Asymptotic Behavior of Marginally Trapped Tubes in Spherically Symmetric Black Hole Spacetimes

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Abstract

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We begin by reviewing some fundamental features of general relativity, then outline the mathematical definitions of black holes, trapped surfaces, and marginally trapped tubes, first in general terms, then rigorously in the context of spherical symmetry. We describe explicitly the reduction of Einstein's equation on a spherically symmetric 4-dimensional Lorentzian manifold to a system of partial differential equations on a subset of 2-dimensional Minkowski space. We discuss the asymptotic behavior of marginally trapped tubes in the Schwarzschild, Vaidya, and Reisner-Nördstrom solutions to Einstein's equations in spherical symmetry, as well as in Einstein-Maxwell-scalar field black hole spacetimes generated by evolving certain classes of asymptotically flat initial data.

Our first main result gives conditions on a general stress-energy tensor $T_{\alpha\beta}$ in a spherically symmetric black hole spacetime that are sufficient to guarantee that the black hole will contain a marginally trapped tube which is eventually achronal, connected, and asymptotic to the event horizon. Here "general" means that the matter model is arbitrary, subject only to a certain positive energy condition. A certain matter field decay rate, known as Price law decay in the literature, is not required per se for this asymptotic result, but such decay does imply that the marginally trapped tube has finite length with respect to the induced metric. In our second main result, we give two separate applications of the first theorem to self-gravitating Higgs field spacetimes, one using weak Price law decay, the other certain strong smallness and monotonicity assumptions.

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Chapter 1
INTRODUCTION

Few features of modern cosmology have captured the popular imagination as vividly as black holes. They have long intrigued those studying general relativity, too; black holes are nearly as old a concept as relativity theory itself. Unfortunately, their mathematical definition requires global information about the spacetime, and this makes them difficult to analyze, physically or mathematically, or to simulate numerically. Considerable effort has therefore been directed toward developing more tractable quasi-local frameworks to describe black hole behavior, especially recently. The most widely accepted such notion uses hypersurfaces called dynamical and isolated horizons to model the surfaces of black holes; these belong to more general class of hypersurfaces known as marginally trapped tubes (MTTs). Investigation of the geometry of such hypersurfaces is now underway, and in the last few years, various existence, uniqueness, and compactness results have been established for them [1, 3, 2]. One remaining open problem concerns the relationship between MTTs and the traditional definition of black holes: it is known that MTTs always lie inside of black holes (in physically reasonably situations), but do the two models coincide asymptotically? That is, must MTTs asymptotically approach black hole event horizons "at late times"? This thesis addresses this question in the special case of spherical symmetry: we give conditions on a black hole spacetime which guarantee the existence and "nice" asymptotic behavior of an MTT. We also show how these conditions may be attained for a certain a type of matter model, the Higgs field, for which the asymptotic result was previously unknown.

1.1 An overview of relativity, black holes, and MTTs

The theory of general relativity postulates that the universe should be described as a Lorentzian 4-manifold (M, g), while the matter and energy in the universe are described by a symmetric 2-tensor $T_{\alpha\beta}$ on M called the stress-energy tensor. Einstein's equations prescribe the interplay between the geometry of the underlying manifold and the matter and energy in it:

$$R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

where $R_{\alpha\beta}$ is the Ricci curvature of the metric g and R is its scalar curvature. Lorentzian manifolds (M,g) satisfying Einstein's equations are called spacetimes.

Of central importance to the study of black holes is the spacetime's causal structure. That is, a vector X tangent to a Lorentzian manifold (M,g) may be classified as spacelike, timelike, or null, depending on whether its Lorentzian inner product with itself is positive, negative, or zero, respectively, and timelike and null vectors (collectively called causal vectors) can be further categorized as future or past-directed, provided M is time orientable (something we will assume). Curves and submanifolds of M inherit these causal characterizations in a natural way. Light and matter sweep out future-directed causal curves called worldlines — null curves describe the paths of light rays, timelike ones the paths of massive objects. A key feature of causal structures on Lorentzian manifolds is that they are preserved by conformal changes of metric.

Mathematically speaking, locating a black hole region entails the following: one begins with a noncompact, asymptotically flat spacetime M — roughly, this means that the complement of some compact region in M is diffeomorphic to a finite union of copies of $\mathbb{R}^4 \setminus \overline{B_1(0)}$ and the metric g decays to the flat metric with respect to a radial coordinate on each copy. One then conformally compactifies each asymptotically flat end and considers the boundary of this new, compact, "unphysical" spacetime to represent the boundary of the physical one "at infinity." Next, one identifies a null portion of this boundary as "future null infinity," denoted $\mathcal{I}^+$, and declares the black hole region $\mathcal{B} \subset M$ to be that portion of the spacetime such that, identifying M with its image in the unphysical spacetime, no future-directed causal curve originating in $\mathcal{B}$ can terminate along $\mathcal{I}^+$. This definition thus aligns with our intuition that light and matter which enter the black hole region must stay in there and can never escape out to infinity. The event horizon $\mathcal{H}$ is the boundary of $\mathcal{B}$ in M; it is a null hypersurface. It is important to note that one must have the entire spacetime in hand in order to make the conformal compactification and hence to locate the black hole.

The cumbersomeness of the preceding definition begs the question: is there any more *local* way to describe how black holes "capture" light? The answer is that there is indeed. Given any spacelike 2-surface $S \subset M$ and a future null vector field ℓ orthogonal to S, one defines the (scalar) expansion of S in the direction ℓ, denoted $\theta_{(\ell)}$, as the infinitesimal change in surface area of S in the direction ℓ. Now, a given spacelike 2-surface S has exactly two orthogonal future null vector fields along it. One can imagine these as directions in which light rays travel as they leave the surface of a glowing bubble: some photons are directed radially out from the surface of the bubble, towards a viewing eyeball, say, while other photons' paths lead radially in toward the center of the bubble. Both directions are orthogonal to S, and because time is included in the Lorentzian picture, they are not collinear. In general, if we determine which of these two orthogonal future null directions points "in" and which points "out" (e.g. if M is asymptotically flat, this may be done in a natural way), then we label the expansions of their corresponding vector fields θ_- and θ_+, respectively. While the magnitudes of $\theta_\pm$ then depend on the scaling of these vector fields, their signs do not. In regions of mild curvature, we generally have $\theta_+ > 0$ and $\theta_- < 0$ — the photon wavefront from the glowing bubble is a larger sphere if the photons are outer-directed, smaller if the photons are inner-directed. In regions of strong curvature, however, this need not be the case. If $\theta_- < 0$ and $\theta_+ < 0$ everywhere on S, then we say the surface S is trapped, while if $\theta_- < 0$ and $\theta_+ = 0$ everywhere on S, we call it marginally trapped. The set of all points in M contained in at least one trapped surface is called the trapped region.

