X_{3,R_i} is a graph over the model Delaunay surface D, where the graphing function is $O(\left(\frac{\cosh t}{\cosh R_i}\right)^{\gamma_2(\tau)})$, which implies that $|\mathcal{L}_D(u_i)(t,\theta)| = O(\left(\frac{\cosh t}{\cosh R_i}\right)^{\hat{\delta}})$ where $\hat{\delta}$ is any positive number such that $\hat{\delta} < \gamma_2(\tau)$, in particular for some $\hat{\delta} \in (\delta, \gamma_2(\tau))$. Thus

$$\|\mathcal{L}_D(u_i)\|_{K^{0,\alpha}_{-\delta}([-R_i+R_0, R_i+R_0]\times \mathcal{S}^1)} = O(\left(\frac{\cosh(R_i - R_0)}{\cosh R_i}\right)^{\hat{\delta}-\delta}) = O(e^{-R_0(\hat{\delta}-\delta)}).$$

Now define $\tilde{u}_i : [-R_i + R_0, R_i - R_0] \times \mathbb{S}^1 \to \mathbb{R}$ by

$$\tilde{u}_i = \hat{\mathcal{G}}(\mathcal{L}_D(u_i))$$

where $\hat{\mathcal{G}}$ is the operator in Lemma 9. By the uniform boundedness of $\hat{\mathcal{G}}$,

$$\|u_i - \tilde{u}_i\|_{K^{2,\alpha}_{\delta}([-R_i+R_0, R_i-R_0]\times \mathbb{S}^1)} = O(e^{-R_0(\hat{\delta}-\delta)}).$$

So

$$\begin{aligned}
\left|\int_{[-R_i+R_0, R_i-R_0]\times\mathbb{S}^1} \mathcal{L}_D(u_i)\Phi^{0,-}\right. &- \left.\mathcal{L}_D(\Phi^{0,-})u_i\right| \\
&\leq \int_{[-R_i+R_0, R_i-R_0]\times\mathbb{S}^1} |\mathcal{L}_D(u_i)\Phi^{0,-} - \mathcal{L}_D(\Phi^{0,-})u_i| \\
&= \int_{[-R_i+R_0, R_i-R_0]\times\mathbb{S}^1} |\mathcal{L}_D(u_i - \tilde{u}_i)| \\
&\leq \int_{[-R_i+R_0, R_i-R_0]\times\mathbb{S}^1} \|\mathcal{L}_D\||u_i - \tilde{u}_i| \\
&\leq \hat{c}\|\mathcal{L}_D\| \int_{-R_i+R_0}^{R_i-R_0} e^{-R_0(\hat{\delta}-\delta)} \left(\frac{\cosh t}{\cosh(R_i - R_0)}\right)^{\delta} \\
&= O(R_0 e^{-R_0(\hat{\delta}-\delta)}),
\end{aligned}$$

which contradicts equation (5.2). This completes the proof that the solution surface $X_{R,\phi}$ is nondegenerate.

Chapter 6

EXAMPLES AND APPLICATIONS

As is [KMP96] we define the moduli space of CMC surfaces below.

Definition 6. *Fix a topological surface Σ of genus $g \geq 0$ and $k \geq 1$ ends. Define $\mathcal{M}_{k,g}$ to be the space of all proper, noncompact CMC embeddings (with the ends labeled) $X : \Sigma \to \mathbb{R}^3$, where embeddings are identified if they differ by a rigid motion which preserves the labeling on the ends or by a reparameterization of Σ. Endow $\mathcal{M}_{k,g}$ with the Hausdorff topology on compact sets. We will make the abbreviation $\mathcal{M}_{k.0} = \mathcal{M}_k$.*

Theorem 10. *If the embeddings X_i with chosen ends E_i are nondegenerate and admit deformation through CMC surfaces which change the asymptotic necksize of $E_i$to first order, and $X_i(t)$ is a curve of such embeddings in moduli space, then for t small the choice of embeddings $X_i(t)$ also admit deformations through CMC surfaces which change the asymptotic necksize of $E_i(t)$ to first order.*

Proof: Geometrically, the reason this theorem holds is that the set of nondegenerate embeddings which do no admit such deformation lie in a closed set. Suppose that at $t = 0$ X_1 admits a deformation through CMC surfaces which changes the asymptotic necksize of E_1. Then $B_{X_1}(0)$ does not lie in the hyperplane $\{\xi_1 = 0\}$. Because $B_{X_1}(t)$ varies continuously with t (it is the kernel of a continuously varying operator with constant rank $3k_1$), $B_{X_1}(t)$ must remain transverse to $\{\xi_1 = 0\}$ for small t. ■

We have yet to show that there exist CMC surfaces which satisfy the gluing criterion. It turns out that many surfaces in $\mathcal{M}_3$ satisfy the gluing criterion for all ends. To see this, we must first sketch the recent classification theorem of Kusner, Grosse-Brauckmann, and Sullivan [KGBS]. They show that $\mathcal{M}_3$ is homeomorphic to the space of triples of distinct points in $\mathbb{S}^2$. The classifying map is given as follows. By a result in [KKS89], each 3-ended CMC surfaces has a plane of reflection symmetry, which we can take to be the xy-plane.

Cut the surface in half along this plane of symmetry. By a construction in [Law70], this yields a conjugate minimal surface in $\mathbb{S}^3$. This minimal surface has three boundary curves, each of which is a great circle. By the construction of the conjugate minimal surface, the boundary great circles are all Hopf circles for the same Hopf fibration. Therefore, the image of the three boundary curves under this fibration is a triple of distinct points in $\mathbb{S}^2$.

For our purposes, the salient feature of the classifying map is that the edge-lengths of the resulting triangle are 2π times the asymptotic necksizes of the surface. Let $X \in \mathcal{M}_3$ and suppose it corresponds under the classifying map to the three points p_1, p_2, and p_3. Then we can change the three side lengths in turn by moving p_2 along the geodesic joining p_1 and p_2, moving p_3 along the geodesic joining p_2 and p_3, and by moving p_3 along the geodesic joining p_1 and p_3. Thus X admits deformations through CMC surfaces which change the asymptotic necksizes of each of its ends in turn, which proves the following theorem.

Theorem 11. *Let $X \in \mathcal{M}_3$ with an end $E \subset X$. Then there exists a deformation of X through CMC surfaces which changes the asymptotic necksize of E to first order. Thus any choice of (X_1, X_2, E_1, E_2) where $X_i \in \mathcal{M}_3$ are nondegenerate and E_1 and E_2 are asymptotic to congruent Delaunay surfaces satisfy the hypotheses for the end-to-end gluing construction.*

Remark 3. *In fact, the proof of this theorem shows the following more general fact: given two three-ended, genus-zero CMC embeddings as above, there is an $R_0 > 0$ such that for $R \geq R_0$ the approximate solution $\bar{X}_{R,\phi}$ of Section 3 is nondegenerate for all relative angles ϕ. Therefore one can glue these surfaces together end-to-end for any relative angle.*

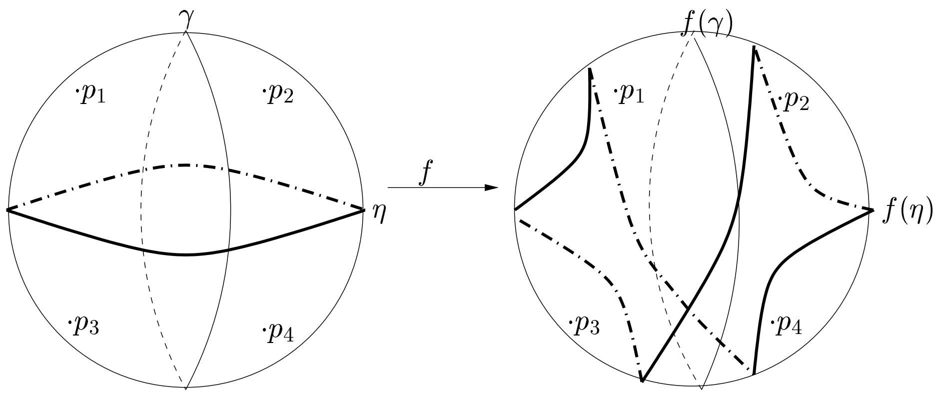
Let $X_1 : \mathbb{S}^2 \backslash \{p_1, p_2, p_3\} \to \mathbb{R}^3$ Be a nondegenerate CMC embedding with no cylindrical ends. Such embeddings exist by the gluing construction of Mazzeo and Pacard [MP01] and the nondegeneracy result of Montiel and Ros [MR91]. By Theorem 11 and Remark 3, one can apply the doubling construction to X_1 for any choice of ends and any choice of relative angle ϕ. Because the approximate solutions depend continuously on ϕ, so do the operators $\mathcal{L}_{\bar{X}_{R,\phi}}$ and $\mathcal{G}_{\bar{X}_{R,\phi}}$. Hence the solution embedding $X(\phi) = X_{R,\phi}$ depends continuously on ϕ. In this construction of $X(\phi)$, one must choose a parameter $R = R(\phi)$ where $R(\phi)$ is large enough so that Proposition 6 applies. Because $\phi \in \mathbb{S}^1$ and $\mathbb{S}^1$ is compact, one can choose an R_0 such that $R_0 \geq R(\phi)$ for all ϕ and then work only with this choice of R_0. Varying ϕ through all values in $\mathbb{S}^1$ yields a continuous closed loop of embeddings $X(\phi)$ in $\mathcal{M}_4$.

Note that each $X(\phi)$ is has the conformal type of a four punctured sphere. Let $\pi : \mathcal{M}_4 \to \mathrm{Conf}(\mathbb{S}^2 \backslash \{p_1, p_2, p_3, p_4\})$ be the natural projection, where $\mathrm{Conf}(\Sigma)$ is the space of conformal structures on Σ, defined by $\mathrm{Conf}(\Sigma) = \mathcal{T}(\Sigma)/\mathrm{PMod}(\Sigma)$. Here Teichmüller space $\mathcal{T}(\Sigma)$ is the space of complete metrics on Σ with constant curvature -1 and the pure mapping class group $\mathrm{PMod}(\Sigma)$ is the group of isotopy classes of diffeomorphisms of Σ which preserve its punctures (or boundary components, setwise). An element $[f] \in \mathrm{PMod}(\Sigma)$ acts on $g \in \mathcal{T}(\Sigma)$ by pulling g back to f^*g. Note that for an arbitrary diffeomorphism f preserving the punctures of Σ, the metric f^*g may not have constant curvature -1, but (by the Uniformization theorem) f is isotopic to some $\tilde{f}$ such that $\tilde{f}^*g$ has constant curvature -1. Thus the action of $\mathrm{PMod}(\Sigma)$ on $\mathcal{T}(\Sigma)$ is well defined. Because $\mathcal{T}(\Sigma)$ is simply connected (contractible, in fact), $\pi_1(\mathrm{Conf}(\Sigma)) = \mathrm{PMod}(\Sigma)$.

