Spherically Symmetric MTTs: Causal Structure and Asymptotic Behavior

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1 Introduction

2 Background

2.1 Spacetime machinery

The theory of general relativity postulates that the universe should be described as a **Lorentzian manifold**, that is, a smooth manifold M^n paired with a metric g of signature $(-, +, \cdots, +)$. Although much of what follows may be carried out for higher dimensions, we will always work in dimension $n = 4$. Matter and energy in the universe are described by a symmetric 2-tensor $T_{\alpha\beta}$ on M called the **stress-energy tensor**. Einstein's equations prescribe the interplay between the geometry of the underlying manifold and the matter and energy in it:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{1}$$

where $R_{\alpha\beta}$ is the Ricci curvature of the metric g and R is its scalar curvature. Lorentzian manifolds (M, g) satisfying Einstein's equations are called **spacetimes**. We will always assume that our spacetimes are connected.

For any spacetime (M, g), the signature of g enables us to partition the tangent vectors of M into three types. A tangent vector $X \in T_pM$ is called **timelike** if $g(X, X) < 0$, **spacelike** if $g(X, X) > 0$, or **null** if $g(X, X) = 0$. If $g(X, X) \leq 0$, that is, X is either timelike or null, then X is said to be **causal**. These characterizations extend to certain curves in M as well. A differentiable curve $\gamma\colon I \to M$ is called timelike (respectively null, spacelike, or causal) if at each $t \in I$, the vector $\gamma'(t) \in T_pM$ is timelike (respectively null, spacelike, or causal). The spacetime M is said to be **time orientable** if it admits a global, continuous, non-vanishing timelike vector field. Given such a vector field, say, V, we may assign M an explicit time orientation: given any causal vector $X \in T_pM$, we declare it to be **future directed** if $g(X, V_p) < 0$ and **past directed** otherwise. Henceforth we will assume that all our spacetimes are time orientable and have been assigned explicit orientations. A submanifold, too, may be characterized as spacelike, timelike, or null if its normal vector is everywhere timelike, spacelike, or null, respectively.

Once we have established a time orientation and the causal character of tangent vectors at each point, we can extend notions of causality to points in the manifold itself. If p is a point in a spacetime M, then we may define the **chronological future** of the point p as the set

$$I^+(p) \;=\; \{q \in M : \exists \text{ a piecewise timelike future directed curve from } p \text{ to } q\}. \tag{2}$$

The **chronological past** of p, denoted $I^-(p)$, is defined similarly, replacing "future" with "past". The **causal future** and **causal past** of p are denoted $J^+(p)$ and $J^-(p)$, respectively, and are defined analogously by replacing "timelike" in (2) with "causal". Note that the point p is not contained in $I^\pm(p)$ but is contained in $J^\pm(p)$, since degenerate curves are by definition causal. A set S in M is said to be **achronal** if no two points $p, q \in S$ may be joined by a piecewise timelike curve, i.e. there do not exist $p, q \in S$ such that $q \in I^+(p)$. The **future domain of dependence** of a set S, denoted $D^+(S)$, is defined to be the set of events in the spacetime that are completely predicted by the events in S. More precisely,

$$D^+(S) \;=\; \{p \in M : \text{ every past inextendible causal curve through } p \text{ intersects } S\},$$

where "inextendible" is defined appropriately. The **past domain of dependence** $D^-(S)$ is defined analogously by replacing "past" with "future". Taken together, they constitute the (full) **domain of dependence** of S:

$$D(S) = D^+(S) \cup D^-(S).$$

If a set S is closed, achronal, and its domain of dependence is all of the spacetime, $D(S) = M$, then S is said to be a **Cauchy surface**. A spacetime (M, g) which admits a Cauchy surface is called **globally hyperbolic**.

An arbitrary observer (*a priori* moving more slowly than the speed of light) traces out a future directed timelike curve in the spacetime manifold called a **worldline**. If ξ^α is the tangent vector to such a worldline, then the (scalar) energy density of matter measured by that observer is given by $T_{\alpha\beta}\xi^\alpha\xi^\beta$, while the quantity $-T^\alpha_\beta\xi^\beta$ represents the full 4-vector energy-momentum density of matter that he or she sees. Motivated by local physical considerations, relativists sometimes impose restrictions on these quantities. In this paper we will be concerned only with the **null energy condition**, which requires that $T_{\alpha\beta}\mu^\alpha\mu^\beta \geq 0$ for all null vectors μ^α, and the **dominant energy condition**, which requires that $-T^\alpha_\beta\xi^\beta$ should be a future directed timelike or null vector.

2.2 Black Holes

Intuitively speaking, a black hole is a region of spacetime curved in such a way that, once an observer passes into the region, he or she can never again escape or even communicate with observers outside it. Such extreme curvature is generally thought to arise from the gravitational collapse of a massive body. Another way of heuristically describing a black hole is to say that the worldline of an observer who has traveled into a black hole will never reach "future infinity"; instead, the worldline is trapped inside the black hole and, it is thought, terminates in "finite time" at some sort of spacetime singularity. Such a notion is difficult to make precise, however; we will have to introduce further machinery in order to make mathematical sense of it, first in general terms here, then later in a fully rigorous way in the context of spherical symmetry.

Roughly speaking, a spacetime (M, g) is said to be **asymptotically flat** if the complement of some compact region in M is diffeomorphic to a finite union of copies of $\mathbb{R}^4 \setminus \overline{B_1(0)}$ and the metric decays to the flat metric with respect to a naturally defined radial coordinate on each copy. Various conditions on the exact decay rates of the metric and its derivatives are usually imposed when one makes the definition rigorous. One can then conformally compactify each asymptotically flat end (similarly to the way that one conformally compactifies Minkowski space) and consider the boundary of this new "unphysical" spacetime to represent the boundary of the physical one "at infinity". Under appropriate conditions, this boundary will contain three components of particular interest: a point i^0, called **spatial infinity**, and sets $\mathcal{I}^+$ and $\mathcal{I}^-$, pronounced "scri plus" and "scri minus", respectively. These sets are defined by the relations $\mathcal{I}^+ = \overline{J^+(i^0)} - i^0$ and $\mathcal{I}^- = \overline{J^-(i^0)} - i^0$, and in particular, under appropriate regularity assumptions, $\mathcal{I}^+$ and $\mathcal{I}^-$ will be null surfaces. They are called **future** and **past null infinity**, respectively. Because a conformal change of metric preserves causal structure, it makes sense to talk about the set $J^-(\mathcal{I}^+)$ in the original physical spacetime M, called the **domain of outer communications**. Roughly speaking, this set consists of all spacetime events which can be seen "from infinity". The complement of this set, $B = M \setminus J^-(\mathcal{I}^+)$, is called the **black hole region**, and its boundary $H = \partial J^-(\mathcal{I}^+)$ is said to be the **event horizon**.