An especially useful extension of the notion of a trapped surface is that of a marginally trapped tube (MTT), denoted $\mathcal{A}$. This is a hypersurface which is foliated by closed marginally trapped (spacelike) 2-surfaces. In physically reasonable black hole spacetimes, trapped and marginally trapped surfaces always lie inside of the black hole region, and hence so do MTTs; the latter in fact act as a kind of boundary between the regular space and the trapped region. Certain MTTs have special names: a dynamical horizon (DH) is an MTT which is itself spacelike, while an isolated horizon (IH) is essentially an MTT which is null. Dynamical and isolated horizons appear to be good models of the surfaces of dynamical and equilibrium black holes, respectively. Many numerical simulations of black holes in fact use DHs and IHs instead, and some of the physics community (e.g. those developing loop quantum gravity) have found them to be well-suited for quantum considerations [21, 4].

Recent results about MTTs paint an intriguing picture. On the one hand, an arbitrary marginally trapped surface satisfying a certain stability condition (akin to that for stable minimal surfaces in Riemannian manifolds) is a leaf of not just one, but infinitely many MTTs [1]. On the other hand, a maximum principle argument shows that the locations of MTTs with respect to each other are in fact highly constrained [3]. In many situations there is a fascinating interplay between physics and geometry: for instance, if one assumes the dominant energy condition (a local notion of positive energy), then the 2-surfaces foliating a generic MTT must have spherical topology. However, the situation in higher dimensions is very different [18, 17].

A convenient setting in which to study MTTs is that of spherical symmetry. In fact, all known analytical (exact) examples of MTTs are spherically symmetric, and all existing analytical theorems concerning their asymptotic behavior assume spherical symmetry. A spherically symmetric spacetime is one which admits an $SO(3)$-action by isometries. Given such a spacetime M, one can work with the 2-dimensional Lorentzian quotient manifold $\mathcal{Q} = M/SO(3)$ instead of M without loss of information.

For concreteness, choose double null coordinates (u,v) on $\mathbb{R}^2$, such that the Minkowski (flat) metric η takes the form $\eta = -du\,dv$. Then we may conformally embed our quotient manifold $\mathcal{Q}$ into $\mathbb{R}^2$ such that the metric on $\mathcal{Q}$ takes the form $-\Omega^2 du\,dv$, and the original metric g may be expressed $g = -\Omega^2 du\,dv + r^2 ds^2$, where $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the standard metric on S^2, and $\Omega = \Omega(u,v)$ and $r = r(u,v)$ are smooth functions on $\mathcal{Q}$, nonnegative away from the center of symmetry (if one exists). Assuming that the stress-energy tensor $T_{\alpha\beta}$ is invariant under the $SO(3)$ action, the Einstein field equations for g on M reduce to a system of three pointwise equations on $\mathcal{Q}$ for the tensor components T_{uu}, T_{uv}, and T_{vv} in terms of r, Ω, and their first- and second-order partial derivatives in u and v. We can pass back and forth between the 2-dimensional $\mathcal{Q}$ and the 4-dimensional M without losing any information about the metric, so we may consider the two to be interchangeable. A straightforward calculation shows that $\theta_- = 2(\partial_u r)r^{-1}$ and $\theta_+ = 2(\partial_v r)r^{-1}$, so for $r > 0$, the signs of θ_+ and θ_- are exactly those of $\partial_v r$ and $\partial_u r$, respectively. Thus, as long as $\partial_u r < 0$ throughout $\mathcal{Q}$, trapped and marginally trapped surfaces in M correspond to points in $\mathcal{Q}$ at which $\partial_v r <$ and $= 0$, respectively, and the MTT $\mathcal{A}$ is precisely the level set $\{\partial_v r \equiv 0\}$.

1.2 Existing and new results on the asymptotic behavior of MTTs

The work described in this thesis involves the MTTs' long term behavior, that is, their behavior "near infinity." In particular, we focus on the question of whether MTTs are, in general, asymptotic to classical black hole event horizons. Recall that, informally speaking, a black hole is a region from which no light rays can escape "to infinity." If an MTT lying inside a black hole *is* asymptotic to the black hole's event horizon, then essentially all the worldlines that enter the black hole must in turn cross through the MTT into the trapped region — that is, the MTT traps all the same light and matter as the black hole. On the other hand, if an MTT is *not* asymptotic to the black hole's event horizon, then some worldlines would be caught by the black hole but not by the MTT. In this sense the resolution of this question completely in the affirmative would be a proof of concept of the dynamical horizon framework as an alternate model for black holes. On the other hand, examples of physically reasonable MTTs not asymptotic to the event horizon could help to shed some light on one of the major open problems in mathematical relativity, the (strong) cosmic censorship conjecture [8, 9].

One major problem in trying to compare the asymptotic behavior of MTTs with classical event horizons is, as has already been pointed out, that the latter are so difficult to locate in general, even in spherical symmetry. One can of course start by examining known (exact) spherically symmetric black hole spacetimes and looking for MTTs there; unfortunately this list of spacetimes is rather quickly exhausted. In the classical spherically symmetric black hole solutions called Schwarzschild and Reisner-Nordström, which describe vacuum and electro-vacuum spacetimes, respectively, the spherically symmetric MTTs are null and coincide exactly with the black hole event horizons. In Vaidya spacetimes, where the matter model is an (ingoing) null fluid, the spherically symmetric MTTs are DHs or IHs and are either asymptotic to or coincide with the event horizon (provided a physically reasonable energy condition holds). This latter example is well-known and has in fact been influential in shaping expectations about MTT behavior, despite the fact that it is not very physical.

Alternately, and more physically, one can recast the Einstein equations as an initial (Cauchy) value problem and *generate* spacetimes from initial data, locate any black holes, and then look for MTTs. This approach has proven successful for several different (spherically symmetric) matter models. In particular, the maximal development of spherically symmetric asymptotically flat initial data for the Einstein equations coupled with a scalar field, the Maxwell equations and a (real) scalar field, or the Vlasov equation (describing a collisionless gas) does indeed contain an MTT which is asymptotic to the event horizon [7, 11, 15].

In order to consider the problem of asymptotic behavior of an MTT for a *general* stress-energy tensor, however, one cannot begin with an asymptotically flat initial data set and generate a black hole spacetime — there are no evolution equations for the matter with which to evolve the initial data. Therefore we *begin* with the interior of a black hole, assuming that it has arisen in the maximal development of such data for some reasonable matter field evolution equations. In practice, generalizing Dafermos's setup in [11], we start with a characteristic rectangle, $[0,u_0]\times[v_0,\infty)$ in coordinates, and assume that initial data has been prescribed on its two past edges $\mathcal{C}_{in} := [0,u_0]\times\{v_0\}$ and $\mathcal{C}_{out} := \{0\}\times[v_0,\infty)$ that agrees with what would be there if $\mathcal{C}_{out}$ coincided with the outer portion of the event horizon of such a black hole. One such requirement is that the radial function $r(u,v)$ satisfy $r(0,v)\to r_+ < \infty$ as $v\to\infty$. We then look at the maximal development $\mathcal{Q}$ of this characteristic initial data inside the rectangle (again assuming that there are matter field evolution equations at play, but without having them explicitly) and try to locate an MTT.