Now examine the induced map

$$\hat{X}(\phi) = \pi \circ X(\phi) : \mathbb{S}^1 \to \mathrm{Conf}(\mathbb{S}^2 \backslash \{p_1, p_2, p_3, p_4\}).$$

The loop $\hat{X}$ in $\mathrm{Conf}(\mathbb{S}^2 \backslash \{p_1, p_2, p_3, p_4\})$ is a Dehn twist in $\mathrm{PMod}(\mathbb{S}^2 \backslash \{p_1, p_2, p_3, p_4\})$ about a loop γ in $\mathbb{S}^2 \backslash \{p_1, p_2, p_3, p_4\}$ which encloses two of the punctures. We can take $\gamma \subset \mathbb{S}^2 \backslash \{p_1, p_2, p_3, p_4\}$ to be the slice $\{x = 0\}$ in the original construction of the approximate solution $\bar{X}(0)$. Such Dehn twists generate $\mathrm{PMod}(\mathbb{S}^2 \backslash \{p_1, p_2, p_3, p_4\})$, and so the loop $\hat{X}$ is homotopically nontrivial in $\mathrm{Conf}(\mathbb{S}^2 \backslash \{p_1, p_2, p_3, p_4\})$. To see that the Dehn twist about γ is a nontrivial element in $\mathrm{PMod}(\mathbb{S}^2 \backslash \{p_1, p_2, p_3, p_4\})$, observe that it does not act trivially on the isotopy class of η (see the figure below). See, e.g., [Iva] or [Bus92] for more details on this point.



In the case where Σ is a four punctured sphere, we can identify the quotient $\mathrm{Conf}(\Sigma) = \mathcal{T}(\Sigma)/\mathrm{PMod}(\Sigma)$ explicitly. Given the choice of punctures $\{p_1, p_2, p_3, p_4\}$, there is a unique Möbius transformation Φ which sends p_1 to 0, p_2 to 1, and p_3 to ∞. The conformal type of $\mathbb{S}^2\backslash\{p_1, p_2, p_3, p_4\}$ is then determined by $\Phi(p_4)$, and so $\mathrm{Conf}(\mathbb{S}^2\backslash\{p_1, p_2, p_3, p_4\})$ is naturally equivalent to $\mathbb{S}^2\backslash\{0, 1, \infty\}$. Moreover, our loop $\hat{X}(\phi)$ describes a loop about one of the punctures $\{0, 1, \infty\}$ and is thus homotopically nontrivial. The loop in question wraps around one of the punctures because it corresponds to the relative positioning of the ends of $X(\phi)$ and by construction the ends of $X(\phi)$ twist around each other when one varies ϕ about a full circle.

If $X(\phi)$ were a homotopically trivial loop in $\mathcal{M}_4$, then it would push forward via π to a homotopically trivial loop $\hat{X}$ in $\mathrm{Conf}(\mathbb{S}^2\backslash\{p_1, p_2, p_3, p_4\})$, contradicting the above argument.

More generally, suppose $X_1 : \Sigma_1 \to \mathbb{R}^3$ is a nondegenerate $(k-1)$ ended CMC surface whose ends are asymptotic to Delaunay surfaces with small necks. Again, such surfaces exist by the gluing theorem of [MP01]. One can pick an end E_1 of $X_1(\Sigma_2)$ and find a nondegenerate three-ended CMC surface (again with all asymptotic necksizes small) $X_2 : \mathbb{S}^2\backslash\{p_1, p_2, p_3\} \to \mathbb{R}^3$ with an end E_2 which is asymptotic to a congruent Delaunay surface. By Theorem 11 and Remark 3 this choice of CMC embeddings and ends is transverse at infinity for all relative angles ϕ, and so the argument above produces a nontrivial loop in $\mathcal{M}_k$ by varying the angle ϕ. Thus we have shown the following theorem.

Theorem 12. *The moduli spaces $\mathcal{M}_k$ for $k \geq 4$ all have connected components which are not simply connected.*

This theorem also follows from recent work of Mazzeo, Pacard, and Pollack in [MPPR]. Here they combine and modify the gluing constructions of [MP01] and [MPP] to produce many new complete surfaces of constant mean curvature. In particular, there are two ways in which the constructions of [MPPR] may be used to produce homotopically nontrivial loops in the respective moduli spaces. First they establish a version of the connected sum theorem of [MPP] which allows them to glue together two complete nondegenerate CMC surfaces, in particular two Delaunay surfaces. This may be done with any choice of relative angle between the surfaces, and again yields a non-contractible loop in the moduli space. The idea for the second type of construction of nontrivial loops is due to Pacard and clearly seems to be the most versatile. Here they glue a half-Delaunay surface (with small necksize) onto a nondegenerate $(k-1)$ ended CMC surface to obtain a family of nondegenerate k-ended CMC surfaces. Since this may be done at any point they obtain one parameter families of surfaces by varying the point at which the gluing is done. In particular, this allows them to construct many distinct homotopy classes of non-contractible loops in $\mathcal{M}_k$.

Appendices

Appendix A

VARIOUS FORMULATIONS OF THE CMC CONDITION

One can formulate the condition that an immersion is of constant mean curvature in various ways. Each is useful to understand some part of the general theory of CMC surfaces.

A.1 The Local Formulation: Principal Curvatures

We start with the local formulation in terms of coordinates (s, θ) on Σ. Then

$$g = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \langle X_s, X_s \rangle & \langle X_s, X_\theta \rangle \\ \langle X_s, X_\theta \rangle & \langle X_\theta, X_\theta \rangle \end{bmatrix},$$

$$A = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \langle X_{ss}, \nu \rangle & \langle X_{s\theta}, \nu \rangle \\ \langle X_{s\theta}, \nu \rangle & \langle X_{\theta\theta}, \nu \rangle \end{bmatrix},$$

and

$$H = \frac{1}{2} \operatorname{tr}_g A = \frac{1}{2} \frac{LG + NE - 2FM}{EG - F^2}.$$

Near any point, we can write Σ as a graph over its tangent plane. Then the immersion X takes the form

$$X(s, \theta) = (s, \theta, f(s, \theta)).$$

If X takes this form, the metric is given by

$$g = \begin{bmatrix} 1 + f_s^2 & f_s f_\theta \\ f_s f_\theta & 1 + f_\theta^2 \end{bmatrix}$$

and the second fundamental form is given by

$$A = \frac{1}{\sqrt{1 + f_s^2 + f_\theta^2}} \begin{bmatrix} f_{ss} & f_{s\theta} \\ f_{s\theta} & f_{\theta\theta} \end{bmatrix}.$$

In particular, the mean curvature is given by

$$H = \frac{f_{ss}(1+f_\theta^2) + f_{\theta\theta}(1+f_s^2) - 2f_{s\theta}f_sf_\theta}{2(1+f_s^2+f_\theta^2)^{\frac{3}{2}}}.$$

Setting $H = 1$ and rearranging yields

$$0 = f_{ss}(1+f_\theta^2) + f_{\theta\theta}(1+f_s^2) - 2f_{s\theta}f_sf_\theta - 2(1+f_s^2+f_\theta^2)^{\frac{3}{2}}. \tag{A.1}$$

Several remarks on equation (A.1) will prove useful. First, this is a quasilinear second order PDE in f. It is strongly elliptic. In fact, the linearization of the second order part the right hand side of equation (A.1) is given by the matrix

$$\begin{bmatrix} 1+f_\theta^2 & -f_sf_\theta \\ -f_sf_\theta & 1+f_s^2 \end{bmatrix}.$$

And so the principal symbol of equation (A.1) is given by

$$\begin{bmatrix} \lambda & \mu \end{bmatrix} \begin{bmatrix} 1+f_\theta^2 & -f_sf_\theta \\ -f_sf_\theta & 1+f_s^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \lambda^2 + \mu^2 + (\lambda f_\theta - \mu f_s)^2 \geq 0$$

with equality only when $\lambda = 0 = \mu$. This implies, among other things, that CMC surfaces are analytic (elliptic regularity) and the function f obeys the strong maximum principle (e.g . f can have no positive interior maxima; see, for example [PW84] or [GT77]).

One can also attach a geometric interpretation to these coordinate computations. In the following paragraph, we will work only at the origin in the (s, θ) coordinates and we will assume that

$$0 = f(0,0) = f_s(0,0) = f_\theta(0,0).$$

This amounts to setting the (s, θ) plane to be the tangent plane to Σ at the point corresponding to $(0,0)$. Then

$$g(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A(0,0) = \begin{bmatrix} f_{ss} & f_{s\theta} \\ f_{s\theta} & f_{\theta\theta} \end{bmatrix}.$$

Recalling the minimax method to find eigenvalues using the Raleigh quotient, we see that the eigenvectors of A point in the direction of steepest descent and ascent for the function f. Call these eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Order them so that their respective eigenvalues k_1 and

k_2 satisfy $k_1 \leq k_2$. Notice that setting $k_1 + k_2 = 2$ and $k_1 \leq k_2$ implies $k_2 \geq 1 > 0$. If we let $h(t) = f(t\vec{v}_2)$ the we see $t = 0$ is a local minimum for h. In fact, the graph of h is concave up and the circle lying above the graph which best fits the graph will have radius $\frac{1}{h''(0)} = \frac{1}{k_2}$. Similar remarks hold for k_1, although one must be careful of signs when treating this case. Thus we see that the eigenvalues k_1 and k_2 correspond to radii of the largest and smallest circles (taking signs into account) fitting curves in Σ one finds by intersecting Σ with a plane normal to Σ at the origin. These eigenvalues k_1 and k_2 are called the principal curvatures of Σ and the eigendirections $\operatorname{span} \vec{v}_1$ and $\operatorname{span} \vec{v}_2$ are called the principal direction. A point on Σ is called umbilic if $k_1 = k_2$. The preceding discussion shows that the mean curvature of a surface at a point p is the average of the curvature of curves in Σ through p in all directions, giving credence to the name "mean curvature".

A.2 The Variational Formulation

The variational set-up described below is the same as in [Li93]. This formulation of the CMC condition is classical. One can find a modern treatment of it in volume IV of [Spi75] (towards the end of Chapter 9) and [Kus91].