Introduce general Penrose diagrams here?

2.3 Quasi-local notions

Although the definition of a black hole given above makes sense intuitively, it has the drawback that one needs to have information about the entire spacetime manifold at hand in order to find its conformal boundary and thus locate the black hole region. One could not, for example, tell by looking at an open subregion of M whether or not it contains a black hole. This restriction makes black holes difficult rather impractical to work with from a physics standpoint. To get around this difficulty, various local notions have been introduced instead, which we will discuss below.

A **congruence** of null geodesics is simply a family of null geodesics which foliates some open region of spacetime. Given a spacelike 2-surface S in M, there are two distinct congruences of future directed null geodesics orthogonal to S, defined up to choice of parametrization. If l^α is the tangent vector field of one these congruences, then we can define $\theta_{(l)}$, the **expansion** of S in the direction l, by

$$\theta_{(l)} = \operatorname{div}_S l_\alpha = h^{\alpha\beta}\nabla_\beta l_\alpha,$$

where ∇ is the Levi-Civita connection on M and h is the induced Riemannian metric on the 2-surface S. Since a null geodesic does not in general admit a canonical parametrization, the vector field l^α and hence its expansion $\theta_{(l)}$ are dependent on the choice of parametrization of the null normal geodesics in the congruence. However, if we rescale l^α by some positive function λ, we can compute that $\theta_{(\lambda l)} = \lambda\theta_{(l)}$, so the *sign* of the expansion $\theta_{(l)}$ is indeed well-defined. Intuitively speaking, $\theta_{(l)}$ measures the infinitesimal change in the area of S in the direction l^α. One typically expects that the expansion will be positive in one of the null normal directions to S and negative in the other (think of the inner- and outer-pointing normals of a standard 2-sphere, for example), but if the ambient manifold is curved enough, that characterization need not hold. In particular, if a 2-surface S has future null directions l and k such that both $\theta_{(l)} < 0$ and $\theta_{(n)} < 0$, then S is called a **trapped surface**; the surface is **marginally trapped** if both expansions are merely nonpositive. If l and n can be distinguished from each other by determining that l is "outer" and n "inner", for example if M is asymptotically flat, then we say that S is **outer marginally trapped** if $\theta_{(l)} \leq 0$, and it is an **apparent horizon** if $\theta_{(l)} \equiv 0$.

A famous result of Penrose shows just why trapped surfaces are important: they signal the development of spacetime singularities often associated with black holes. In particular, in 1965, he proved the following

Theorem. *Let (M, g) be a connected, globally hyperbolic spacetime whose Cauchy surface is noncompact and which satisfies the null energy condition. If M contains a trapped surface S, then there exists at least one inextendible future directed orthogonal null geodesic emanating from S and having finite affine length in M.*

The existence of an inextendible geodesic of finite affine length signals either that some sort of singularity occurs at its "endpoint" or that global hyperbolicity fails there. In either case, the trapped surface acts as a local indication of a nearby pathology in the spacetime. Furthermore, in an asymptotically flat spacetime M, which is the only type of spacetime in which black holes may even be defined, under a certain extra asymptotic condition ("strong asymptotic predictability"), one can show that any trapped surface must lie inside a black hole region [4].

Trapped surfaces still do not quite provide a local model for what physicists call a black hole. But in recent years, a new object has been proposed to provide a quasi-local model for a black hole: a **dynamical horizon** H in a spacetime (M, g) is a spacelike hypersurface H foliated by closed spacelike 2-surfaces such that, on each leaf S, the expansion $\theta_{(l)}$ of one null normal l^α vanishes, and the expansion $\theta_{(n)}$ of the other null normal n^α is strictly negative. If M is asymptotically flat or some other notion of spatial infinity can be applied, then one always takes $\theta_{(l)} = \theta_+$ and $\theta_{(n)} = \theta_-$, where the plus and minus denote the "outward" and "inward" directions, respectively. Note that each of the foliating 2-surfaces is thus a marginally trapped surface as well as an apparent horizon. It turns out that dynamical horizons provide a good local model for an evolving black hole, and physicists have been able to extend notions of black hole thermodynamics and entropy to them with great success [1].

Dynamical horizons have timelike and null analogs as well, called **timelike membranes** and **isolated horizons**, defined by replacing spacelike with timelike or null, respectively, as the hypothesis on the hypersurface H (but not on the foliating 2-surfaces). The latter is thought to model the asymptotic state of a dynamical black hole settling down to an equilibrium state, while the former has no concrete physical meaning; since future directed timelike curves can pass through a timelike membrane in either direction, it is not good candidate for a model of a black hole. Collectively, dynamical and isolated horizons and timelike membranes are called **marginally trapped tubes**.

3 Spherical Symmetry

The study of spherically symmetric spacetimes has a venerable history. One of the first exact solution to Einstein's equations, found by the physicist Karl Schwarzschild in the same year Einstein published his theory of general relativity, is spherically symmetric. The **Schwarzschild solution** describes the (vacuum) exterior gravitational field of a static, spherically symmetric body. In spherical coordinates (t, r, θ, ϕ), in which t corresponds to time, r is a radial coordinate, and θ and ϕ are the usual spherical coordinates on S^2, the metric takes the form

$$g = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2,$$

where M is a constant (typically interpreted as the mass of the spherically symmetric body), and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the usual round metric on S^2. Notice that as $r \to \infty$, g tends to $-dt^2 + dr^2 + d\Omega^2$, the Minkowski metric; indeed, g is asymptotically flat.

Besides the Schwarzschild solution, there are many other important spherically symmetric exact solutions to Einstein's equations, such as the Kerr, Kerr-Newman, Reisner-Nordström, and Vaidya spacetimes. Such solutions are the model spaces for many aspects of relativity theory and provide a testing ground for a wide range of theories. But even aside from such exact solutions, imposing an assumption of spherical symmetry in general casts the theory into a vastly simpler setting while still providing (one hopes) heuristics representative of generic non-spherically symmetric solutions. In what follows, we will describe in detail the reduction of the spherically symmetric 3+1-manifold setting to a 1+1-setting.