Our first main result, presented in Section 5.3, says that if the stress-energy tensor satisfies the dominant energy condition (which amounts to T_{uu}, T_{vv}, and $T_{uv}\geq 0$ in spherical symmetry), and if there exists a $\delta > 0$ such that four particular conditions involving r, Ω, T_{uu}, T_{uv} and their derivatives are satisfied in the region of $\mathcal{Q}$ where $\partial_v r > 0$ and $r(u,v)\geq r_+ - \delta$, then the rectangle contains an MTT which is asymptotic to the event horizon and which is connected and spacelike or null at late times. These four conditions are nontrivial, but they are satisfied in Vaidya black holes and in the Einstein-scalar field and Einstein-Maxwell-scalar field black holes considered in [7] and [11], respectively. Our second set of results, given in Sections 6.1 and 6.2, shows that these conditions can also be satisfied in a large class of Higgs field spacetimes, where the matter is described by a scalar field ϕ and a potential function $V(\phi)$ satisfying $\Box\phi = V'(\phi)$. In particular, we give two examples in which physically reasonable conditions on ϕ and V imply that the conditions and hence conclusions of the main theorem hold. All of these results were presented first in [24] and are reproduced here essentially unchanged.

Chapter 2

BACKGROUND

2.1 Spacetime machinery

As noted in the introduction, general relativity theory postulates that the universe should be described as a *Lorentzian manifold*, that is, a smooth manifold M^n paired with a metric g of signature $(-,+,\cdots,+)$. Although much of what follows may be carried out for higher dimensions, we will always work in dimension $n = 4$. Matter and energy in the universe are described by a symmetric 2-tensor $T_{\alpha\beta}$ on M called the *stress-energy tensor*. The geometry of this manifold and the matter and energy in it are coupled via Einstein's equations:

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{2.1}$$

where $R_{\alpha\beta}$ is the Ricci curvature of the metric g and R is its scalar curvature. Lorentzian manifolds (M,g) satisfying Einstein's equations are called *spacetimes*. We will always assume that our spacetimes are connected.

For any spacetime (M,g), the signature of g enables us to partition the tangent vectors of M into three types. A tangent vector $X \in T_pM$ is called *timelike* if $g(X,X) < 0$, *spacelike* if $g(X,X) > 0$, or *null* if $g(X,X) = 0$. If $g(X,X) \leq 0$, that is, X is either timelike or null, then X is said to be *causal*. These characterizations extend to certain curves in M as well. A differentiable curve $\gamma\colon I \to M$ is called timelike (respectively null, spacelike, or causal) if at each $t \in I$, the vector $\gamma'(t) \in T_pM$ is timelike (respectively null, spacelike, or causal). The spacetime M is said to be *time orientable* if it admits a global, continuous, non-vanishing timelike vector field. Given such a vector field, say, V, we may assign M an explicit time orientation: given any causal vector $X \in T_pM$, we declare it to be *future directed* if $g(X,V_p) < 0$ and *past directed* otherwise. Henceforth we will assume that all our spacetimes are time orientable and have been assigned explicit orientations. A submanifold, too, may be characterized as spacelike, timelike, or null if the pullback of g on it is everywhere Riemannian, Lorentzian, or degenerate, respectively.

Once we have established a time orientation and the causal character of tangent vectors at each point, we can extend notions of causality to points in the manifold itself. If p is a point in a spacetime M, then we may define the *chronological future* of the point p as the set

$$I^+(p) = \{q \in M : \exists \text{ a piecewise timelike future directed curve from } p \text{ to } q\}.$$

The *chronological past* of p, denoted $I^-(p)$, is defined similarly, replacing "future" with "past". The *causal future* and *causal past* of p are denoted $J^+(p)$ and $J^-(p)$, respectively, and are defined analogously by replacing the word "timelike" in the definition with "causal". Note that the point p is not contained in $I^\pm(p)$ but is contained in $J^\pm(p)$, since degenerate curves are by definition causal. A set S in M is said to be *achronal* if no two points $p,q \in S$ may be joined by a piecewise timelike curve, i.e. there do not exist $p,q \in S$ such that $q \in I^+(p)$. The *future domain of dependence* of a set S, denoted $D^+(S)$, is defined to be the set of events in the spacetime that are completely predicted by the events in S. More precisely,

$$D^+(S) = \{p \in M : \text{ every past inextendible causal curve through } p \text{ intersects } S\},$$

where "inextendible" is defined appropriately. The *past domain of dependence* $D^-(S)$ is defined analogously by replacing "past" with "future". Taken together, they constitute the (full) *domain of dependence* of S:

$$D(S) = D^+(S) \cup D^-(S).$$

If a set S is closed, achronal, and its domain of dependence is all of the spacetime, $D(S) = M$, then S is said to be a *Cauchy surface*. A spacetime (M,g) which admits a Cauchy surface is called *globally hyperbolic*.

An arbitrary observer (*a priori* moving more slowly than the speed of light) traces out a future directed timelike curve in the spacetime manifold called a *worldline*. If ξ^α is the tangent vector to such a worldline, then the (scalar) energy density of matter measured by that observer is given by $T_{\alpha\beta}\xi^\alpha\xi^\beta$, while the quantity $-T^\alpha_\beta\xi^\beta$ represents the full 4-vector energy-momentum density of matter that he or she sees. Motivated by local physical considerations, relativists often impose restrictions on these quantities. In this paper we will be concerned only with the *null energy condition*, which requires that $T_{\alpha\beta}\mu^\alpha\mu^\beta \geq 0$ for all null vectors μ^α, and the *dominant energy condition*, which requires that for any future directed timelike ξ^α, $-T^\alpha_\beta\xi^\beta$ should be a future directed timelike or null vector.

2.2 Black holes

Intuitively speaking, a black hole is a region of spacetime curved in such a way that, once an observer passes into the region, he or she can never again escape or even communicate with observers outside it. Such extreme curvature is generally thought to arise from the gravitational collapse of a massive body. Another way of heuristically describing a black hole is to say that the worldline of an observer who has traveled into a black hole will never reach "future infinity"; instead, the worldline is trapped inside the black hole and, it is thought, terminates in "finite time" at some sort of spacetime singularity. Such a notion is difficult to make precise, however; we will need to introduce further machinery in order to make mathematical sense of it, first in general terms here, then later in a fully rigorous way in the context of spherical symmetry.

Roughly speaking, a spacetime (M,g) is said to be *asymptotically flat* if the complement of some compact region in M is diffeomorphic to a finite union of copies of $\mathbb{R}^4 \setminus \overline{B_1(0)}$ and the metric decays to the flat metric with respect to any radial coordinate on each copy. Various conditions on the exact decay rates of the metric and its derivatives are usually imposed when one makes the definition rigorous [5]. One can then conformally compactify each asymptotically flat end (similarly to the way that one conformally compactifies Minkowski space) and consider the boundary of this new "unphysical" spacetime to represent the boundary of the physical one "at infinity". Under appropriate conditions, this boundary will contain three components of particular interest: a point i^0, called *spatial infinity*, and sets $\mathcal{I}^+$ and $\mathcal{I}^-$, pronounced "scri plus" and "scri minus", respectively. These sets are defined by the relations $\mathcal{I}^+ = \overline{J^+(i^0)} - i^0$ and $\mathcal{I}^- = \overline{J^-(i^0)} - i^0$, and in particular, under appropriate regularity assumptions, $\mathcal{I}^+$ and $\mathcal{I}^-$ will be null surfaces. They are called *future* and *past null infinity*, respectively. Because a conformal change of metric preserves causal structure, it makes sense to talk about the set $J^-(\mathcal{I}^+)$ in the original physical spacetime M, called the *domain of outer communications*. Roughly speaking, this set consists of all spacetime events which can be seen "from infinity". The complement of this set, $\mathcal{B} = M \setminus J^-(\mathcal{I}^+)$, is called the *black hole region*, and its boundary $\mathcal{H} = \partial J^-(\mathcal{I}^+)$ is said to be the *event horizon*.