On can also formulate the condition that X is a CMC immersion in variational terms. First consider a one parameter family of immersions $X_t : \Sigma \to \mathbb{R}^3$ with $X_0 = X$. Then the first variation of area $\frac{d}{dt}\Big|_{t=0} \operatorname{Area}(X_t(\Sigma)) = \frac{d}{dt}\Big|_{t=0} \int_\Sigma X_t^*(dV)$ is given by

$$\frac{d}{dt}\bigg|_{t=0} \operatorname{Area}(X_t(\Sigma)) = \int_\Sigma \langle \frac{d}{dt}\bigg|_{t=0} X_t, H\nu \rangle.$$

Now consider the following situation. Let X be a CMC immersion of Σ as above and let $U \subset \mathbb{R}^3$ be a bounded open set with $\partial U = Q \cup S$ where S is an open subset of $X(\Sigma)$ and $\partial Q = \partial S = ?$ is a smooth closed curve in $X(\Sigma)$. Let V be a vector field supported in $U \backslash \bar{Q}$ and denote its flow by ϕ_t. This vector field yields a one parameter family of surfaces $S_t = \phi_t(S)$ and a one parameter family of solids $U_t = \phi_t(U)$. Pick a real constant H and let h denote the mean curvature of X. Then the formula for the first variation of volume yields

$$\frac{d}{dt}\bigg|_{t=0} (\operatorname{Area}(S_t) - H \operatorname{Vol}(U_t)) = (h - H) \operatorname{Area}(S).$$

Thus we see that surfaces with mean curvature identically H are critical points of the functional Area $-H$ Vol.

A.3 The Hopf Differential, the Sinh-Gordon Equation, and Harmonicity of the Gauss Map

Much of this formulation can be found in [Woo94].

For this section we will work in conformal coordinates on Σ. In other words, we will let (s,θ) be coordinates on Σ such that $E = G = 2e^{2\omega}$ and $F = 0$. Then $z = s + i\theta$ is a complex coordinate on Σ. Define the vector fields

$$\partial_z = \frac{1}{2}(\partial_s - i\partial_\theta) \text{ and } \partial_{\bar{z}} = \frac{1}{2}(\partial_s + i\partial_\theta).$$

Notice that

$$\partial_z \partial_{\bar{z}} = \frac{1}{4}\Delta.$$

Consider the immersion X restricted to a simply connected region Ω on the surface. The condition that $z = s + i\theta$ is a conformal coordinate with conformal factor $2e^{2\omega}$ is equivalent to

$$\langle X_z, X_z\rangle = 0 \qquad \langle X_z, X_{\bar{z}}\rangle = e^{2\omega}.$$

In addition, we also have

$$\langle \nu, X_z\rangle = 0 \qquad \langle \nu, X_{\bar{z}}\rangle = 0.$$

Taking derivatives of these equations yields

$$\langle X_{zz}, X_z\rangle = 0 \quad \langle X_{z\bar{z}}, X_z\rangle = 0 \quad \langle X_{zz}, X_{\bar{z}}\rangle = 2\omega_z e^{2\omega}$$

and

$$\langle \nu, X_{zz}\rangle + \langle \nu_z X_z\rangle = 0 \quad \langle \nu, X_{z\bar{z}}\rangle + \langle \nu_{\bar{z}}, X_z\rangle = 0.$$

If we let $\langle \nu, X_{zz}\rangle = Q$ and note $\langle \nu, X_{z\bar{z}}\rangle = \frac{1}{4}\langle \nu, \Delta X\rangle = \frac{1}{2}e^{2\omega}H$, then the above equations imply

$$X_{zz} = 2\omega_z X_z + Q\nu \quad X_{z\bar{z}} = \frac{1}{2}e^{2\omega}H\nu \quad \nu_z = -\frac{1}{2}HX_z - Qe^{-2\omega}X_{\bar{z}}. \tag{A.2}$$

As a side note, Q is the coefficient of a quadratic differential form Qdz^2. The function Q itself is only locally defined, but Qdz^2 is a globally defined quadratic differential form. This quadratic form is called the Hopf differential.

We can rewrite equations (A.2) as

$$\begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix}_z = \begin{bmatrix} 2\omega_z & 0 & Q \\ 0 & 0 & \frac{1}{2}e^{2\omega}H \\ -\frac{1}{2}H & -Qe^{-2\omega} & 0 \end{bmatrix} \begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix} = U \begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix}. \tag{A.3}$$

Similarly,

$$\begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix}_{\bar{z}} = \begin{bmatrix} 0 & 0 & \frac{1}{2}e^{2\omega}H \\ 0 & 2\omega_{\bar{z}} & \bar{Q} \\ -\bar{Q}e^{-2\omega} & -\frac{1}{2}H & 0 \end{bmatrix} \begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix} = V \begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix}. \tag{A.4}$$

Setting $\partial_{\bar{z}}$ of equation (A.3) equal to ∂_z of equation (A.4) yields

$$U_{\bar{z}} - V_z + [U, V] = 0. \tag{A.5}$$

One can compute that $U_{\bar{z}} - V_z + [U, V]$ is given by

$$\begin{bmatrix} 2\omega_{z\bar{z}} - |Q|^2e^{-2\omega} + \frac{1}{4}H^2e^{2\omega} & 0 & Q_{\bar{z}} - \frac{1}{2}e^{2\omega}H_z \\ 0 & -2\omega_{z\bar{z}} + |Q|^2e^{-2\omega} - \frac{1}{4}H^2e^{2\omega} & -\bar{Q}_z + \frac{1}{2}e^{2\omega}H_{\bar{z}} \\ -\frac{1}{2}H_{\bar{z}} + e^{-2\omega}\bar{Q}_z & \frac{1}{2}H_z - e^{-2\omega}Q_{\bar{z}} & 0 \end{bmatrix}.$$

Setting this quantity to zero yields the following two equations:

$$\Delta\omega + \frac{1}{2}H^2e^{2\omega} - 2|Q|^2e^{-2\omega} = 0 \tag{A.6}$$

and

$$Q_{\bar{z}} - \frac{1}{2}e^{2\omega}H_z = 0. \tag{A.7}$$

Recalling that H is real-valued (and so $H_{\bar{z}} = (\bar{H_z}))$, we see that the latter equation implies H is constant if and only if Q is holomorphic. From this Hopf (see [Hop56]) proved

Theorem 13. (Hopf's Theorem)*: Let Σ be a compact simply connected immersed CMC surface. Then Σ is a round sphere.*

First note that we can rewrite Q as

$$Q = \frac{L - N}{2} - iM.$$

From this formulation we conclude that zeroes of the Hopf differential are umbilic points. By uniformization, if Σ is a compact simply connected surface then Σ is conformally equivalent to a sphere. From the fact that Σ is CMC we conclude that Qdz^2 is a holomorphic differential on the sphere. This forces $Qdz^2 = 0$ on all of Σ, and so all points of Σ are umbilic. From this fact it is easy to show that Σ must be a round sphere.

If X is a mean curvature one immersion of a torus, then one can extend the function Q from a small patch Ω about the origin to be a doubly periodic function on the entire plane $\mathbb{C}$. In particular, Q is a bounded holomorphic function on $\mathbb{C}$ and hence must be constant. After multiplying by an appropriate number in the domain, we can choose $Q = \frac{1}{2}$. With $Q = \frac{1}{2}$ and $H = 1$, equation (A.6) now becomes

$$\Delta\omega + \sinh 2\omega = 0, \tag{A.8}$$

which is known as the Sinh-Gordon equation. Notice that the rescaling to set $Q = \frac{1}{2}$ is a rescaling in the parameter space and the rescaling to set $H = 1$ is a rescaling in the target space. In particular, these rescalings can be done independently.

Further computation shows

$$\nu_{z\bar{z}} = -[\frac{1}{2}H_{\bar{z}}X_z + Q_{\bar{z}}e^{-2\omega}X_{\bar{z}} + (\frac{1}{4}H^2e^{2\omega} + |Q|^2e^{-2\omega})\nu].$$

Thus X is a CMC immersion if and only if $\nu_{z\bar{z}}$ is a multiple of ν. Recalling that $\nu : \Omega \to \mathbb{S}^2$, we see that $\Delta\nu = \lambda\nu$ is precisely the condition that ν is a harmonic map into $\mathbb{S}^2$. Thus X is a CMC immersion if and only if the Gauss map ν is harmonic.

Appendix B

EXAMPLES OF CMC SURFACES

As mentioned above, the unit sphere and the cylinder of radius $\frac{1}{2}$ are both CMC. The Delaunay surfaces provide the next example of embedded CMC surfaces. One can think of these surfaces as interpolating between spheres and cylinders.

B.1 Delaunay Surfaces

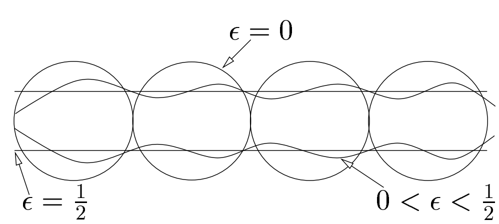
We seek an embedding of the form

$$D(t,\theta) = (\rho(t)\cos\theta, \rho(t)\sin\theta, t) : \mathbb{R}\times\mathbb{S}^1 \to \mathbb{R}^3$$

with mean curvature 1. An embedding of this form is rotationally symmetric about the z axis. The condition that D is an embedding implies $\rho > 0$. The CMC condition implies that ρ satisfies the equation

$$\rho_{tt} - \frac{1}{\rho}(1+\rho_t^2) + 2(1+\rho_t^2)^{\frac{3}{2}} = 0. \tag{B.1}$$

One particular solution is $\rho = \frac{1}{2}$. This solution corresponds to the cylinder. Normalize ρ so that ρ assumes a local minimum of ϵ at $t = 0$ (this amounts to a translation in the t variable). One can then show that ρ is periodic and in fact ϵ is a global minimum for ρ. Critical points for ρ alternate between minima and maxima. The minimum value (ϵ) for ρ is called the necksize of the embedding. One can show that as $\epsilon \to 0$ the embedding D tends to a string of unit spheres $\{x^2 + y^2 + (z-2n)^2 = 1\}$ for $n \in \mathbb{Z}$.



We will change variables, first to make D into a conformal embedding. To this end, we must replace t with $k(s)$ where k satisfies the equation

$$\rho(k(s)) = k'(s)(\rho'(k(s)) + 1).$$

Now let $\tau = 2\epsilon - \epsilon^2$ and define $\sigma(s)$ by $\rho(k(s)) = \tau e^{\sigma(s)}$. One can show that τ is a first integral of equation (B.1), see section C.2. Then one can show

$$\sigma_{ss} + \frac{\tau^2}{2} \sinh 2\sigma = 0 \qquad \frac{dk}{ds} = \frac{\tau^2}{2}(e^{2\sigma} + 1).$$

In fact, finding solutions to the above equations is equivalent to finding an embedded Delaunay surface.