3.1 Structure and energy assumptions

In general, a spacetime (M, g) is said to be **spherically symmetric** if the Lie group $SO(3)$ acts on it by isometries with orbits which are either fixed points or spacelike 2-spheres. In order to make use of this concept in practice, however, we will need to impose a large number of very specific additional conditions on (M, g). The end goal is to transfer all of the important causal and asymptotic data of M to a 1+1-dimensional quotient manifold which we can then conformally embed into Minkowski space; all of the assumptions we make here are necessary to ensure that this conformally embedded quotient manifold and its boundary are sufficiently well-behaved. The remainder of the setup described in this section is lifted directly from [3].

In what follows, we will always take (M, g) to be a globally hyperbolic spacetime satisfying the dominant energy condition which admits an $SO(3)$-action by isometries. We will assume that the quotient manifold $\mathcal{Q} = M/SO(3)$ inherits the structure of a 1+1 Lorentzian manifold with a boundary corresponding to the points fixed by the $SO(3)$-action, the center of symmetry. We further assume M is the maximal development of its Cauchy surface Σ and that the quotient $\mathcal{Q}^+$ of its causal future $J^+(\Sigma)$ may be conformally embedded into a bounded subset of Minkowski space $(\mathbb{R}^2, \eta)$. We assume that $\mathcal{Q}^+$ contains just one of its connected boundary components and that this boundary component has the form $\Gamma \cup S$, where Γ, the center of symmetry in $\mathcal{Q}^+$, is a connected timelike curve comprising the points in $\mathcal{Q}^+$ fixed by the $SO(3)$-action, $S = \Sigma/SO(3)$ is a connected spacelike curve, and Γ and S intersect in a single point p.

Suppose we choose double null coordinates (u, v) on $\mathbb{R}^2$, such that the Minkowski metric η takes the form $\eta = -du\, dv$ and the positive u- and v-axes are at 135° and 45° from the usual positive x-axis, respectively. We assume that $(\mathbb{R}^2, \eta)$ is time oriented in the usual way such that u and v are both increasing toward the future. Then with respect to the conformal embedding, the metric on $\mathcal{Q}^+$ takes the form $-\Omega^2 du\, dv$, and suppressing pullback notation, the original metric g may be expressed

$$g = -\Omega^2 du\, dv + r^2\gamma,$$

where $\gamma = d\theta^2 + \sin^2\theta\, d\phi^2$ is the standard metric on S^2, and $\Omega = \Omega(u,v)$ and $r = r(u,v)$ are smooth functions on $\mathcal{Q}^+$ such that $\Omega > 0$, $r \geq 0$, and $r(q) = 0$ if and only if $q \in \Gamma$. Introducing the quantity

$$m = m(u,v) = \frac{r}{2}(1 + 4\Omega^{-2}\partial_u r\, \partial_v r),$$

the so-called **Hawking mass**, we require that m be uniformly bounded along S. Finally, we assume that $\mathcal{Q}^+$ is foliated by connected constant u curves with past endpoint on $\Gamma \cup S$ and also by connected constant v curves with past endpoint on S, called "outgoing" and "ingoing", respectively.

By direct computation (see appendix B), we find that the Einstein field equations (2.1) for the metric $g = -\Omega^2 du\, dv + r^2\gamma$ on M reduce to the following system:

$$\begin{aligned}
8\pi T_{uu} &= -2r_{,uu}r^{-1} + 4\,\Omega_{,u}r_{,u}(\Omega r)^{-1} && (3)\\
8\pi T_{uv} &= 2r_{,vu}r^{-1} + 2r_{,u}r_{,v}r^{-2} + \tfrac{1}{2}r^{-2}\Omega^2 && (4)\\
8\pi T_{vv} &= -2r_{,vv}r^{-1} + 4\,\Omega_{,v}r_{,v}(\Omega r)^{-1} && (5)\\
8\pi T\big|_{S_r} &= (-4rr_{,uv}\Omega^{-2} - 4r^2\,\Omega_{,vu}\Omega^{-3} + 4r^2\,\Omega_{,u}\Omega_{,v}\Omega^{-4})\gamma. && (6)
\end{aligned}$$

Each of equations (3), (4), and (5) holds pointwise at all $p = (u, v, \theta, \phi) \in M$, but the right-sides only depend on the u and v coordinates. Thus, assuming that this system of equations is satisfied, i.e. that (M, g) is indeed a space-time, the component functions T_{uu}, T_{uv}, and T_{vv} of the stress-energy tensor descend to functions on the quotient manifold $\mathcal{Q}^+$ and satisfy (3), (4), and (5) there as well. In fact, henceforth we will consider equations (3), (4), and (5) *only* as pointwise equations on $\mathcal{Q}^+$. We can restate them more nicely in terms of the Hawking mass m:

$$\begin{aligned}
\partial_u(\Omega^{-2}\partial_u r) &= -4\pi r\Omega^{-2}T_{uu} && (7)\\
\partial_v(\Omega^{-2}\partial_v r) &= -4\pi r\Omega^{-2}T_{vv} && (8)\\
\partial_u m &= 8\pi r^2\Omega^{-2}(T_{uv}\partial_u r - T_{uu}\partial_v r) && (9)\\
\partial_v m &= 8\pi r^2\Omega^{-2}(T_{uv}\partial_u r - T_{vv}\partial_v r). && (10)
\end{aligned}$$

Recall that one of the initial assumptions was that (M, g) satisfy the dominant energy condition. In terms of the component functions of the stress-energy tensor on $\mathcal{Q}^+$, this implies (and is even equivalent to???) the statement that

$$T_{uu} \geq 0, \quad T_{uv} \geq 0, \quad \text{and } T_{vv} \geq 0$$

at all points $(u, v) \in \mathcal{Q}^+$.

3.2 Penrose diagrams

Because $\mathcal{Q}^+$ is just a bounded domain in $\mathbb{R}^2$ and the conformal embedding does not change causal relationships, it is very useful to consider this image graphically. Such a picture is called a **Penrose diagram** of M. Since the null directions u and v are at 135° and 45° from the horizontal, we can read off causal information very easily. Consider figure (figure) and describe the various bits of causal information it provides. (NEED FIGURE + MORE DESCRIPTION HERE.)