2.3 Trapped surfaces

Although the definition of a black hole given above makes sense both intuitively and mathematically, it has the drawback that one needs to have information about the entire spacetime manifold at hand in order to find its conformal boundary and thus locate the black hole region. One could not, for example, tell by looking at an open subregion of M whether or not it contains a black hole. So physicists have looked for local and quasi-local notions to use along with or in place of the standard definition of black holes, hoping to capture their essence but make the mathematics and physics more manageable. Various ideas have been proposed over time, some of which we discuss below.

A *congruence* of null geodesics is simply a family of null geodesics which foliates some open region of spacetime. Given a point p on a spacelike 2-surface S in M, there are exactly two distinct future null directions orthogonal to S at p, and we can always find two distinct congruences of future directed null geodesics orthogonal to S, defined up to choice of parametrization, whose tangent vectors coincide with these two future null directions along S. (Away from S, the congruences are not uniquely specified.) If ℓ^α is the tangent vector field of one these congruences, then we can define $\theta_{(\ell)}$, the *expansion* of S in the direction ℓ, by

$$\theta_{(\ell)} = \mathrm{div}_S\,\ell_\alpha = h^{\alpha\beta}\nabla_\beta \ell_\alpha,$$

where ∇ is the Levi-Civita connection on M and h is the induced Riemannian metric on the 2-surface S. Since a null geodesic does not in general admit a canonical parametrization, the vector field ℓ^α and hence its expansion $\theta_{(\ell)}$ are dependent on the choice of parametrization of the null normal geodesics in the congruence. However, if we rescale ℓ^α by some positive function λ, we can compute that $\theta_{(\lambda\ell)} = \lambda\theta_{(\ell)}$, so the *sign* of the expansion $\theta_{(\ell)}$ is indeed well-defined. Intuitively speaking, $\theta_{(\ell)}$ measures the infinitesimal change in the area of S in the direction ℓ^α. One typically expects that the expansion will be positive in one of the null normal directions to S and negative in the other (think of the inner- and outer-pointing normals of a standard 2-sphere, for example), but if the ambient manifold is sufficiently curved, that characterization need not hold. In particular, if a 2-surface S has future null directions ℓ and n such that both $\theta_{(\ell)} < 0$ and $\theta_{(n)} < 0$, then S is called a *trapped surface*; the surface is *marginally trapped* if both expansions are merely nonpositive. If ℓ and n can be distinguished from each other by determining that ℓ is “outer” and n “inner”, for example if M

is asymptotically flat (in which case ℓ is chosen to point "towards future null infinity"), then we say that S is *outer marginally trapped* if $\theta_{(\ell)} \leq 0$, and it is an *apparent horizon* if $\theta_{(\ell)} \equiv 0$.

A famous result of Penrose shows just why trapped surfaces are important: they signal the development of spacetime singularities often associated with black holes. In particular, in 1965, he proved the following

Theorem. *Let (M,g) be a connected, globally hyperbolic spacetime whose Cauchy surface is non-compact and which satisfies the null energy condition. If M contains a closed trapped surface S, then there exists at least one inextendible, future directed, orthogonal null geodesic emanating from S and having finite affine length in M.* [19]

When a geodesic γ is given an affine parametrization, i.e. γ: $(0,T) \to M$ satisfies $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ for all $t \in (0,T)$, then its *affine length* is just the value T. The existence of an inextendible geodesic of finite affine length signals either that some sort of singularity occurs at its "endpoint" or that global hyperbolicity fails there. In either case, the trapped surface acts as a local indication of a pathology in the spacetime. Furthermore, in an asymptotically flat spacetime M, which is the only type of spacetime in which black holes may even be defined, under a certain extra asymptotic condition ("strong asymptotic predictability"), one can show that any trapped surface must lie inside a black hole region [22].

2.4 Marginally trapped tubes

Trapped surfaces still do not quite provide a local model for what physicists call a black hole. But another object has recently been proposed to provide a quasi-local model for a black hole: a *dynamical horizon* (DH) in a spacetime (M,g) is a spacelike hypersurface foliated by closed spacelike 2-surfaces such that, on each leaf S, the expansion $\theta_{(l)}$ of one null normal ℓ^α vanishes, and the expansion $\theta_{(n)}$ of the other null normal n^α is strictly negative. If M is asymptotically flat or some other notion of spatial infinity can be applied, then one always takes $\theta_{(l)} = \theta_+$ and $\theta_{(n)} = \theta_-$, where the plus and minus denote the "outward" and "inward" directions, respectively. Note that each of the foliating 2-surfaces is thus a marginally trapped surface as well as an apparent horizon. It turns out that dynamical horizons provide a good local model for an evolving black hole, and physicists have been able to extend notions of black hole thermodynamics and entropy to them with great success

[4].

Dynamical horizons have timelike and null analogs as well, called *timelike membranes* (TMs) and *non-expanding horizons* (NEHs). These are defined by replacing spacelike with timelike or null, respectively, as the hypothesis on the hypersurface (but not on the foliating 2-surfaces), and in the case of non-expanding horizons, one requires in addition that $-T_{\alpha\beta}\xi^{\beta}$ be future causal for any future directed null normal ξ^{α} [4, 3]. In spacetimes in which the dominant energy condition holds, this extra condition for non-expanding horizons is satisfied a priori. *Isolated horizons* (IHs), which are non-expanding horizons satisfying certain other physical and regularity conditions, are thought to model black holes in an equilibrium state, while no concrete physical meaning is associated with timelike membranes; since future directed timelike curves can pass through a timelike membrane in either direction, it is not good candidate for a model of the surface of a black hole. Collectively, dynamical and non-expanding horizons and timelike membranes are called *marginally trapped tubes*.

Regardless of whether one wants to use some flavor of marginally trapped tube as a replacement model for a black hole, however, the study of their causal and asymptotic behavior is of interest in its own right. Since MTTs always lie inside of black holes in physically reasonable spacetimes, any new insight into their geometry provides a window into the black hole interior, a notoriously difficult region to study.

Chapter 3
SPHERICAL SYMMETRY

The study of spherically symmetric spacetimes has a venerable history. One of the first exact solutions to Einstein's equations, found by the physicist Karl Schwarzschild in the same year Einstein published his theory of general relativity, is spherically symmetric. The Schwarzschild solution describes the gravitational field in the vacuum surrounding a spherically symmetric body and was used to provide some of the first experimental confirmation of Einstein's theory. Besides Schwarzschild's, there are other important spherically symmetric exact solutions to Einstein's equations, such as the Reisner-Nordström and Vaidya spacetimes. Such solutions are the model spaces for many aspects of relativity theory and provide a testing ground for a wide range of theories. But even aside from such exact solutions, imposing an assumption of spherical symmetry in general casts the theory into a vastly simpler setting while still providing (one hopes) heuristics representative of generic non-spherically symmetric solutions. In this section, we will describe in detail the reduction of the spherically symmetric 3+1-manifold setting to a 1+1-setting.