Geometrically, one can think of the Delaunay surfaces as interpolating between the cylinder and the string of spheres. First place an ellipse tangent to the z axis in the $x - z$ plane so that one of the foci is on the x axis and is positioned to that it is as close to the z axis as possible. Now roll the ellipse along the z axis. One can show that the focal point which started on the x axis traces out the profile curve of a Delaunay surface (see [Eel87]). Varying the eccentricity of the ellipse corresponds to varying the necksize of the surface. The cylinder corresponds to rolling a circle of radius $\frac{1}{2}$ (the center is the only focal point and stays at constant height $\frac{1}{2}$). The string of spheres corresponds to rolling a line segment of length 1 (this is the degenerate case where the eccentricity goes to ∞ and the focal points go to the endpoints of the line segment).

B.2 CMC Tori

One might think to look for CMC immersions of compact surfaces. In the 1950's Hopf proved that any simply connected CMC immersion of a compact surface has to be a round sphere (see Hopf's theorem above, or [Hop56]). Around the same time, Alexandrov proved that any embedded compact CMC surface must be the round sphere (see Theorem 15 below). If one were looking for compact CMC immersions, then given these two results one might next look for CMC tori. Below we will regard a torus as $\mathbb{R}^2/?$ where ? is a lattice.

To find CMC tori, we look for doubly periodic immersions $\mathbb{R}^2 \to \mathbb{R}^3$. We can reduce this problem as follows. First, note that any immersion of a surface is determined up to rigid motions by its metric and its second fundamental form. Also notice that locally, the metric is determined by its conformal factor and that equations (A.3) and (A.4) determine the second fundamental form. Therefore, the conformal exponent ω will locally determine the immersion. Finally, notice that if Σ is a torus then ω must in fact be a doubly periodic function on $\mathbb{R}^2$. Thus the task of finding a CMC torus is the same as finding a doubly periodic solution to the Sinh-Gordon equation (equation (A.8)). In 1986, Wente proved [Wen86] that such doubly periodic solutions exist.

In 1987, Aubresch ([Aub87]) found many CMC tori by requiring that one line of curvature be planar. The condition that A has distinct eigenvalues allows us to simultaneously diagonalize A and g, so away from umbilic points we can choose coordinate lines which are also lines of curvature. The condition that the θ coordinate line in planar is equivalent to

$$\omega_{s\theta}\cosh\omega - \omega_s\omega_\theta\sinh\omega = 0.$$

We combine this equation with equation (A.8) to get an overdetermined system of equations. Under the change of variables $W = \cosh\omega$ this system becomes

$$\left\{\begin{array}{rcl} (W^2-1)\Delta W - W|\nabla W|^2 + W(W^2-1)^2 & = & 0 \\ (W^2-1)W_{s\theta} - 2WW_sW_\theta & = & 0. \end{array}\right.$$

Theorem 14. (Aubresch): *The real analytic solutions of the above system are given by*

$$W = \frac{f_s + g_\theta}{1 + f^2 + g^2}$$

where $f(s)$ and $g(\theta)$ are elliptic functions. Moreover, one can recover f and g by

$$\left\{\begin{array}{rcl} W_s & = & -f(s)(W^2-1) \\ W_\theta & = & -g(\theta)(W^2-1). \end{array}\right.$$

However, one still has to find conditions so that W is doubly periodic (these are called closing conditions). This is a rationality condition on the initial conditions c and d of f and g. Thus the CMC tori with one planar line of curvature are parameterized by the two parameters c and d. Aubresch then finds closing conditions on c and d (assuring that the solution W is in fact doubly periodic).

In 1989 Pinkhall and Sterling classified all CMC tori in [PS89]. Their idea is to write solutions to equation (A.8) as the flows of two commuting vector fields. Then one can integrate to get solutions and show that there exist only finitely many independent integrals. They then embed the ODE system in the Jacobian variety of the torus and find the closing conditions.

B.3 Kapouleas' Surfaces

In [Kap90] Kapouleas produced many examples of noncompact embedded CMC surfaces. As a first step, he creates a central graph, consisting of vertices, edges, rays, and weights for each vertex. He requires that the edges of these graphs have lengths that are even integers and that the graphs are balanced around each vertex (see section C.2). About each vertex he places a sphere of radius one. He places half a Delaunay surface about each ray, with necksize determined by the weight at the starting vertex of the ray. About the edges of length greater than 2, he places a piece of a Delaunay surface to connect the two spheres centered at the vertices which are the endpoints of the edge in question. Again, the necksize of this joining piece of Delaunay surface is determined by the weights of the vertices (which must be the same by balancing).

Next Kapouleas pieces all the surfaces together to form a smooth approximate solution. He pastes the spheres and pieces of Delaunay surfaces together with appropriately chosen cut-off functions. However, all the parts do not quite fit together without some sort of perturbation. For instance, the period of a Delaunay surface with small necksize is almost, but not quite, 2. So the Delaunay piece joining the two spheres mentioned above does not quite fit. To remedy this problem, Kapouleas first slightly perturbs the graph, and then slightly perturbs the necksizes of the Delaunay surfaces. After this step, he has a surface which has mean curvature one everywhere except for small bands near each neck of the Delaunay pieces. In these bands about the Delaunay necks the mean curvature is close to one.

Then Kapouleas solves the linearized problem (locally) on each bulge between the Delaunay necks. However, in these regions he must avoid the spherical harmonics which arise from eigenfunctions of the operator $\Delta + 2$ on $\mathbb{S}^2$. Thus he solves the linearized problem orthogonal to a finite dimensional "substitute kernel" on each bulge. A further difficulty in piecing together a global solution to the linearized operator from all these local solutions is that the global solution must be orthogonal to each of the substitute kernels mentioned above. This means that one must find a solution to the linearized problem which is orthogonal of to an infinite dimensional subspace. Finally, he must solve the nonlinear problem. To do this, Kapouleas shows one can find appropriate solutions for the linear problem after perturbing the graph mentioned above, and then uses a Leray-Schauder fixed point argument to show that a solution to the nonlinear problem for one of the perturbed graphs must exist.

Appendix C

GENERAL PROPERTIES OF ALMOST EMBEDDED CMC SURFACES

As stated above, we are concerned here with embedded CMC surfaces. However, many of the theorems still hold for a wider class of immersions, called *almost embeddings.*

Definition 7. *An immersion $X : \Sigma \to \mathbb{R}^3$ is called an almost embedding (or an Alexandrov embedding) if one can write Σ as the boundary of a solid handle-body Ω and X extends to be an immersion of Ω.*

One can think of this property as distinguishing an "outside" and an "inside" for the surface (the inside corresponding to the interior of the solid handle-body). Roughly speaking, the condition that a surface is almost embedded is the weakest condition one can place on the surface such that one can apply the Alexandrov reflection argument below.

C.1 Alexandrov Reflection

Alexandrov reflection is really an application of the maximum principle. To see how it works, we will first apply it to a compact CMC surface.

Let $X : \Sigma \to \mathbb{R}^3$ be a CMC embedding of a compact surface. Fix a large negative T so that the Σ lies completely above the plane $\pi = \{z = T\}$ (one can do this because Σ is compact). Let $\pi_t = \pi + (0, 0, t)$ be the translate of π by t in the z direction. Let Σ_t be the part of Σ which lies below π_t and let $\tilde{\Sigma}_t$ be the reflection of Σ_t through the plane π_t. For t small, both Σ_t and $\tilde{\Sigma}_t$ will be empty. If t_0 is the first time of contact of π_t with Σ, then (locally) one can write Σ as a graph over π_{t_0}. Thus for $t = t_0 + \delta$, with $\delta > 0$ small, the reflected surface $\tilde{\Sigma}_t$ will lie completely inside Σ. In other words, for those values of t slightly larger that t_0, the reflected surface $\tilde{\Sigma}_t$ lies in the bounded component of $\mathbb{R}^3 \backslash \Sigma$. We pause to note that this is where we need Σ to be embedded.

Note that for t sufficiently large, Σ will lie completely below π_t (again, by the compactness of Σ), and so $\tilde{\Sigma}_t$ cannot be contained in the bounded component of $\mathbb{R}^3\backslash\Sigma$ for all t. Let t_1 be the infimum of $t > t_0$ such that $\tilde{\Sigma}_t$ is not contained in the bounded component of $\mathbb{R}^3\backslash\Sigma$ and let $\tilde{\Sigma}$ be the reflection of Σ through the plane π_{t_1}. Then in fact Σ and $\tilde{\Sigma}$ are tangent at some point p.

If the tangency at p is not a vertical tangency, write Σ and $\tilde{\Sigma}$ as graphs of u and u_1 (respectively) over the plane π_{t_1}. Let the tangency point p have coordinates (x, y) in this plane. Then $u(x, y) = u_1(x, y)$ and $\nabla u(x, y) = \nabla u_1(x, y)$. Also, u and u_1 both satisfy the same strongly elliptic equation (equation (A.1)). By the maximum principle, $u = u_1$, and therefore Σ locally agrees with $\tilde{\Sigma}$. Both surfaces are analytic and connected, so $\Sigma = \tilde{\Sigma}$.

If the tangency is at p is a vertical tangency, one needs to apply the Hopf boundary lemma (see Theorem 10 of Chapter 2 of [PW84]). In either case, we see that Σ has a plane of symmetry parallel to the $x - y$ plane. However, the $x - y$ plane had no special relation to the original surface Σ, and so we conclude that Σ has a plane of symmetry in every direction. Alexandrov used this to conclude

Theorem 15. (Alexandrov's Theorem): *Let $\Sigma \hookrightarrow \mathbb{R}^3$ be a compact embedded CMC surface. Then Σ is the round sphere.*

In the case where Σ is noncompact, a similar construction (found in [KKS89]) still works. Let $\pi \subset \mathbb{R}^3$ be a plane with unit normal v. Let L be the line parameterized by $L(t) = tv$. For $t \in \mathbb{R}$ and $p \in \pi$ define

$$\pi_t = \pi + tv \qquad \Pi_t = \cup_{s \geq t} \pi_s \qquad L_p = L + p.$$

For any set $G \subset \mathbb{R}^3$ let

$$G_t = G \cap \Pi_t \qquad \tilde{G}_t = \{p + (t - r)v \mid p \in \pi, p + (t + r)v \in G_t\}.$$

Let Σ be an almost embedded surface, with $\Sigma = \partial\Omega$. First we restrict to a piece of Σ by taking an open set $W \subset \Omega$ and letting $S = \partial W \cap \Sigma$. Note that neither W nor S need be connected nor bounded. Suppose $p + tv \notin W$ for sufficiently large t. Let t_1 be the supremum of t such that $P + tv \in W$. Then $P_1 = p + t_1 v$ is the point of first contact of L_p with W.