3.3 Black hole spacetimes

In our 2-dimensional setting, we can now make rigorous the definition of a black hole as suggested in section 2.2. First, however, we translate some of our trapped surface machinery into this quotient manifold setting. Now, each point (u, v) of $\mathcal{Q}^+$ represents a two-sphere of radius $r = r(u, v)$ in the original manifold M, and the two future null directions orthogonal to this sphere are precisely ∂_u and ∂_v. Since we have labeled u as the "ingoing" direction and v the "outgoing" direction, we will use θ_- and θ_+ to denote the expansions in the directions ∂_u and ∂_v, respectively. The induced Riemannian metric on this two-sphere is of course just $h_{ab} = r^2\gamma_{ab}$. A straightforward calculation now shows (see Appendix C) that $\theta_- = 2(\partial_u r)r^{-1}$ and $\theta_+ = 2(\partial_v r)r^{-1}$. Since r is strictly positive away from the center of symmetry Γ, the signs of θ_+ and θ_- are exactly those of $\partial_v r$ and $\partial_u r$, respectively.

Now define three regions of spacetime: the **regular region**

$$\mathcal{R} = \{(u, v) \in \mathcal{Q}^+ : \partial_v r > 0 \text{ and } \partial_u r < 0\},$$

the **trapped region**

$$\mathcal{T} = \{(u, v) \in \mathcal{Q}^+ : \partial_v r < 0 \text{ and } \partial_u r < 0\},$$

and the **marginally trapped region**,

$$\mathcal{T} = \{(u, v) \in \mathcal{Q}^+ : \partial_v r = 0 \text{ and } \partial_u r < 0\}.$$

In order to gain some necessary control over the quotient manifold $\mathcal{Q}^+$, we now introduce the new assumption, called "no anti-trapped surfaces initially" in [3], that $\partial_u r < 0$ along S. With this assumption, we have the following result, originally due to Christodoulou:

Proposition 1. *$\mathcal{Q}^+ = \mathcal{R} \cup \mathcal{T} \cup \mathcal{A}$, that is, anti-trapped surfaces cannot evolve if none are present initially.*

Proof. We have assumed that all ingoing null curves, constant-v curves, have past end-point on S. We integrate equation (7) along any such curve: for any $(u_0, v_0) \in S$, we have

$$\Omega^{-2}(\partial_u r)(u, v_0) = \Omega^{-2}(\partial_u r)(u_0, v_0) - \int_{u_0}^{u} 4\pi r\, \Omega^{-2} T_{uu}(\overline{u}, v_0)\, d\overline{u}.$$

Since we have assumed that $\partial_u r < 0$ along S and that $T_{uu} \geq 0$ everywhere, the righthand side of this equation is strictly negative, and hence so is the left-hand side. $\square$

We are now in a position to rigorously define future null infinity. First observe that the boundary curve S must have a unique endpoint in $\overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+$; by analogy with the asymptotically flat case, call it i^0. Next, let

$$\mathcal{U} = \{u \ : \sup_{v:(u,v)\in\mathcal{Q}^+} r(u,v) = \infty\}.$$

This set may well be empty, even if r goes to infinity along S. If $u \in \mathcal{U}$, however, then there exists a unique $v = v^*(u)$ such that $(u, v^*(u)) \in \overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+$. Now define

$$\mathcal{I}^+ = \bigcup_{u\in\mathcal{U}} (u, v^*(u)).$$

Then, if it is not empty, $\mathcal{I}^+$ is called **future null infinity**.

Proposition 2. *If it is not empty, $\mathcal{I}^+$ is a connected ingoing null ray with past limit point i^0.*

Proof. Suppose $i^0 = (U, V)$. By Proposition 1, we know that $\partial_u r < 0$ throughout $\mathcal{Q}^+$, so r decreases along each ingoing null ray. It follows that for any $v_0 < V$, r is bounded above on $\{v \leq v_0\} \cap \mathcal{Q}^+$ by its supremum on $\{v \leq v_0\} \cap S$, which is necessarily finite. Thus if $(u, v) \in \mathcal{I}^+$, we must have $v \geq V$. On the other hand, we have assumed that $\mathcal{Q}^+$ is foliated by ingoing null rays with past endpoint on S, and so we must have $\overline{\mathcal{Q}^+} \subset \{v \leq V\}$; thus $\mathcal{I}^+ \subset \{v = V\}$.

Now suppose that $(u_0, V) \in \mathcal{I}^+$. Then since $\partial_u r < 0$ in $\mathcal{Q}^+$, for any $u < u_0$ we have $r(u_0, v) < r(u, v)$ for all $v < V$. On the other hand, by definition of $\mathcal{I}^+$, we have $\sup_{v<V} r(u_0, v) = \infty$. Thus we must have $\sup_{v<V} r(u, v) = \infty$, so $(u, V) \in \mathcal{I}^+$ as well, and hence we must have $(U, u_0] \times \{V\} \subset \mathcal{I}^+$. This proves the proposition. □

We now make one final assumption, that $\mathcal{I}^+$ is not empty. Define the **domain of outer communications** as before to be $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+$, and the **black hole region** to be $\mathcal{Q}^+ \setminus J^-(\mathcal{I}^+)$. Recall that in our earlier discussion, we mentioned results which indicated that any trapped surface must lie inside a black hole; here there are no extra technical hypotheses, and we have just

Proposition 3. *The domain of outer communications is contained in the regular region, i.e. $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+ \subset \mathcal{R}$. In other words, any spherically symmetric trapped surface (corresponding to a point $(u, v) \in \mathcal{Q}^+$) must lie inside the black hole region.*

Proof. Fix some point $(u_0, v_0) \in \mathcal{Q}^+$ and integrate equation (8) along an outgoing null ray, say the curve $u = u_0$. As in the proof of Proposition 1, we get

$$\Omega^{-2}(\partial_v r)(u_0, v) = \Omega^{-2}(\partial_v r)(u_0, v_0) - \int_{v_0}^{v} 4\pi r \, \Omega^{-2} T_{vv}(u_0, \overline{v}) \, d\overline{v}.$$