3.1 Structure and energy assumptions

In general, a spacetime (M,g) is said to be *spherically symmetric* if the Lie group $SO(3)$ acts on it by isometries with orbits which are either fixed points or spacelike 2-spheres. In order to make use of this concept in practice, however, we will need to impose a large number of very specific additional conditions on (M,g). The end goal is to transfer all of the important causal and asymptotic data of M to a 1+1-dimensional quotient manifold which we can then conformally embed into Minkowski space; all of the assumptions we make here are necessary to ensure that this conformally embedded quotient manifold and its boundary are sufficiently well-behaved. The remainder of the setup described in this section is essentially from [13].

In what follows, we take (M,g) to be a globally hyperbolic spacetime satisfying the dominant energy condition which admits an $SO(3)$-action by isometries. We assume that the quotient manifold

$\mathcal{Q} = M/SO(3)$ inherits the structure of a 1+1 Lorentzian manifold with a boundary corresponding to the points fixed by the $SO(3)$-action, the center of symmetry. We further assume M is the maximal development of its Cauchy surface Σ and that the quotient $\mathcal{Q}^+$ of its causal future $J^+(\Sigma)$ may be conformally embedded into a bounded subset of Minkowski space $(\mathbb{R}^2, \eta)$. We assume that $\mathcal{Q}^+$ contains just one of its connected boundary components and that this boundary component has the form $\Gamma \cup S$, where Γ, the center of symmetry in $\mathcal{Q}^+$, is a connected timelike curve comprising the points in $\mathcal{Q}^+$ fixed by the $SO(3)$-action, $S = \Sigma/SO(3)$ is a connected spacelike curve, and Γ and S intersect in a single point p.

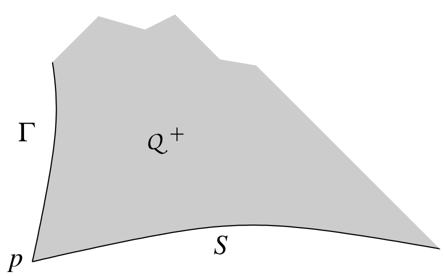

Figure 3.1: *The 2-dimensional Lorentzian quotient manifold* $\mathcal{Q}^+$.

Suppose we choose double null coordinates (u, v) on $\mathbb{R}^2$, such that the Minkowski metric η takes the form $\eta = -du\,dv$ and the positive u- and v-axes are at 135° and 45° from the usual positive x-axis, respectively. We assume that $(\mathbb{R}^2, \eta)$ is time oriented in the usual way such that u and v are both increasing toward the future. Then with respect to the conformal embedding, the metric on $\mathcal{Q}^+$ takes the form $-\Omega^2 du\,dv$, and suppressing pullback notation, the original metric g may be expressed

$$g = -\Omega^2 du\,dv + r^2 ds^2, \tag{3.1}$$

where as before $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the standard metric on S^2, and $\Omega = \Omega(u, v)$ and $r = r(u, v)$ are smooth functions on $\mathcal{Q}^+$ such that $\Omega > 0$, $r \geq 0$, and $r(q) = 0$ if and only if $q \in \Gamma$. Define the *Hawking mass* by

$$m = m(u, v) = \frac{r}{2}(1 + 4\Omega^{-2}\partial_u r \partial_v r); \tag{3.2}$$

we require that m be uniformly bounded along S. Finally, we assume that $\mathcal{Q}^+$ is foliated by connected constant u curves with past endpoint on $\Gamma \cup S$ and also by connected constant v curves with past endpoint on S, called "outgoing" and "ingoing" null curves, respectively.

For convenience, let us choose units such that Einstein's equations (2.1) instead take the form

$$R_{ab} - \tfrac{1}{2} R g_{ab} = 2T_{\alpha\beta}, \tag{3.3}$$

By direct computation (see appendix B), we find that equation (3.3) for the metric (3.1) on M reduces to the following system:

$$2T_{uu} = -2(\partial^2_{uu} r) r^{-1} + 4(\partial_u \Omega)(\partial_u r)(\Omega r)^{-1} \tag{3.4}$$

$$2T_{uv} = 2(\partial^2_{uv} r) r^{-1} + 2(\partial_u r)(\partial_v r) r^{-2} + \tfrac{1}{2} r^{-2} \Omega^2 \tag{3.5}$$

$$2T_{vv} = -2(\partial^2_{vv} r) r^{-1} + 4(\partial_v \Omega)(\partial_v r)(\Omega r)^{-1} \tag{3.6}$$

$$2T\big|_{S_r} = \left[-4r(\partial^2_{uv} r)\Omega^{-2} - 4r^2(\partial^2_{uv}\Omega)\Omega^{-3} + 4r^2(\partial_u \Omega)(\partial_v \Omega)\Omega^{-4}\right] g_{S^2}. \tag{3.7}$$

Clearly in order for equation (3.7) to hold, we must require that the stress-energy tensor T_{ab} be invariant under the $SO(3)$ action. Each of equations (3.4), (3.5), and (3.6) holds pointwise at all $p = (u, v, \theta, \phi) \in M$, but the right-hand sides only depend on the u and v coordinates. Thus, assuming that this system of equations is satisfied, i.e. that (M, g) is indeed a spacetime, the component functions T_{uu}, T_{uv}, and T_{vv} of the stress-energy tensor descend to functions on the quotient manifold $\mathcal{Q}^+$ and satisfy (3.4), (3.5), and (3.6) there as well. In fact, henceforth we consider equations (3.4), (3.5), and (3.6) *only* as pointwise equations on $\mathcal{Q}^+$. We can restate them in terms of the Hawking mass m:

$$\partial_u(\Omega^{-2}\partial_u r) = -r\Omega^{-2} T_{uu} \tag{3.8}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -r\Omega^{-2} T_{vv} \tag{3.9}$$

$$\partial_u m = 2r^2\Omega^{-2}(T_{uv}\partial_u r - T_{uu}\partial_v r) \tag{3.10}$$

$$\partial_v m = 2r^2\Omega^{-2}(T_{uv}\partial_u r - T_{vv}\partial_v r). \tag{3.11}$$

Observe that we have now dropped equation (3.7) from our system in the move to the quotient setting. (The addition of equation (3.2) to the system accounts for there still being four equations.) The idea is that, if we can find r, m, and Ω solving equations (3.8)-(3.11) on $\mathcal{Q}^+$ for prescribed functions T_{uu}, T_{vv}, and T_{uv}, then we can *define* the remaining components of the stress-energy tensor via (3.7) when we move back upstairs and thus obtain a full solution to the Einstein field equations. Thus, because we can pass back and forth between the 2-dimensional manifold and the 4-dimensional one without losing any information about the metric, we may consider the two to be interchangeable. Without risk of confusion, then, we will refer to both M and $\mathcal{Q}^+$ as the spacetime.