If this first contact is transverse, let t_2 be the supremum of $t < t_1$ such that $p + tv \notin W$. Then $P_2 = p + t_2 v$ is the point where L_p first leaves W. Otherwise, let $P_1 = P_2$. If P_1 and P_2 are both in S, then (as in [KKS89]) we define

$$\alpha_1(p) = \frac{t_1 + t_2}{2}.$$

Notice that α_1 is not defined for all $p \in \pi$.

Lemma 16. (Korevaar, Kusner, Solomon): *Fix a plane π and its normal v. If, with $W \subset \Omega$ and $S \subset \Sigma$ as above, α_1 has a local interior maximum value z at $p \in \pi$ then the plane π_z is a plane of symmetry for Σ.*

Proof: First notice that $P_1(p)$ reflects to $P_2(p)$ through π_z, by construction. Pick a nearby q. Then by maximality $t_1(q) + t_2(q) \leq 2z$, and so

$$z - (t_1(q) - z) \geq t_2(q).$$

This means the reflection of $P_1(q)$ through π_z lies above $P_2(q)$. This implies a neighborhood of $P_2(p)$ in $\tilde{S}_z$ lies inside W. If $P_1(p) \neq P_2(p)$,Then S and $\tilde{S}_z$ are tangent at $P_2(p)$ with nonvertical tangent. If $P_1(p) = P_2(p)$, then S and $\tilde{S}_z$ are tangent with vertical tangent. In either case, argue as above using the maximum principle to see that π_z is a plane of symmetry for Σ. ■

The ends of Σ are the unbounded connected components of $\Sigma \backslash \mathbb{B}_r$ for sufficiently large r. Consider an end of Σ contained in a solid cylinder $C^+_{a,R}(P) = \{p + ta \mid |p - P| < R; \langle p - P, a\rangle = 0; t > 0\}$. We take $W = \Omega \cap C^+_{a,R}(P)$ and $S = \partial W \cap \Sigma$. Meeks proved in [Mee88] that any end of a complete embedded CMC surface is contained in such a solid half-cylinder. Choose a plane π and normal v as above with $a - v$. Let $x(p) = \langle p, a\rangle$ and define

$$\alpha(x) = \max_{\langle p,a\rangle = x \geq 0} \alpha_1(p).$$

Then one can use this Alexandrov function and similar arguments as in the above Lemma to show:

Theorem 17. (Korevaar, Kusner, Solomon): *If Σ is a properly embedded CMC surface contained in a solid cylinder, then Σ has a rotational axis of symmetry parallel to the axis* *of the cylinder. Also, if Σ has finitely many ends and is contained in a half-space Π_0 for some plane π, then Σ has a plane of symmetry parallel to π and is thus contained in a solid slab.*

C.2 The Balancing Formula

CMC surfaces must also obey a balancing condition. This means that the ends of the surface Σ must be arranged to balance each other. To see this, we start with the following general proposition found in [Kus91].

Theorem 18. (Kusner): *Let M be a 3 dimensional Riemannian manifold with $H_1(M)$ and $H_2(M)$ trivial. Let G be the isometry group of M and let $\mathfrak{g}$ be its Lie algebra. For some constant H, let Σ be a surface in M with mean curvature H. Then there is a natural cohomology class $\mu \in H^1(\Sigma) \otimes \mathfrak{g}^*$ defined as follows: let ? be a 1-cycle in Σ with $\Delta \subset M$ such that $\partial\Delta = ?$. Let ν be the oriented normal to Δ and η the oriented conormal to ?. Let $Y \in \mathfrak{g}$. Then*

$$\langle \mu(?), Y\rangle = \int_\Gamma \langle \eta, Y\rangle - H\int_\Delta \langle \nu, Y\rangle.$$

The content of this theorem is that the formula above depends only on Y and the homology class of ?. Let $\tilde{?}$ be another 1-cycle homologous to ? in Σ. Because $H_1(M) = 0$ there are surfaces Δ and $\tilde{\Delta}$ in M with $\partial\Delta = ?$ and $\partial\tilde{\Delta} = \tilde{?}$. Also, $? - \tilde{?}$ forms the boundary of some surface $S \subset \Sigma$. Then $\Delta - \tilde{\Delta} + S$ forms a 2-cycle in M. Because $H_2(M) = 0$, there is an open set $U \subset M$ such that $\partial U = \Delta - \tilde{\Delta} + S$. Now take $Y \in \mathfrak{g}$. Note $\phi_t = e^{tY}$ is a one-parameter family of isometries. In fact, the Killing field associated to ϕ_t is just the left-invariant vector field associated to Y. Therefore,

$$0 = \left.\frac{d}{dt}\right|_{t=0} [\mathrm{Area}(\partial(\phi_t(U))) - H\mathrm{Vol}(\phi_t(U))].$$

Applying Stokes' Theorem, the right hand side becomes

$$\int_\Gamma \langle \eta, Y\rangle - \int_{\tilde{\Gamma}} \langle \eta, Y\rangle - H\int_\Delta \langle \nu, Y\rangle + H\int_{\tilde{\Delta}} \langle \nu, Y\rangle,$$

which shows

$$\int_\Gamma \langle \eta, Y\rangle - H\int_\Delta \langle \nu, Y\rangle = \int_{\tilde{\Gamma}} \langle \eta, Y\rangle - H\int_{\tilde{\Delta}} \langle \nu, y\rangle.$$

Now consider the case $M = \mathbb{R}^3$ and take $Y = e_1, e_2, e_3$, the constant translational vector fields in the directions of the coordinate axes. Let $W \subset \Omega$ as above and $\partial W = S \cup Q$, where $S = \partial W \cap \Sigma$. Then the above theorem implies

$$\int_{\partial S} \eta - H \int_Q \nu = 0.$$

One useful choice of W is to take $W = \Omega \cap \mathbb{B}^3_R$ for some large R. Let $S = \partial W \cap \Sigma$. As mentioned above, Meeks proved in [Mee88] that any end of Σ must be contained in a solid cylinder. So we can take R large enough so that ∂S is k disjoint simple closed curves, where Σ has k ends. Then we define the weight vector of an end as follows.

Definition 8. *For an end E which is contained in a solid half-cylinder $C^+_{a,r}(P)$, define the weight of the end E as*

$$w(E) = \int_{E\cap\pi} \eta - \int_{\pi\cap W} \nu$$

where $\pi = a^-$, arranged so that π intersects E transversally, ν is the normal to π, and η is the conormal to $\pi \cap E$.

By the balancing formula, the weights of all the ends of Σ must sum to the zero vector.

Consider the case of a Delaunay end. We can take $a = (0, 0, 1)$ and

$$E(t, \theta) = (\rho(t)\cos\theta, \rho(t)\sin\theta, t),$$

and π any plane $\pi = \{z = z_0\}$. By symmetry, $w(E)$ must point along the z axis. Moreover, $\langle a, \nu\rangle = 1$ and $\langle a, \eta\rangle = (1 + \rho_t^2)^{-\frac{1}{2}}$. Using length$(\pi \cap E) = 2\pi\rho$ and Area$(\pi \cap W) = \pi\rho^2$, we get

$$w = (\frac{2\pi\rho}{\sqrt{1+\rho_t^2}} - \pi\rho^2)(0, 0, 1).$$

One can check that

$$\frac{d}{dt}[\frac{2\pi\rho}{\sqrt{1+\rho_t^2}} - \pi\rho^2] = -\frac{\pi\rho\rho_t}{(1+\rho_t^2)^{\frac{3}{2}}}[\rho_{tt} - \frac{1}{\rho}(1+\rho_t^2) + 2(1+\rho_t^2)^{\frac{3}{2}}] = 0,$$

and so the coefficient of the above weight vector is a first integral of equation (B.1). In fact, if we normalize so that $\rho(0) = \epsilon$ is a minimum, then evaluating this constant at $t = 0$ shows $\frac{2\rho}{\sqrt{1+\rho_t^2}} - \rho^2 = 2\epsilon - \epsilon^2 = \tau$. Thus $\tau = 2\epsilon - \epsilon^2$ determines the weight of a Delaunay end of necksize ϵ.

Appendix D

SOME USEFUL COMPUTATIONS INVOLVING G, A, AND ν

The following computations involving g, A, and ν will prove useful later. The general setting is that we have an immersed surface $X : \Sigma \to \mathbb{R}^3$ with coordinates s and θ. The metric is given by $g = Eds^2 + 2Fdsd\theta + Gd\theta^2$ where $E = \|\partial_s X\|^2$, etc.

First note

$$\langle \partial_s^2 X, \partial_s X \rangle = \frac{1}{2} E_s \tag{D.1}$$

and

$$\langle \partial_\theta^2 X, \partial_\theta X \rangle = \frac{1}{2} G_\theta. \tag{D.2}$$

Also,

$$\langle X_{s\theta}, X_s \rangle = E_\theta - \langle X_s, X_{s\theta} \rangle$$

and so

$$\langle X_{s\theta}, X_s \rangle = \frac{1}{2} E_\theta.$$

This implies

$$\langle X_{ss}, X_\theta \rangle = F_s - \langle X_s, X_{s\theta} \rangle = F_s - \frac{1}{2} E_\theta.$$

Similarly,

$$\langle X_{s\theta}, X_\theta \rangle = \frac{1}{2} G_s$$

and

$$\langle X_{\theta\theta}, X_s \rangle = F_\theta - \frac{1}{2} G_s.$$

In conformal coordinates ($g = E(ds^2 + d\theta^2)$) these reduce to

$$\langle X_{ss}, X_\theta \rangle = -\frac{1}{2} E_\theta = -\langle X_{s\theta}, X_s \rangle \tag{D.3}$$

and

$$\langle X_{\theta\theta}, X_s \rangle = -\frac{1}{2} E_s = -\langle X_{s\theta}, X_\theta \rangle. \tag{D.4}$$

Also, we can write

$$\nu_s = \frac{1}{E}\langle \partial_s X, \nu_s\rangle \partial_s X + \frac{1}{G}\langle \partial_\theta X, \nu_s\rangle \partial_\theta X = -\frac{L}{E}\partial_s X - \frac{M}{G}\partial_\theta X. \tag{D.5}$$