If $(u_0, v_0) \in \mathcal{T} \cup \mathcal{A}$, then by definition $(\partial_v r)(u_0, v_0) \leq 0$, which in turn implies that the right-hand side of the equation is nonpositive, since $T_{vv} \geq 0$ from the dominant energy condition. Thus $(\partial_v r)(u_0, v)$ is a non-increasing function of v, so the whole outgoing null ray must lie entirely in $\mathcal{T} \cup \mathcal{A}$. But this in turn implies that $r(u_0, v)$ itself is a nonincreasing function of v along the ray, and so $\sup_{v:(u_0,v)\in\mathcal{Q}^+} r(u_0, v) \leq r(u_0, v_0) < \infty$. From Proposition (2), we know that $\mathcal{I}^+$ is connected and contains i^0, so we can conclude that no portion of it can extend into the causal future of the ray $\{u = u_0\}$, and hence $(u_0, v_0) \notin J^-(\mathcal{I}^+)$. This completes the proof. (INCLUDE PENROSE DIAGRAM?) □

3.4 Proposed problems

3.5 Existing results

Appendix A: Christoffel symbols

Here we explicitly compute the Christoffel symbols for the metric

$$g = -\Omega^2 du\, dv + r^2\gamma.$$

Recall that the functions r and Ω depend only on the coordinates u and v. Assign the coordinates u, v, θ, ϕ labels 1, 2, 3, 4 respectively. Then we have

$$\begin{aligned} g_{12} = g_{21} &= -\frac{\Omega^2}{2} & \text{and} \quad g^{12} = g^{21} &= -\frac{2}{\Omega^2} \\ g_{33} &= r^2 & g^{33} &= \frac{1}{r^2} \\ g_{44} &= r^2 \sin^2\theta, & g^{44} &= \frac{1}{r^2 \sin^2\theta}, \quad \text{all others 0.} \end{aligned}$$

In order to compute this metric's Christoffel symbols, we first write down all of the partial derivatives of its components that are non-zero. These are:

$$\begin{aligned} g_{12,1} = g_{21,1} &= -\Omega\, \Omega_{,u} & g_{44,1} &= 2rr_{,u} \sin^2\theta \\ g_{12,2} = g_{21,2} &= -\Omega\, \Omega_{,v} & g_{44,2} &= 2rr_{,v} \sin^2\theta \\ g_{33,1} &= 2rr_{,u} & g_{44,3} &= 2r^2 \sin\theta \cos\theta \\ g_{33,2} &= 2rr_{,v}. \end{aligned}$$

Now, we have the coordinate formula for the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}).$$

So we can compute:

$$\begin{aligned} \Gamma^1_{ij} &= \tfrac{1}{2} g^{1l}(g_{il,j} + g_{jl,i} - g_{ij,l}) \\ &= \tfrac{1}{2} g^{12}(g_{i2,j} + g_{j2,i} - g_{ij,2}) \\ &= -\tfrac{1}{\Omega^2}(g_{i2,j} + g_{j2,i} - g_{ij,2}); \end{aligned}$$

thus

$$\begin{aligned} \Gamma^1_{11} &= -\tfrac{1}{\Omega^2}(g_{12,1} + g_{12,1} - g_{11,2}) = -\tfrac{1}{\Omega^2}(2\, g_{12,1}) = -\tfrac{1}{\Omega^2}(-2\,\Omega\,\Omega_{,u}) = \tfrac{2\,\Omega_{,u}}{\Omega}, \\ \Gamma^1_{12} = \Gamma^1_{21} &= -\tfrac{1}{\Omega^2}(g_{12,2} + g_{22,1} - g_{12,2}) = 0, \\ \Gamma^1_{33} &= -\tfrac{1}{\Omega^2}(g_{32,3} + g_{32,3} - g_{33,2}) = \tfrac{1}{\Omega^2} g_{33,2} = \tfrac{2}{\Omega^2} rr_{,v} \\ \Gamma^1_{44} &= -\tfrac{1}{\Omega^2}(g_{42,4} + g_{42,4} - g_{44,2}) = \tfrac{1}{\Omega^2} g_{44,2} = \tfrac{2}{\Omega^2} rr_{,v} \sin^2\theta. \end{aligned}$$

An identical procedure gives us the Christoffel symbols of the form Γ^2_{ij}, and we have

$$\begin{aligned}
\Gamma^2_{12} = \Gamma^2_{21} &= -\tfrac{1}{\Omega^2}(g_{11,2} + g_{21,1} - g_{12,1}) = 0 \\
\Gamma^2_{22} &= -\tfrac{1}{\Omega^2}(g_{21,2} + g_{21,2} - g_{22,1}) = -\tfrac{1}{\Omega^2}(2g_{21,2}) = -\tfrac{1}{\Omega^2}(-2\,\Omega\,\Omega_{,v}) = \tfrac{2\Omega_{,v}}{\Omega} \\
\Gamma^2_{33} &= -\tfrac{1}{\Omega^2}(g_{31,3} + g_{31,3} - g_{33,1}) = \tfrac{1}{\Omega^2} g_{33,1} = \tfrac{2}{\Omega^2} r r_{,u} \\
\Gamma^2_{44} &= -\tfrac{1}{\Omega^2}(g_{41,4} + g_{41,4} - g_{44,1}) = \tfrac{1}{\Omega^2} g_{44,1} = \tfrac{2}{\Omega^2} r r_{,u} \sin^2\theta.
\end{aligned}$$

And

$$\begin{aligned}
\Gamma^3_{ij} &= \tfrac{1}{2} g^{3l}(g_{il,j} + g_{jl,i} - g_{ij,l}) \\
&= \tfrac{1}{2} g^{33}(g_{i3,j} + g_{j3,i} - g_{ij,3}) \\
&= \tfrac{1}{2r^2}(g_{i3,j} + g_{j3,i} - g_{ij,3});
\end{aligned}$$

thus

$$\begin{aligned}
\Gamma^3_{13} = \Gamma^3_{31} &= \tfrac{1}{2r^2}(g_{13,3} + g_{33,1} - g_{13,3}) = \tfrac{1}{2r^2}\, g_{33,1} = \tfrac{1}{2r^2} 2 r r_{,u} = \tfrac{r_{,u}}{r} \\
\Gamma^3_{23} = \Gamma^3_{32} &= \tfrac{1}{2r^2}(g_{23,3} + g_{33,2} - g_{23,3}) = \tfrac{1}{2r^2}\, g_{33,2} = \tfrac{1}{2r^2} 2 r r_{,v} = \tfrac{r_{,v}}{r} \\
\Gamma^3_{44} &= \tfrac{1}{2r^2}(g_{43,4} + g_{43,4} - g_{44,3}) = -\tfrac{1}{2r^2}\, g_{44,3} = -\sin\theta\cos\theta.
\end{aligned}$$