Recall that one of the initial assumptions was that (M,g) satisfy the dominant energy condition. In terms of the component functions of the stress-energy tensor on $\mathcal{Q}^+$, this taken to mean that

$$T_{uu} \geq 0, \;\; T_{uv} \geq 0, \;\; \text{and } T_{vv} \geq 0 \tag{3.12}$$

at all points $(u,v) \in \mathcal{Q}^+$ [13].

3.2 Penrose diagrams

Our setup now includes a 2-dimensional Lorentzian manifold $\mathcal{Q}^+$ conformally embedded into Minkowski space $(\mathbb{R}^2,\eta)$. The conformal embedding does not change causal relationships between points, so its image still carries the full causal structure of the original manifold. A graphical depiction of this conformal image $\mathcal{Q}^+$ in the plane is called a *Penrose diagram* of M. In particular, since the null directions u and v are at 135° and 45° from the horizontal, we can easily determine global geometric and causal information by inspection; e.g. in Figure 3.2 below, we can read off the following causal and incidence relations: $p,q,x \in S$, $\gamma_1,\gamma_2,\gamma_3 \in S$, $q \notin J^-(p) \cup J^+(p)$, $x \in I^+(p)$, $p = \gamma_1 \cap \gamma_2 \cap \gamma_3$, and that γ_1, γ_2, and γ_3 are spacelike, timelike, and null, respectively.

In the next section, we will identify and make assumptions about certain subsets of the boundary $\overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+$ of the quotient manifold; as we shall see, this boundary information can also be read off of the associated Penrose diagram. Indeed, Penrose diagrams may be considered to be expressions of fully rigorous mathematical statements about causal and incidence relationships in $\mathcal{Q}^+$ and its conformal boundary.

3.3 Black hole spacetimes

In our 2-dimensional setting, we can now make rigorous the definition of a black hole as suggested in Section 2.2. First, however, we translate some of our trapped surface notions into this quotient manifold setting. Each point (u,v) of $\mathcal{Q}^+$ represents a two-sphere of radius $r = r(u,v)$ in the original

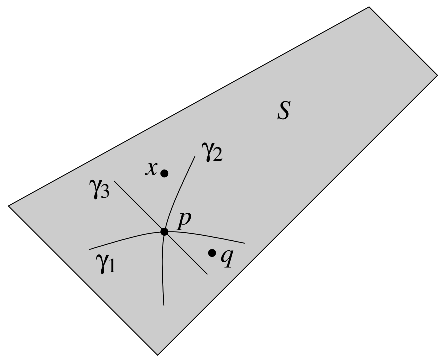

Figure 3.2: *A basic Penrose diagram.*

manifold M, and the two future null directions orthogonal to this sphere are precisely ∂_u and ∂_v. Since we have labeled u as the "ingoing" direction and v the "outgoing" direction, we will use θ_- and θ_+ to denote the expansions in the directions ∂_u and ∂_v, respectively. The induced Riemannian metric on this two-sphere is of course just $h_{ab} = r^2(g_{S^2})_{ab}$. A straightforward calculation now shows that $\theta_- = 2(\partial_u r)r^{-1}$ and $\theta_+ = 2(\partial_v r)r^{-1}$; see Appendix C. Since r is strictly positive away from the center of symmetry Γ, the signs of θ_+ and θ_- are exactly those of $\partial_v r$ and $\partial_u r$, respectively.

We define three regions of spacetime: the *regular region*

$$\mathcal{R} = \{(u,v) \in \mathcal{Q}^+ : \partial_v r > 0 \text{ and } \partial_u r < 0\},$$

the *trapped region*

$$\mathcal{T} = \{(u,v) \in \mathcal{Q}^+ : \partial_v r < 0 \text{ and } \partial_u r < 0\},$$

and the *marginally trapped set*,

$$\mathcal{A} = \{(u,v) \in \mathcal{Q}^+ : \partial_v r = 0 \text{ and } \partial_u r < 0\}.$$

An *anti-trapped surface* is one for which $\partial_u r \geq 0$. In order to gain some necessary control over the quotient manifold $\mathcal{Q}^+$, we now introduce the new assumption that there are "no anti-trapped surfaces initially", i.e. that $\partial_u r < 0$ along S. With this assumption, we have the following result of Christodoulou's (see [8, 13]):

Proposition 1. *$\mathcal{Q}^+ = \mathcal{R} \cup \mathcal{T} \cup \mathcal{A}$, that is, anti-trapped surfaces cannot evolve if none are present initially.*

Proof. We have assumed that all ingoing null curves, constant-v curves, have past end-point on S. We integrate equation (3.8) along any such curve: for any $(u_0, v_0) \in S$, we have

$$\Omega^{-2}(\partial_u r)(u, v_0) = \Omega^{-2}(\partial_u r)(u_0, v_0) - \int_{u_0}^{u} r\Omega^{-2} T_{uu}(\overline{u}, v_0)\, d\overline{u}.$$

Since we have assumed that $\partial_u r < 0$ along S and that $T_{uu} \geq 0$ everywhere, the righthand side of this equation is strictly negative, and hence so is the left-hand side. □

We are now in a position to define future null infinity rigorously. First observe that the boundary curve S must have a unique endpoint in $\overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+$; by analogy with the asymptotically flat case, call it i^0. Next, let

$$\mathcal{U} = \{u : \sup_{v:(u,v) \in \mathcal{Q}^+} r(u, v) = \infty\}.$$

This set may well be empty, even if r goes to infinity along S. If $u \in \mathcal{U}$, however, then there exists a unique value of the v-coordinate, say $v^*(u)$, such that $(u, v^*(u)) \in \overline{\mathcal{Q}^+} \setminus \mathcal{Q}^+$, i.e. $\lim_{v \to v^*(u)} r(u, v) = \infty$. Define

$$\mathcal{I}^+ = \bigcup_{u \in \mathcal{U}} (u, v^*(u)).$$

Then, if it is not empty, $\mathcal{I}^+$ is called *future null infinity*.

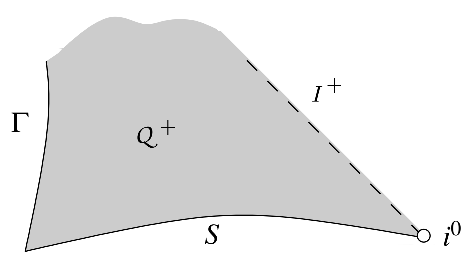

Figure 3.3: *Future null infinity, $\mathcal{I}^+$.*

Proposition 2. *If it is not empty, $\mathcal{I}^+$ is a connected ingoing null ray with past limit point i^0.*

Proof. Suppose $i^0 = (U,V)$. By Proposition 1, we know that $\partial_u r < 0$ throughout $\mathcal{Q}^+$, so r decreases along each ingoing null ray. It follows that for any $v_0 < V$, r is bounded above on $\{v \le v_0\} \cap \mathcal{Q}^+$ by its supremum on $\{v \le v_0\} \cap S$, which is necessarily finite. Thus if $(u,v) \in \mathcal{I}^+$, we must have $v \ge V$. On the other hand, we have assumed that $\mathcal{Q}^+$ is foliated by ingoing null rays with past endpoint on S, and so we must have $\overline{\mathcal{Q}^+} \subset \{v \le V\}$; thus $\mathcal{I}^+ \subset \{v = V\}$.