Similarly,

$$\nu_\theta = -\frac{M}{E}\partial_s X - \frac{N}{G}\partial_\theta X. \tag{D.6}$$

So

$$\|\nu_s\|^2 = \frac{L^2}{E} + \frac{M^2}{G} + 2\frac{FLM}{EG}$$

and

$$\|\nu_\theta\|^2 = \frac{M^2}{E} + \frac{N^2}{G} + 2\frac{FMN}{EG}.$$

Notice that in conformal coordinates, these reduce to

$$\|\nu_s\|^2 = \frac{1}{E}(L^2 + M^2) \tag{D.7}$$

and

$$\|\nu_\theta\|^2 = \frac{1}{E}(M^2 + N^2). \tag{D.8}$$

We can put all of this together to read

$$\begin{pmatrix} X_s \\ X_\theta \\ \nu \end{pmatrix}_s = \begin{bmatrix} \frac{E_s}{2E} & \frac{F_\theta - \frac{1}{2}E_\theta}{G} & L \\ \frac{E_\theta}{2E} & \frac{G_s}{2G} & M \\ -\frac{L}{E} & -\frac{M}{G} & 0 \end{bmatrix} \begin{pmatrix} X_s \\ X_\theta \\ \nu \end{pmatrix} = U \begin{pmatrix} X_s \\ X_\theta \\ \nu \end{pmatrix}$$

and

$$\begin{pmatrix} X_s \\ X_\theta \\ \nu \end{pmatrix}_\theta = \begin{bmatrix} \frac{E_\theta}{2E} & \frac{G_s}{2G} & M \\ \frac{F_\theta - \frac{1}{2}G_s}{E} & \frac{G_\theta}{2G} & N \\ -\frac{M}{E} & -\frac{N}{G} & 0 \end{bmatrix} \begin{pmatrix} X_s \\ X_\theta \\ \nu \end{pmatrix} = V \begin{pmatrix} X_s \\ X_\theta \\ \nu \end{pmatrix}.$$

In conformal coordinates, U and V reduce to

$$U = \frac{1}{E}\begin{bmatrix} \frac{1}{2}E_s & -\frac{1}{2}E_\theta & LE \\ \frac{1}{2}E_\theta & \frac{1}{2}E_s & ME \\ -L & -M & 0 \end{bmatrix}$$

and

$$V = \frac{1}{E}\begin{bmatrix} \frac{1}{2}E_\theta & \frac{1}{2}E_s & ME \\ -\frac{1}{2}E_s & \frac{1}{2}E_\theta & NE \\ -M & -N & 0 \end{bmatrix}.$$

Appendix E

THE LINEARIZED OPERATOR

To explicitly compute $\mathcal{L}_{\bar{X}}$ we first need to write out $\Delta_{\bar{X}}$ in coordinates. To compute the Laplacian, first let $A = EG - F^2$, $B = E_sG + EG_s - 2FF_s$, and $C = E_\theta G + EG_\theta - 2FF_\theta$. Then

$$\begin{aligned}
\Delta_{\bar{X}} &= \frac{1}{\sqrt{\det g}}\partial_i g^{ij}\sqrt{\det g}\partial_j \\
&= \frac{1}{\sqrt{A}}\left[\partial_s \frac{G}{\sqrt{A}}\partial_s - \partial_s \frac{F}{\sqrt{A}}\partial_\theta - \partial_\theta \frac{F}{\sqrt{A}}\partial_s + \partial_\theta \frac{E}{\sqrt{A}}\partial_\theta\right] \\
&= \frac{1}{A}\left[G\partial_s^2 - 2F\partial_s\partial_\theta + E\partial_\theta^2\right] + \frac{1}{A}\left[G_s\partial_s - \frac{1}{2}\frac{GB}{A}\partial_s - F_\theta\partial_s + \frac{1}{2}\frac{FC}{A}\partial_s\right] \\
&\quad + \frac{1}{A}\left[-F_s\partial_\theta + \frac{1}{2}\frac{FB}{A}\partial_\theta + E_\theta\partial_\theta - \frac{1}{2}\frac{EC}{A}\partial_\theta\right].
\end{aligned}$$

Notice that in conformal coordinates ($g = E(ds^2 + d\theta^2)$) this expression reduces to

$$\Delta_{\bar{X}} = \frac{1}{E}(\partial_s^2 + \partial_\theta^2). \tag{E.1}$$

If we replace w by tw, then the formal Taylor expansion above becomes a Taylor expansion in t. Thus we see that

$$\mathcal{L}_{\bar{X}}(w) = \left.\frac{d}{dt}H(tw)\right|_{t=0}.$$

Thus our next task is to compute this derivative. Kapouleas computes the linearization in appendix C of [Kap90] using a slightly different method. Both computations are straightforward but somewhat involved. Below we will sometimes use a dot to indicate differentiation with respect to t. Recall that all the unbarred quantities depend on t. We will suppress this dependence. We have

$$H = \frac{1}{2}\frac{EN + GL - 2FM}{EG - F^2}.$$

So

$$\left.\frac{dH}{dt}\right|_0 = -\frac{1}{2}(\bar{E}\bar{G}-\bar{F}^2)^{-2}(\dot{E}\bar{G}+\bar{E}\dot{G}-2\bar{F}\dot{F})(\bar{E}\bar{N}+\bar{G}\bar{L}-2\bar{F}\bar{M})$$
$$+\frac{1}{2}(\bar{E}\bar{G}-\bar{F})^{-1}(\dot{E}\bar{N}+\dot{G}\bar{L}-2\dot{F}\bar{M}+\bar{E}\dot{N}+\bar{G}\dot{L}-2\bar{F}\dot{M}).$$

We can significantly simplify the task of finding $\mathcal{L}_{\bar{X}}$ is we work in conformal coordinates for $\bar{X}$. So now assume that $\bar{g}=\bar{E}(ds^2+d\theta^2)$. Then the above expression for $\frac{dH}{dt}$ simplifies to

$$\begin{aligned}\left.\frac{dH}{dt}\right|_0 &= -\frac{1}{2}\bar{E}^{-4}(\bar{E}(\dot{E}+\dot{G}))(\bar{E}(\bar{L}+\bar{N}))+\frac{1}{2}\bar{E}^{-2}(\dot{E}\bar{N}+\dot{E}\bar{L}-2\dot{F}\bar{M}+\bar{E}(\dot{L}+\dot{N}))\\ &= \frac{1}{2\bar{E}^2}[-(\dot{E}+\dot{G})(\bar{N}+\bar{L})+(\dot{E}\bar{N}+\dot{G}\bar{L}-2\dot{F}\bar{M}+\bar{E}(\dot{L}+\dot{N}))]\\ &= \frac{1}{2\bar{E}^2}[\bar{E}(\dot{N}+\dot{L})-\dot{E}\bar{L}-\dot{G}\bar{N}-2\dot{F}\bar{M}].\end{aligned}\tag{E.2}$$

To identify this beast, we will need some preliminary computations. First,

$$\begin{aligned}\left.\frac{d}{dt}\right|_0 E &= \left.\frac{d}{dt}\right|_0 \langle \bar{X}_s+tw_s\bar{\nu}+tw\bar{\nu}_s, \bar{X}_s+tw_s\bar{\nu}+tw\bar{\nu}_s\rangle\\ &= \left.\frac{d}{dt}\right|_0 (\bar{E}+2tw\langle\bar{X}_s,\bar{\nu}_s\rangle+t^2w_s^2+t^2w^2\|\bar{\nu}_s\|^2)\\ &= -2w\bar{L}.\end{aligned}\tag{E.3}$$

Similarly,

$$\left.\frac{d}{dt}\right|_0 F = -2w\bar{M}\tag{E.4}$$

and

$$\left.\frac{d}{dt}\right|_0 G = -2w\bar{N}.\tag{E.5}$$

Now we can compute the part of $\frac{dH}{dt}$ which has derivatives of components of the metric. Plugging equations (E.3), (E.4), and (E.5) into (E.2) yields

$$-\frac{1}{2\bar{E}^2}(-2w)(\bar{L}^2+\bar{N}^2+2\bar{M}^2)=\|A_{\bar{X}}\|^2w.\tag{E.6}$$

It remains to compute

$$\frac{1}{2\bar{E}}(\dot{L}+\dot{N}).$$

We have that $\dot{L} = \frac{d}{dt}\big|_0 \langle \partial_s^2 X, \nu\rangle = \langle \frac{d}{dt}\big|_0 \partial_s^2 X, \bar{\nu}\rangle + \langle \partial_s^2 \bar{X}, \frac{d\nu}{dt}\big|_0\rangle$. To do this computation we need to know $\frac{d\nu}{dt}$. First write $\nu = (EG - F^2)^{-\frac{1}{2}} \partial_s X \times \partial X_\theta$. So

$$\begin{aligned}
\left.\frac{d\nu}{dt}\right|_0 &= -\frac{1}{2}(\bar{E}\bar{G} - \bar{F}^2)^{-\frac{3}{2}}(\dot{E}\bar{G} + \bar{E}\dot{G} - 2\dot{F}\bar{F})\partial_s\bar{X} \times \partial_\theta\bar{X} + \frac{1}{\bar{E}}\left.\frac{d}{dt}\right|_0 \partial_s X \times \partial_\theta X \\
&= \frac{w}{\bar{E}^3}(\bar{L}\bar{G} + \bar{N}\bar{E} - 2\bar{F}\bar{M})\partial_s\bar{X} \times \partial_\theta\bar{X} + \frac{1}{\bar{E}}\left.\frac{d}{dt}\right|_0 \partial_s X \times \partial_\theta X \\
&= 2w\bar{H}\bar{\nu} + \frac{1}{\bar{E}}\left.\frac{d}{dt}\right|_0 \partial_s X \times \partial_\theta X.
\end{aligned}$$

Using equations (D.5) and (D.6), this last term is

$$\begin{aligned}
\left.\frac{d}{dt}\right|_0 \partial_s X \times \partial_\theta X &= \left.\frac{d}{dt}\right|_0 (\partial_s\bar{X} + tw\bar{\nu}_s + tw_s\bar{\nu}) \times (\partial_\theta\bar{X} + tw\bar{\nu}_\theta + tw_\theta\bar{\nu}) \\
&= w(\partial_s\bar{X} \times \bar{\nu}_\theta + \bar{\nu}_s \times \partial_\theta\bar{X}) + w_s\bar{\nu} \times \partial_\theta\bar{X} + w_\theta\partial_s\bar{X} \times \bar{\nu} \\
&= -\frac{w}{\bar{E}}[\partial_s\bar{X} \times (\bar{M}\partial_s\bar{X} + \bar{N}\partial_\theta\bar{X}) + (\bar{L}\partial_s\bar{X} + \bar{M}\partial_\theta\bar{X}) \times \partial_\theta\bar{X}] \\
&\quad -w_s\partial_s\bar{X} - w_\theta\partial_\theta\bar{X} \\
&= -w(\bar{N} + \bar{L})\bar{\nu} - w_s\partial_s\bar{X} - w_\theta\partial_\theta\bar{X}.
\end{aligned}$$