Likewise,

$$\Gamma^4_{ij} = \tfrac{1}{2r^2\sin^2\theta}(g_{i4,j} + g_{j4,i} - g_{ij,4}),$$

so

$$\begin{aligned}
\Gamma^4_{14} = \Gamma^4_{41} &= \tfrac{1}{2r^2\sin^2\theta}(g_{14,4} + g_{44,1} - g_{14,4}) = \tfrac{1}{2r^2\sin^2\theta}\, g_{44,1} = \tfrac{r_{,u}}{r} \\
\Gamma^4_{24} = \Gamma^4_{42} &= \tfrac{1}{2r^2\sin^2\theta}(g_{24,4} + g_{44,2} - g_{24,4}) = \tfrac{1}{2r^2\sin^2\theta}\, g_{44,2} = \tfrac{r_{,v}}{r} \\
\Gamma^4_{34} = \Gamma^4_{43} &= \tfrac{1}{2r^2\sin^2\theta}(g_{34,4} + g_{44,3} - g_{34,4}) = \tfrac{1}{2r^2\sin^2\theta}\, g_{44,3} = \tfrac{\cos\theta}{\sin\theta}.
\end{aligned}$$

To summarize: all Christoffel symbols are zero except

$$\begin{aligned}
\Gamma^1_{11} &= 2\,\Omega_{,u}\Omega^{-1} & \Gamma^2_{22} &= 2\,\Omega_{,v}\Omega^{-1} \\
\Gamma^1_{33} &= 2 r r_{,v}\Omega^{-2} & \Gamma^2_{33} &= 2 r r_{,u}\Omega^{-2} \\
\Gamma^1_{44} &= 2 r r_{,v}\sin^2\theta\,\Omega^{-2} & \Gamma^2_{44} &= 2 r r_{,u}\sin^2\theta\,\Omega^{-2}
\end{aligned}$$

$$\begin{aligned}
\Gamma^3_{13} = \Gamma^3_{31} &= r_{,u} r^{-1} & \Gamma^4_{14} = \Gamma^4_{41} &= r_{,u} r^{-1} \\
\Gamma^3_{23} = \Gamma^3_{32} &= r_{,v} r^{-1} & \Gamma^4_{24} = \Gamma^4_{42} &= r_{,v} r^{-1} \\
\Gamma^3_{44} &= -\sin\theta\cos\theta & \Gamma^4_{34} = \Gamma^4_{34} = \Gamma^4_{43} &= (\tan\theta)^{-1}.
\end{aligned}$$

Appendix B: Deriving the Einstein field equations

In order to write down the field equations, we must first find the Ricci and scalar curvatures for the metric $g = \Omega^2 du\, dv + r^2\gamma$. Now, the components of the curvature tensor are

$$R_{ijkl} = g_{ml}(\Gamma^m_{jk,i} - \Gamma^m_{ik,j}) + g_{pl}(\Gamma^m_{jk}\Gamma^p_{im} - \Gamma^m_{ik}\Gamma^p_{jm}),$$

and so the Ricci tensor is given by

$$R_{jk} = \Gamma^i_{jk,i} - \Gamma^i_{ik,j} + \Gamma^m_{jk}\Gamma^i_{im} - \Gamma^m_{ik}\Gamma^i_{jm}.$$

Thus, using our coordinates (u, v, θ, ϕ) (labeled 1, 2, 3, and 4 respectively) and using the results of Appendix A, we have

$$\begin{aligned}
R_{11} &= \Gamma^i_{11,i} - \Gamma^i_{i1,1} + \Gamma^m_{11}\Gamma^i_{im} - \Gamma^m_{i1}\Gamma^i_{1m} \\
&= \Gamma^1_{11,1} - (\Gamma^1_{11,1} + \Gamma^3_{31,1} + \Gamma^4_{41,1}) + \Gamma^1_{11}(\Gamma^1_{11} + \Gamma^3_{31} + \Gamma^4_{41}) \\
&\quad -(\Gamma^1_{11}\Gamma^1_{11} + \Gamma^3_{31}\Gamma^3_{13} + \Gamma^4_{41}\Gamma^4_{14}) \\
&= -(\Gamma^3_{31,1} + \Gamma^4_{41,1}) + \Gamma^1_{11}(\Gamma^3_{31} + \Gamma^4_{41}) - (\Gamma^3_{31}\Gamma^3_{13} + \Gamma^4_{41}\Gamma^4_{14}) \\
&= -2(r_{,u}r^{-1})_{,u} + 2\,\Omega_{,u}\Omega^{-1}(2r_{,u}r^{-1}) - 2(r_{,u}r^{-1})^2 \\
&= -2r_{,uu}r^{-1} + 2(r_{,u})^2r^{-2} + 2\,\Omega_{,u}\Omega^{-1}(2r_{,u}r^{-1}) - 2(r_{,u}r^{-1})^2 \\
&= -2r_{,uu}r^{-1} + 4\,\Omega_{,u}r_{,u}(\Omega r)^{-1};
\end{aligned}$$

$$\begin{aligned}
R_{12} = R_{21} &= \Gamma^i_{12,i} - \Gamma^i_{i2,1} + \Gamma^m_{12}\Gamma^i_{im} - \Gamma^m_{i2}\Gamma^i_{1m} \\
&= -(\Gamma^2_{22,1} + \Gamma^3_{32,1} + \Gamma^4_{42,1}) - (\Gamma^3_{32}\Gamma^3_{13} + \Gamma^4_{42}\Gamma^4_{14}) \\
&= -(2\,\Omega_{,v}\Omega^{-1} + 2r_{,v}r^{-1})_{,u} - 2(r_{,v}r^{-1})(r_{,u}r^{-1}) \\
&= -2\,\Omega_{,vu}\Omega^{-1} + 2\,\Omega_{,u}\Omega_{,v}\Omega^{-2} - 2r_{,vu}r^{-1} + 2r_{,u}r_{,v}r^{-2} - 2r_{,u}r_{,v}r^{-2} \\
&= -2\,\Omega_{,vu}\Omega^{-1} + 2\,\Omega_{,u}\Omega_{,v}\Omega^{-2} - 2r_{,vu}r^{-1};
\end{aligned}$$