Now suppose that $(u_0,V) \in \mathcal{I}^+$. Then since $\partial_u r < 0$ in $\mathcal{Q}^+$, for any $u < u_0$ we have $r(u_0,v) < r(u,v)$ for all $v < V$. On the other hand, by definition of $\mathcal{I}^+$, we have $\sup_{v<V} r(u_0,v) = \infty$. Thus we must have $\sup_{v<V} r(u,v) = \infty$, so $(u,V) \in \mathcal{I}^+$ as well, and hence we must have $(U,u_0] \times \{V\} \subset \mathcal{I}^+$. This proves the proposition. $\square$

We now make one final assumption, that $\mathcal{I}^+$ is not empty. This assumption insures that $\mathcal{Q}^+$ represents a (possible) black hole spacetime. Define the domain of outer communications as before to be $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+$, and the black hole region to be $\mathcal{Q}^+ \setminus J^-(\mathcal{I}^+)$ (note that it could be empty). The event horizon of the black hole is then $\mathcal{H} = \partial(J^-(\mathcal{I}^+)) \cap \mathcal{Q}^+$. See Figure 3.4 for a representative Penrose diagram.

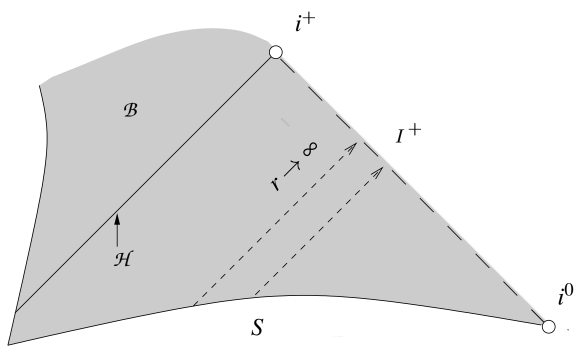

Figure 3.4: *The black hole region* $\mathcal{B} = \mathcal{Q}^+ \setminus J^-(\mathcal{I}^+)$.

Recall that in our discussion in Section 2.3, we mentioned results which indicated that any trapped surface must lie inside a black hole, but only when an additional technical condition was imposed; here there are no extra hypotheses, and we have just

Proposition 3. *The domain of outer communications is contained in the regular region, i.e.* $J^-(\mathcal{I}^+) \cap$

$\mathcal{Q}^+ \subset \mathcal{R}$. *In other words, any spherically symmetric trapped surface (corresponding to a point* $(u,v) \in \mathcal{Q}^+$*) must lie inside the black hole region.*

Proof. Fix some point $(u_0, v_0) \in \mathcal{Q}^+$ and integrate equation (3.9) along an outgoing null ray, say the curve $u = u_0$. As in the proof of Proposition 1, we get

$$\Omega^{-2}(\partial_v r)(u_0, v) = \Omega^{-2}(\partial_v r)(u_0, v_0) - \int_{v_0}^{v} r\Omega^{-2} T_{vv}(u_0, \overline{v})\, d\overline{v}.$$

If $(u_0, v_0) \in \mathcal{T} \cup \mathcal{A}$, then by definition $(\partial_v r)(u_0, v_0) \leq 0$, which in turn implies that the right-hand side of the equation is nonpositive, since $T_{vv} \geq 0$ from (3.12), the dominant energy condition. Thus $(\partial_v r)(u_0, v)$ is a non-increasing function of v, so the whole outgoing null ray must lie entirely in $\mathcal{T} \cup \mathcal{A}$. But this in turn implies that $r(u_0, v)$ itself is a nonincreasing function of v along the ray, and so $\sup_{v:(u_0,v)\in\mathcal{Q}^+} r(u_0, v) \leq r(u_0, v_0) < \infty$. From Proposition (2), we know that $\mathcal{I}^+$ is connected and has past limit point i^0, so we can conclude that no portion of it can extend into the causal future of the ray $\{u = u_0\}$, and hence $(u_0, v_0) \notin J^-(\mathcal{I}^+)$. This completes the proof. □

Chapter 4

EXAMPLES OF MARGINALLY TRAPPED TUBES

In our 2-dimensional quotient manifold $\mathcal{Q}^+$, the marginally trapped region $\mathcal{A}$ forms the boundary between the trapped and regular regions, $\mathcal{T}$ and $\mathcal{R}$, and since it is the zero set of $\partial_v r$, it is a hypersurface in $\mathcal{Q}^+$. Since each point $(u,v) \in \mathcal{A}$ has one negative expansion and one expansion identically zero, $\mathcal{A}$ is in fact (the quotient of) a marginally trapped tube (MTT). We saw in the previous section that, with the dominant energy condition imposed, the domain of outer communications $J^-(\mathcal{I}^+) \cap \mathcal{Q}^+$ lies in the regular region $\mathcal{R}$, so in particular this MTT $\mathcal{A}$ must lie within the black hole region. The examples and problems we will consider here concern the both the asymptotic behavior as well as the causal character of $\mathcal{A}$.

4.1 Schwarzschild spacetime

As mentioned earlier, the *Schwarzschild solution* describes the vacuum exterior gravitational field of a static, spherically symmetric body. In spherical coordinates (t,r,θ,ϕ), in which t corresponds to time, r is a radial coordinate, and θ and ϕ are the usual spherical coordinates on S^2, the metric takes the form

$$g = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 g_{S^2}, \tag{4.1}$$

where M is a constant (typically interpreted as the mass of the spherically symmetric body), and g_{S^2} is the usual round metric on S^2. Notice that as $r \to \infty$, g tends to $-dt^2 + dr^2 + g_{S^2}$, the Minkowski metric, so g is indeed asymptotically flat.

In the vacuum Schwarzschild spacetime, one finds the simplest possible configuration of the regions $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{A}$. The regular region $\mathcal{R}$ coincides exactly with the domain of outer communications, the trapped region $\mathcal{T}$ is exactly the black hole region, and the marginally trapped tube $\mathcal{A}$ is the event horizon. To see this, let us find double-null coordinates u and v for the Schwarzschild metric. We begin by defining the so-called Eddington-Finkelstein tortoise coordinate,

$$r^* = r + 2M \ln\left|\frac{r}{2M} - 1\right|, \tag{4.2}$$

and then set $u = t - r^*$ and $v = t + r^*$. One easily computes that

$$-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 = -\left(1 - \frac{2M}{r}\right) du\, dv.$$

Here it appears as though the metric will become singular as r approaches $2M$, but the apparent singularity is just an artifact of the choice of coordinates; in the Eddington-Finkelstein coordinates (v, r, θ, ϕ), the metric instead becomes

$$g = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2 g_{S^2},$$

and it is clear that there is no true metric singularity at $r = 2M$.