Adding these together, we get

$$\left.\frac{d\nu}{dt}\right|_0 = 2w\bar{H}\bar{\nu} - \frac{1}{\bar{E}}(w(\bar{L} + \bar{N})\bar{\nu} + w_s\partial_s\bar{X} + w_\theta\partial_\theta\bar{X}). \tag{E.7}$$

By equations (E.7), (D.1), (D.3), and (D.7)

$$\begin{aligned}
\left.\frac{dL}{dt}\right|_0 &= \langle w_{ss}\bar{\nu} + 2w_s\bar{\nu}_s + w\bar{\nu}_{ss}, \bar{\nu}\rangle + \langle \partial_s^2\bar{X}, 2w\bar{H}\bar{\nu} - \frac{1}{\bar{E}}(w(\bar{L} + \bar{N})\bar{\nu} + w_s\partial_s\bar{X} + w_\theta\bar{X})\rangle \\
&= w_{ss} - \frac{w}{\bar{E}}(\bar{L}^2 + \bar{M}^2) + 2w\bar{H}\bar{L} - \frac{w\bar{L}}{\bar{E}}(\bar{L} + \bar{N}) - \frac{1}{2}w_s\frac{\bar{E}_s}{\bar{E}} + \frac{1}{2}w_\theta\frac{\bar{E}_\theta}{\bar{E}}.
\end{aligned}$$

Similarly, by equations (E.7), (D.2), (D.4), and (D.8)

$$\begin{aligned}
\left.\frac{dN}{dt}\right|_0 &= \langle w_{\theta\theta}\bar{\nu} + 2w_\theta\bar{\nu}_\theta + w\bar{\nu}_{\theta\theta}, \bar{\nu}\rangle \\
&\quad +\langle \partial_\theta^2\bar{X}, 2w\bar{H}\bar{\nu} - \frac{1}{\bar{E}}(w(\bar{L} + \bar{N})\bar{\nu} + \frac{1}{2}w_s\partial_s\bar{X} + \frac{1}{2}w_\theta\partial_\theta\bar{X}\rangle \\
&= w_{\theta\theta} - \frac{w}{\bar{E}}(\bar{M}^2 + \bar{N}^2) + 2w\bar{H}\bar{N} - \frac{w\bar{N}}{\bar{E}}(\bar{L} + \bar{N}) + \frac{1}{2}w_s\bar{E}_s - \frac{1}{2}w_\theta\bar{E}_\theta.
\end{aligned}$$

Adding these together, we get

$$\begin{aligned}
\dot{L}+\dot{N} &= w_{ss}+w_{\theta\theta}-\frac{1}{2}w_s\frac{\bar{E}_s}{\bar{E}}+\frac{1}{2}w_\theta\frac{\bar{E}_\theta}{\bar{E}}+\frac{1}{2}w_s\frac{\bar{E}_s}{\bar{E}}-\frac{1}{2}w_\theta\frac{\bar{E}_\theta}{\bar{E}}-\frac{w}{\bar{E}}(\bar{L}^2+2\bar{M}^2+\bar{N}^2) \\
&\quad +2w\bar{H}(\bar{L}+\bar{N})-\frac{w}{\bar{E}}(\bar{L}^2+2\bar{L}\bar{N}+\bar{N}^2) \\
&= w_{ss}+w_{\theta\theta}-\frac{w}{\bar{E}}(2\bar{L}^2+2\bar{M}^2+2\bar{N}^2+2\bar{L}\bar{N})+\frac{w}{\bar{E}}(\bar{L}+\bar{N})^2 \\
&= w_{ss}+w_{\theta\theta}-\frac{w}{\bar{E}}(\bar{L}^2+2\bar{M}^2+\bar{N}^2). \qquad \text{(E.8)}
\end{aligned}$$

Adding together equations (E.6) and (E.8) and using equation (E.1) yields

$$\begin{aligned}
\mathcal{L}_{\bar{X}}w &= \left.\frac{dH}{dt}\right|_0 \\
&= \|A_{\bar{X}}\|^2 w+\frac{1}{2\bar{E}}(w_{ss}+w_{\theta\theta})-\frac{w}{2\bar{E}^2}(\bar{L}^2+2\bar{M}^2+\bar{N}^2) \\
&= \frac{1}{2}(\Delta_{\bar{X}}+\|A_{\bar{X}}\|^2)w. \qquad \text{(E.9)}
\end{aligned}$$

Appendix F

FREDHOLM PROPERTIES AND GROWTH RATES

In this chapter we prove some general mapping properties of $\mathcal{L}_X$ on weighted Hölder spaces where $X : \Sigma \to \mathbb{R}^3$ is an embedding with asymptotically Delaunay ends. The two main results will be the characterization of when $\mathcal{L}_X$ is Fredholm (i.e. for which weights) and the Linear Decomposition theorem. The analysis in this chapter is essentially contained in [MPU96]. Below we will use $C^{k,\alpha}_{\delta,\mathcal{D}}(M)$ to denote the Hölder space of functions with Dirichlet boundary data. (We will omit the weighting if M is compact with boundary.)

Lemma 19. *Let $X = K \cup (\cup_1^k E_j)$ be a properly embedded, noncompact surface with ends written as graphs over cylinders. Suppose*

$$\mathcal{L}_X : C^{k+2,\alpha}_{\mathcal{D}}(K) \to C^{k,\alpha}_{\mathcal{D}}(K)$$

is Fredholm and

$$\mathcal{L}_X : C^{k+2,\alpha}_{\delta,\mathcal{D}}(E_j) \to C^{k,\alpha}_{\delta,\mathcal{D}}(E_j)$$

is Fredholm for all j. Then

$$\mathcal{L}_X : C^{k+2,\alpha}_{\delta}(X) \to C^{k,\alpha}_{\delta}(X)$$

is Fredholm.

The proof of this lemma follows that of Proposition 11 in [MPP], and we refer the reader to the latter proof for more details.

Proof: Let χ be a smooth compactly supported function with $\chi(x) = 1$ for $|x| \leq r$ and $\chi(x) = 0$ for $|x| \geq r + 1$. By assumption, there are Greens operators

$$\mathcal{G}_K : C^{k,\alpha}_{\mathcal{D}}(K) \to C^{k+2,\alpha}_{\mathcal{D}}(K)$$

and

$$\mathcal{G}_j : C^{k,\alpha}_{\delta,\mathcal{D}}(E_j) \to C^{k+2,\alpha}_{\delta,\mathcal{D}}(E_j)$$

for the restrictions of $\mathcal{L}_X$. Now define

$$\mathcal{G}(f) = \mathcal{G}_K(\chi f) + \sum_j^k \mathcal{G}_j((1-\chi)f).$$

To check that $\mathcal{G}\mathcal{L} - \mathrm{Id}$ is a compact operator, take a sequence $u_i \in C_\delta^{k+2,\alpha}(X)$ with $\|u_i\| = 1$ and let $w_i = (\mathcal{G}\mathcal{L} - \mathrm{Id})u_i$. Note that w_i is supported in the compact set $\mathbb{B}_{r+1} \backslash \mathbb{B}_r$. By local elliptic regularity for the operator $\mathcal{G}\mathcal{L} - \mathrm{Id}$,

$$\|w_i\|_{C_\delta^{k+2,\alpha}(B)} \leq c\|u_i\|_{C_\delta^{k+2,\alpha}(B)} = c$$

for any bounded set set B, and so by Arzela-Ascoli a subsequence w_i converges uniformly on compact sets. However, w_i is supported in the fixed compact set $\mathbb{B}_{r+1} \backslash \mathbb{B}_r$, so

$$\|w_i - w_j\|_{C_\delta^{k+2,\alpha}(X)} \to 0.$$

■

Because $\mathcal{L}_X$ is elliptic and K is compact,

$$\mathcal{L}_X : C_{\mathcal{D}}^{k+2,\alpha}(K) \to C_{\mathcal{D}}^{k,\alpha}(K)$$

is always Fredholm. One can construct a Greens operator microlocally by inverting the symbol. The operator

$$\mathcal{L}_X : C_{\delta,\mathcal{D}}^{k+2,\alpha}(E_j) \to C_{\delta,\mathcal{D}}^{k,\alpha}(E_j)$$

is Fredholm if and only if the operator $\mathcal{L}_D$ for the model Delaunay surface is Fredholm (the difference between the two is exponentially decaying). Notice $\mathcal{L}_D$ is a periodic operator. To simplify notation, we will assume below that the period is 1. To find out when $\mathcal{L}_D$ is Fredholm, we introduce the Fourier-Laplace transform. Let $u \in C_\delta^{k,\alpha}([0,\infty) \times \mathbb{S}^1)$ and let $\zeta \in \mathbb{C}$. Then

$$\mathcal{F}(u)(\zeta, t, \theta) = \hat{u}(\zeta, t, \theta) = \sum_{-\infty}^{\infty} e^{-i\zeta k} u(t+k, \theta).$$

The sum above converges uniformly and absolutely for $\Im\zeta < -\delta$, and so

$$\hat{u} \in \mathrm{Holo}(\{\Im\zeta < -\delta\}; C^{k,\alpha}([0,\infty) \times \mathbb{S}^1)).$$

One can invert $\mathcal{F}$: if $\zeta = \mu + i\nu$ and $t = l + \tilde{t}$ where $l \in \mathbb{Z}$ and $0 \leq \tilde{t} < 1$, then

$$\mathcal{F}^{-1}(u)(t,\theta) = \check{u}(t,\theta) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(\mu+i\nu)l} u(\mu+i\nu,\tilde{t},\theta)d\mu.$$

Notice that $\nu = \Im\zeta$ is a parameter in the inversion formula above. Changing ν amounts to changing the weighting of $\check{u}$. One can check that $\check{u} \in C^{k,\alpha}_{-\nu}([0,\infty)\times\mathbb{S}^1)$, but $\check{u} \notin C^{k,\alpha}_{-\nu-\epsilon}([0,\infty)\times\mathbb{S}^1)$ for any $\epsilon > 0$. The reason for introducing the Fourier-Laplace transform build a one-parameter family of operators out of $\mathcal{L}_D$ where the parameter changes the weight. To this end, define

$$\tilde{\mathcal{L}}_D(\zeta)(u) = e^{-i\zeta t}\mathcal{F}\circ\mathcal{L}_D\circ\mathcal{F}^{-1}(e^{i\zeta t}u).$$

$\tilde{\mathcal{L}}_D(\zeta) : C^{k+2,\alpha}([0,\infty)\times\mathbb{S}^1) \to C^{k,\alpha}([0,\infty)\times\mathbb{S}^1)$ and it depends holomorphically on ζ.