by the symmetry between u and v,

$$R_{22} = -2r_{,vv}r^{-1} + 4\,\Omega_{,v}r_{,v}(\Omega r)^{-1};$$

$$\begin{aligned}
R_{13} &= \Gamma^i_{13,i} - \Gamma^i_{i3,1} + \Gamma^m_{13}\Gamma^i_{im} - \Gamma^m_{i3}\Gamma^i_{1m} \\
&= \Gamma^3_{13,3} - \Gamma^4_{43,1} + \Gamma^3_{13}\Gamma^4_{43} - \Gamma^4_{43}\Gamma^4_{14} \\
&= 0 - 0 + (r_{,u}r^{-1})(\tan\theta)^{-1} - (\tan\theta)^{-1}(r_{,u}r^{-1}) \\
&= 0,
\end{aligned}$$

and similarly $R_{23} = 0$;

$$\begin{aligned} R_{14} &= \Gamma^i_{14,i} - \Gamma^i_{i4,1} + \Gamma^m_{14}\Gamma^i_{im} - \Gamma^m_{i4}\Gamma^i_{1m} \\ &= \Gamma^4_{14,4} \\ &= 0, \end{aligned}$$

and similarly $R_{24} = 0$;

$$\begin{aligned} R_{33} &= \Gamma^i_{33,i} - \Gamma^i_{i3,3} + \Gamma^m_{33}\Gamma^i_{im} - \Gamma^m_{i3}\Gamma^i_{3m} \\ &= \Gamma^1_{33,1} + \Gamma^2_{33,2} - \Gamma^4_{43,3} + \Gamma^1_{33}(\Gamma^1_{11} + \Gamma^3_{31} + \Gamma^4_{41}) + \Gamma^2_{33}(\Gamma^2_{22} + \Gamma^3_{32} + \Gamma^4_{42}) \\ &\quad -(2\Gamma^3_{13}\Gamma^1_{33} + 2\Gamma^2_{33}\Gamma^3_{32} + \Gamma^4_{43}\Gamma^4_{34}) \\ &= \Gamma^1_{33,1} + \Gamma^2_{33,2} - \Gamma^4_{43,3} + \Gamma^1_{33}\Gamma^1_{11} + \Gamma^2_{33}\Gamma^2_{22} - \Gamma^4_{43}\Gamma^4_{34} \\ &= (2rr_{,v}\Omega^{-2})_{,u} + (2rr_{,u}\Omega^{-2})_{,v} - ((\tan\theta)^{-1})_{,\theta} + (2rr_{,v}\Omega^{-2})(2\,\Omega_{,u}\Omega^{-1}) \\ &\quad +(2rr_{,u}\Omega^{-2})(2\,\Omega_{,v}\Omega^{-1}) - (\tan\theta)^{-2} \\ &= 2r_{,u}r_{,v}\Omega^{-2} + 2rr_{,vu}\Omega^{-2} - 4rr_{,v}\Omega_{,u}\Omega^{-3} + 2r_{,v}r_{,u}\Omega^{-2} + 2rr_{,uv}\Omega^{-2} \\ &\quad -4rr_{,u}\Omega_{,v}\Omega^{-3} + 1 + 4rr_{,v}\Omega_{,u}\Omega^{-3} + 4rr_{,u}\Omega_{,v}\Omega^{-3} \\ &= 4r_{,u}r_{,v}\Omega^{-2} + 4rr_{,uv}\Omega^{-2} + 1; \end{aligned}$$

$$\begin{aligned} R_{34} \;=\; R_{43} &= \Gamma^i_{34,i} - \Gamma^i_{i4,3} + \Gamma^m_{34}\Gamma^i_{im} - \Gamma^m_{i4}\Gamma^i_{3m} \\ &= 0; \end{aligned}$$

and finally,

$$\begin{aligned} R_{44} &= \Gamma^i_{44,i} - \Gamma^i_{i4,4} + \Gamma^m_{44}\Gamma^i_{im} - \Gamma^m_{i4}\Gamma^i_{4m} \\ &= \Gamma^1_{44,1} + \Gamma^2_{44,2} + \Gamma^3_{44,3} + \Gamma^1_{44}(\Gamma^1_{11} + \Gamma^3_{31} + \Gamma^4_{41}) + \Gamma^2_{44}(\Gamma^2_{22} + \Gamma^3_{32} + \Gamma^4_{42}) \\ &\quad +\Gamma^3_{44}\Gamma^4_{43} - (2\Gamma^1_{44}\Gamma^4_{41} + 2\Gamma^2_{44}\Gamma^4_{42} + 2\Gamma^3_{44}\Gamma^4_{43}) \\ &= \Gamma^1_{44,1} + \Gamma^2_{44,2} + \Gamma^3_{44,3} + \Gamma^1_{44}\Gamma^1_{11} + \Gamma^2_{44}\Gamma^2_{22} - \Gamma^3_{44}\Gamma^4_{43} \\ &= (2rr_{,v}\sin^2\theta\,\Omega^{-2})_{,u} + (2rr_{,u}\sin^2\theta\,\Omega^{-2})_{,v} + (-\sin\theta\cos\theta)_{,\theta} \\ &\quad +(2rr_{,v}\sin^2\theta\,\Omega^{-2})(2\,\Omega_{,u}\Omega^{-1}) + (2rr_{,u}\sin^2\theta\,\Omega^{-2})(2\,\Omega_{,v}\Omega^{-1}) \\ &\quad -(-\sin\theta\cos\theta)(\tan\theta)^{-1} \\ &= 2\sin^2\theta\, r_{,u}r_{,v}\,\Omega^{-2} + 2\sin^2\theta\, rr_{,vu}\,\Omega^{-2} - 4\sin^2\theta\, rr_{,v}\Omega_{,u}\Omega^{-3} \\ &\quad +2\sin^2\theta\, r_{,v}r_{,u}\,\Omega^{-2} + 2\sin^2\theta\, rr_{,uv}\,\Omega^{-2} - 4\sin^2\theta\, rr_{,u}\Omega_{,v}\Omega^{-3} + \sin^2\theta \\ &\quad +4\sin^2\theta\, rr_{,v}\Omega_{,u}\Omega^{-3} + 4\sin^2\theta\, rr_{,u}\Omega_{,v}\Omega^{-3} \\ &= 4\sin^2\theta\, r_{,u}r_{,v}\,\Omega^{-2} + 4\sin^2\theta\, rr_{,vu}\,\Omega^{-2} + \sin^2\theta. \end{aligned}$$