Proposition 20. *The operator*

$$\mathcal{L}_D = \partial_s^2 + \partial_\theta^2 + \tau^2\cosh 2\sigma : C^{k+2,\alpha}_{\delta,\mathcal{D}}([0,\infty)\times\mathbb{S}^1) \to C^{k,\alpha}_{\delta,\mathcal{D}}([0,\infty)\times\mathbb{S}^1)$$

is Fredholm if and only if

$$\delta \notin ? = \{\dots,-\gamma_3,-\gamma_2,0,\gamma_2,\gamma_3,\dots\}$$

where $0 < \gamma_j < \gamma_{j+1} \to \infty$.

In fact, these γ_j's are the indicial roots of the Delaunay surface D. One can see that they correspond to rates of growth from the contour-shifting argument that yields the Linear Decomposition Theorem (see below).

Proof: As above, consider

$$\tilde{\mathcal{L}}_D(\zeta)(u) = e^{-i\zeta t}\mathcal{F}\circ\mathcal{L}_D\circ\mathcal{F}^{-1}(e^{i\zeta t}u).$$

By the analytic Fredholm theorem, $\tilde{\mathcal{L}}_D$ is either Fredholm for all values of ζ but a discrete set, or it is never Fredholm. Because $\tilde{\mathcal{L}}_D$ is formally self adjoint, it suffices to show that $\tilde{\mathcal{L}}_D(\zeta)$ is injective for some value of ζ. Suppose $\tilde{\mathcal{L}}_D(\zeta)(u) = 0$. Then $\mathcal{L}_D(\mathcal{F}^{-1}(e^{i\zeta t}u)) = 0$.

By definition,

$$\begin{aligned}\mathcal{F}^{-1}(e^{i\zeta t}u)(t,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} e^{i(\mu+i\nu)l}e^{i(\mu+i\nu)\tilde{t}}u(\tilde{t},\theta)d\mu \\ &= \frac{e^{-\nu t}u(\tilde{t},\theta)}{2\pi}\int_0^{2\pi} e^{i\mu t}d\mu \\ &= \frac{e^{-\nu t}u(\tilde{t},\theta)}{2\pi i t}(e^{2\pi i t}-1).\end{aligned}$$

Notice that $\mathcal{F}^{-1}(e^{i\zeta t}u)(t,\theta)$ is zero whenever t is an integer. For the particular choice of $\zeta = \frac{1}{2} + \frac{i}{2}$, the above computation shows that the solution u would either have to be identically zero or $u \in C^{k+2,\alpha}_{-\frac{1}{2}}$ but $u \notin C^{k+2,\alpha}_{-\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$. We have already shown that this behavior is impossible for solutions to the equation $\mathcal{L}_D u = 0$. Therefore $\tilde{\mathcal{L}}_D(\zeta)$ is injective and hence (because it is formally self adjoint) an isomorphism for all but a discrete set of ζ. Note that $\tilde{\mathcal{L}}_D(\zeta) : C^{k+2,\alpha} \to C^{k,\alpha}$ is not Fredholm if and only if $\mathcal{L}_D : C^{k+2,\alpha}_{-\nu} \to C^{k,\alpha}_{-\nu}$, where $\nu = \Im\zeta$, is not Fredholm.

Let $\tilde{?}$ be the poles of the Greens operator of $\tilde{\mathcal{L}}_D(\zeta)$ ($\zeta \in \tilde{?}$ if and only if $\tilde{\mathcal{L}}(\zeta) : C^{k+2,\alpha}_{\mathcal{D}} \to C^{k,\alpha}_{\mathcal{D}}$ is not Fredholm). It remains to see that $\{\nu \mid \Im\zeta = -\nu, \zeta \in \tilde{?}\}$ does not have any accumulation points. A priori, it is possible that a sequence $\zeta_n \in \tilde{?}$ could look like $\zeta_n = n + \frac{i}{n}$. However, $\tilde{\mathcal{L}}_D(\zeta) = \tilde{\mathcal{L}}_D(\zeta + 2\pi)$ and the operators $\tilde{\mathcal{L}}_D(\zeta)$ and $\tilde{\mathcal{L}}_D(\zeta + 2\pi i)$ are unitarily equivalent (the unitary isomorphism which transforms one operator to the other is multiplication by $e^{2\pi i t}$). Thus $\tilde{?}$ is invariant under translations by $2k\pi + 2l\pi i$, where k and l are any integers. If $? = \{\nu \mid \Im\zeta = -\nu, \zeta \in \tilde{?}\}$ had an accumulation point, then by translation invariance $\tilde{?}$ would have infinitely many points, and hence an accumulation point, in $[-\pi,\pi]\times[-\pi,\pi]$. This would contradict the fact that $\tilde{?}$ has no accumulation points. Thus $?$ cannot have an accumulation point. Moreover, $\tilde{\mathcal{L}}_D(\zeta)$ is Fredholm if and only if $\tilde{\mathcal{L}}_D(-\zeta)$ is. In fact the two operators are conjugate under multiplication by $e^{-2\zeta t}$. This shows $\gamma \in ?$ if and only if $-\gamma \in ?$, and completes the proof. ■

Our next task is to try to understand the behavior of the kernel of $\mathcal{L}_D : C^{0,\alpha}_{\delta} \to C^{2.\alpha}_{\delta}$ for $0 < \delta < \gamma_1$. Let $\tilde{\mathcal{G}}_D(\zeta)$ be the Greens operator of $\tilde{\mathcal{L}}_D(\zeta)$ chosen above. Then $\tilde{\mathcal{G}}_D$ is meromorphic in $\mathbb{C}$ with poles at $\zeta \in \tilde{?}$. Functions in the kernel of $\tilde{\mathcal{L}}_D(\zeta)$ for $\zeta \in \tilde{?}$ can then be recovered from the residue of $\tilde{\mathcal{G}}_D$ at ζ. In particular, we are interested in the "tempered" solutions (those with subexponential growth), which arise from the residue of $\mathcal{G}_D$ at $\zeta = 0$.

These functions, collectively labeled B_D above, correspond to solutions to $\mathcal{L}_D(u) = 0$ where $u \in C^{2,\alpha}_{\delta}$ but $u \notin C^{2,\alpha}_{-\delta}$ for all $\delta > 0$. Recall $\mathcal{L}_D$ is nondegenerate. Let $f \in C^{0,\alpha}_{-\delta} \subset C^{0,\alpha}_{\delta}$. Then, because $\mathcal{L}_D : C^{2,\alpha}_{\delta} \to C^{0,\alpha}_{\delta}$ is surjective, we can find $u \in C^{2,\alpha}_{\delta}$ such that $\mathcal{L}_D u = f$.

In fact

$$\begin{aligned} u(s,\theta) &= \mathcal{F}^{-1}(\tilde{\mathcal{G}}(\zeta)(\mathcal{F}(f)(s,\theta,\zeta))) \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{\mu - i\delta}\tilde{\mathcal{G}}(\mu - i\delta)(\mathcal{F}(f))d\mu \\ &= \frac{2n+1}{2\pi}\int_{-2n\pi}^{2n\pi} e^{\mu - i\delta}\tilde{\mathcal{G}}(\mu - i\delta)(\mathcal{F}(f))d\mu. \end{aligned}$$

It turns out that if $\delta' > 0$ then $u \in C^{2,\alpha}_{\delta'}$ as well. Let η_n be oriented contours which trace the perimeter of the rectangle $2\pi(n-1) \leq \mu \leq 2\pi n$, $-\delta \leq \nu \leq -\delta'$ in a counterclockwise direction. Then by Cauchy's theorem,

$$\begin{aligned} 0 &= \sum_{-n}^{n}\int_{\eta_k} e^{\zeta}\tilde{\mathcal{G}}(\zeta)(\mathcal{F}(f))d\zeta \\ &= \int_{-2\pi n}^{2\pi n} e^{\mu - i\delta}\tilde{\mathcal{G}}(\mu - i\delta)(\mathcal{F}(f))d\mu - \int_{-2\pi n}^{2\pi n} e^{\mu - i\delta'}\tilde{\mathcal{G}}(\mu - i\delta')(\mathcal{F}(f))d\mu + O(\frac{1}{n}). \end{aligned}$$

Taking n large, this shows u actually lies in $C^{s+2}_{\delta'}$. However, this computation does not work if $\delta' < 0$, because in this case $\tilde{\mathcal{G}}_D$ has a pole in each η_n. If we now take $\delta' = -\delta$, we get a new solution $u = v + w$ where

$$v + w = \frac{1}{2\pi}\int_0^{2\pi} e^{\mu + i\delta}\tilde{\mathcal{G}}_D(\mathcal{F}(f))d\mu.$$

The difference is given by

$$\begin{aligned} w &= \frac{2n+1}{2\pi}\sum_{-n}^{n}\int_{\eta_k} e^{\zeta}\tilde{\mathcal{G}}(\zeta)\mathcal{F}(f)d\zeta \\ &= \frac{2n+1}{2\pi}\sum_{-n}^{n}\mathrm{Res}_{\zeta=2\pi k}(e^{\zeta}\tilde{\mathcal{G}}(\zeta)\mathcal{F}(f)) \\ &= \frac{2n+1}{2\pi}\sum_{-n}^{n}\mathrm{Res}_{\zeta=0}e^{\zeta}\tilde{\mathcal{G}}(\zeta)\mathcal{F}(f) \\ &= \frac{1}{2\pi}\mathrm{Res}_{\zeta=0}e^{\zeta}\tilde{\mathcal{G}}(\zeta)\mathcal{F}(f) \end{aligned}$$

In the second to last equality we used the fact that residue of this pole is the same for each contour. The above computation yields the Linear Decomposition theorem stated above, and in fact yields a more general asymptotic expansion for solutions to the equation $\mathcal{L}_D u = f$ by further contour shifting. Notice all these computations only use the asymptotic behavior of $\mathcal{L}_X$, and so all the linear analysis results we have for CMC embeddings X also apply to the approximate solutions $\bar{X}_{R,\phi}$ we constructed.

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