To summarize: all components of the Ricci tensor are zero except

$$\begin{aligned} R_{11} &= -2r_{,uu}r^{-1} + 4\,\Omega_{,u}r_{,u}(\Omega r)^{-1} \\ R_{12} = R_{21} &= -2\,\Omega_{,vu}\Omega^{-1} + 2\,\Omega_{,u}\Omega_{,v}\Omega^{-2} - 2r_{,vu}r^{-1} \\ R_{22} &= -2r_{,vv}r^{-1} + 4\,\Omega_{,v}r_{,v}(\Omega r)^{-1} \\ R_{33} &= 4r_{,u}r_{,v}\Omega^{-2} + 4rr_{,uv}\Omega^{-2} + 1 \\ R_{44} &= \sin^2\theta\,(4r_{,u}r_{,v}\Omega^{-2} + 4rr_{,uv}\Omega^{-2} + 1). \end{aligned}$$

Now the scalar curvature $R = g^{jk}R_{jk}$ is just

$$\begin{aligned} R &= 2g^{12}R_{12} + g^{33}R_{33} + g^{44}R_{44} \\ &= -4\,\Omega^{-2}(-2\,\Omega_{,vu}\Omega^{-1} + 2\,\Omega_{,u}\Omega_{,v}\Omega^{-2} - 2r_{,vu}r^{-1}) \\ &\quad +r^{-2}(4r_{,u}r_{,v}\Omega^{-2} + 4rr_{,uv}\Omega^{-2} + 1) \\ &\quad +(r\sin\theta)^{-2}(\sin^2\theta\,(4r_{,u}r_{,v}\Omega^{-2} + 4rr_{,uv}\Omega^{-2} + 1)) \\ &= 8\,\Omega_{,vu}\Omega^{-3} - 8\,\Omega_{,u}\Omega_{,v}\Omega^{-4} + 16r_{,vu}r^{-1}\Omega^{-2} + 8r_{,u}r_{,v}r^{-2}\Omega^{-2} + 2r^{-2}, \end{aligned}$$

and so the Einstein tensor $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ has the following non-zero components:

$$G_{11} = R_{11} = -2r_{,uu}r^{-1} + 4\,\Omega_{,u}r_{,u}(\Omega r)^{-1};$$

$$G_{22} = R_{22} = -2r_{,vv}r^{-1} + 4\,\Omega_{,v}r_{,v}(\Omega r)^{-1};$$

$$\begin{aligned} G_{12} &= R_{12} - \tfrac{1}{2}R\,g_{12} \\ &= -2\,\Omega_{,vu}\Omega^{-1} + 2\,\Omega_{,u}\Omega_{,v}\Omega^{-2} - 2r_{,vu}r^{-1} \\ &\quad +\tfrac{1}{4}\Omega^2(8\,\Omega_{,vu}\Omega^{-3} - 8\,\Omega_{,u}\Omega_{,v}\Omega^{-4} + 16r_{,vu}r^{-1}\Omega^{-2} \\ &\qquad +8r_{,u}r_{,v}r^{-2}\Omega^{-2} + 2r^{-2}) \\ &= 2r_{,vu}r^{-1} + 2r_{,u}r_{,v}r^{-2} + \tfrac{1}{2}r^{-2}\Omega^2; \end{aligned}$$

$$\begin{aligned} G_{33} &= R_{33} - \tfrac{1}{2}R\,g_{33} \\ &= 4r_{,u}r_{,v}\Omega^{-2} + 4rr_{,uv}\Omega^{-2} + 1 \\ &\quad -\tfrac{1}{2}r^2(8\,\Omega_{,vu}\Omega^{-3} - 8\,\Omega_{,u}\Omega_{,v}\Omega^{-4} + 16r_{,vu}r^{-1}\Omega^{-2} \\ &\qquad +8r_{,u}r_{,v}r^{-2}\Omega^{-2} + 2r^{-2}) \\ &= -4rr_{,uv}\Omega^{-2} - 4r^2\,\Omega_{,vu}\Omega^{-3} + 4r^2\,\Omega_{,u}\Omega_{,v}\Omega^{-4}; \end{aligned}$$

and by inspection,

$$G_{44} = \sin^2\theta(-4rr_{,uv}\Omega^{-2} - 4r^2\,\Omega_{,vu}\Omega^{-3} + 4r^2\,\Omega_{,u}\Omega_{,v}\Omega^{-4}).$$

Plugging these components into the Einstein field equation $G = 8\pi T$ now clearly yields equations (3) through (6).

Appendix C: Computing $\theta_\pm$ in spherical symmetry

Fix coordinates $u = u_0$ and $v = v_0$ and consider the two-sphere of radius $r = r(u_0, v_0)$ comprising the points $\{(u_0, v_0, \theta, \phi)\} \subset M$. We previously defined ∂_u as the "ingoing" direction and ∂_v as the "outgoing" one; denoting these vectors by k^a and l^a, respectively, we then have

$$\theta_+ = \theta_{(l)} = h^{ab}\nabla_b l_a = h^{ab}(l_{a,b} - l_c\Gamma^c_{ab}),$$

where $h_{ab} = r^2\gamma_{ab}$ is the induced metric on the given two-sphere and h^{ab} is its inverse. Using the coordinate labeling as in Appendices A and B, we then have that $h^{33} = r^{-2}$, $h^{44} = (r\sin\theta)^{-2}$, and all other components are zero. Also,

$$l_a = g_{ab}l^b = g_{a2},$$

so $l_1 = -\frac{1}{2}\Omega^2$ and the other three components are zero. In particular, it is now clear that $h^{ab}l_{a,b} = 0$ for all $a, b = 1, 2, 3, 4$, and we are left with

$$\begin{aligned}
\theta_+ &= -h^{ab}l_c\Gamma^c_{ab} \\
&= -h^{33}l_1\Gamma^1_{33} - h^{44}l_1\Gamma^1_{44} \\
&= -r^{-2}\cdot(-\tfrac{1}{2}\Omega^2)\cdot(2rr_{,v}\Omega^{-2}) - (r\sin\theta)^{-2}\cdot(-\tfrac{1}{2}\Omega^2)\cdot(2rr_{,v}\sin^2\theta\Omega^{-2}) \\
&= 2r_{,v}r^{-1}.
\end{aligned}$$

Similarly, one computes that $\theta_- = \theta_{(k)} = 2r_{,u}r^{-1}$.

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