Setting $r^* = r^*(u, v)$, on the one hand we compute from (4.2) that

$$\partial_v r^* = \left(1 - \frac{2M}{r}\right)^{-1} \partial_v r,$$

and on the other hand, since $t = u + r^*$, we have $v = u + 2r^*$, and hence

$$\partial_v r^* = \frac{1}{2}.$$

Thus

$$\partial_v r = \frac{1}{2}\left(1 - \frac{2M}{r}\right),$$

and using the Eddington-Finkelstein coordinates (v, r) on $\mathcal{Q}^+$, we have

$$\mathcal{R} = \{(v, r) : r > 2M\},$$

$$\mathcal{T} = \{(v, r) : r < 2M\},$$

and

$$\mathcal{A} = \{(v, r) : r = 2M\}.$$

Using these coordinates, it is not easy to actually compute the location of the black hole in the sense of Section 3.3. It is well established, however, that the event horizon $\mathcal{H}$ of the Schwarzschild solution is precisely the null hypersurface $\{(v, r) : r = 2M\}$ and the black hole is $\mathcal{B} = \{(v, r) : r < 2M\}$ (see, e.g., [22]). Thus we see that, in this spacetime, the marginally trapped tube is a non-expanding horizon coinciding with the event horizon, $\mathcal{A} = \mathcal{H}$, and the trapped region is precisely the black hole, $\mathcal{T} = \mathcal{B}$.

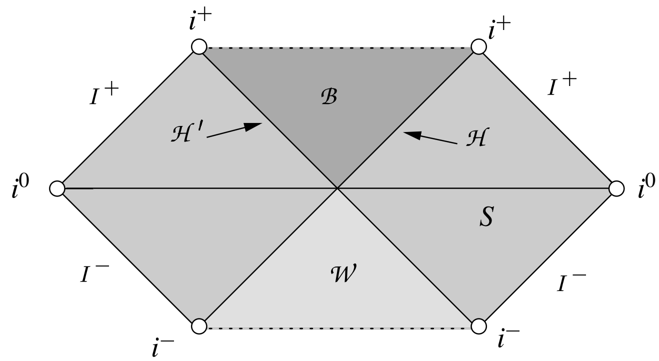

Figure 4.1: *The fully extended Schwarzschild spacetime with Cauchy surface S. The extension has two asymptotically flat ends, each containing an event horizon for the black hole $\mathcal{B}$. The region labeled $\mathcal{W}$ is a* white hole*, which has exactly the time-reversed properties of $\mathcal{B}$.*

4.2 Vaidya spacetimes

The simplest example of a black hole spacetime in which one sees an MTT which does not coincide with the event horizon is in the *Vaidya spacetime*. This spacetime is a generalization of Schwarzschild which models null dust in spherical symmetry. The metric is given in coordinates (v, r, θ, ϕ) by

$$g = -\left(1 - \frac{2M(v)}{r}\right) dv^2 + 2\,dvdr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right),$$

for some smooth function $M(v)$. Then g satisfies the Einstein field equations with stress-energy tensor

$$T = \frac{\dot{M}(v)}{4\pi r^2}\, dv^2,$$

and one can check that the dominant energy condition is satisfied if and only if $\dot{M}(v) \geq 0$. Furthermore, if $M(v)$ is constant, $M(v) \equiv M$, then g reduces to the usual Schwarzschild metric. Now, it is not possible to transform the metric from these so-called radiation coordinates (v, r) into double-null coordinates for general functions $M(v)$ [23]. But we can still obtain information about the regular, marginally trapped, and trapped regions of the spacetime by working with the radiation coordinates. In particular, we compute that each 2-sphere (v, r) has future directed null vector fields

$$\ell = \partial_v + \left(\frac{1}{2} - \frac{M(v)}{r}\right)\partial_r \qquad \text{(the outer null normal)},$$

and

$$n = -\partial_r \qquad \text{(the inner null normal).}$$

We compute further that

$$\theta_{(\ell)} = \theta_+ = \frac{1}{r} - \frac{2M(v)}{r^2}$$

and

$$\theta_{(n)} = \theta_- = -\frac{2}{r}.$$

Thus, in the quotient manifold with metric $g = -\left(1 - \frac{2M(v)}{r}\right) dv^2 + 2\,dvdr$, we have

$$\mathcal{R} = \{(v,r) : r > 2M(v)\},$$

$$\mathcal{T} = \{(v,r) : r < 2M(v)\},$$

and

$$\mathcal{A} = \{(v,r) : r = 2M(v)\}.$$

The induced metric on $\mathcal{A}$ is

$$h = 2\,drdv = 2\,(2\dot{M}(v)dv)\,dv = 4\dot{M}(v)\,dv^2,$$

so $\mathcal{A}$ is a null hypersurface wherever $\dot{M}(v) = 0$ and spacelike wherever $\dot{M}(v) > 0$, i.e. a non-expanding horizon or a dynamical horizon, respectively.

The cases we are interested in are those in which $M(v)$ is zero until some finite v, say $v = 0$, and then grows monotonically, either reaching an asymptotic value M_0 as v approaches ∞, or reaching this value at some finite value $v = v_0$ and remaining constant thereafter. Each of these two cases is depicted in Figure 4.2. In particular, the former case is the marginally trapped tube $\mathcal{A}$ is a dynamical horizon, thought to be asymptotic to the event horizon; in the latter case, $\mathcal{A}$ is a dynamical horizon up until $v = v_0$, after which it becomes a non-expanding horizon, thought to coincide with the event horizon [4]. In Section 5.4.2, we show as an application of the main theorem that the marginally trapped tube does indeed have the asymptotic behavior ascribed to it in the former case.

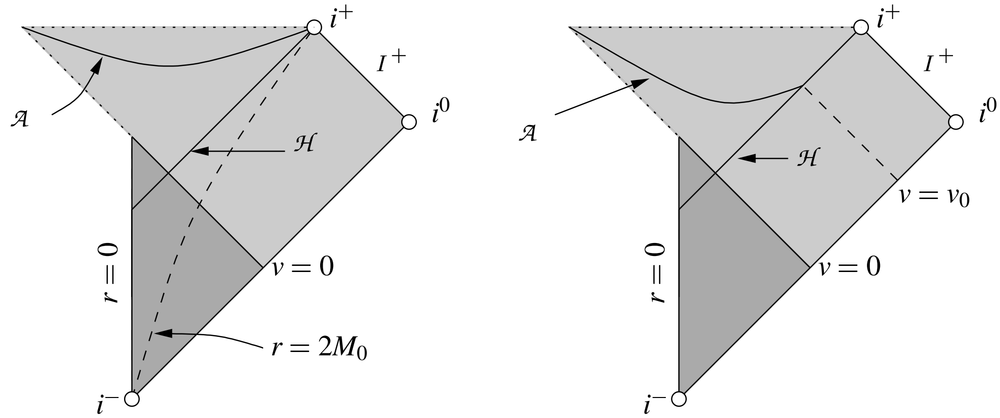

Figure 4.2: *Penrose diagrams for the Vaidya spacetime in which $M(v) \equiv 0$ for $v \leq 0$. In both panels, the region to the past of $v = 0$ is flat. In the left panel, as v tends to infinity, M tends to a constant value M_0; the spacelike marginally trapped tube, the null event horizon H and the timelike surface $r = 2M_0$ all meet tangentially at i^+. In the right panel, $M \equiv M_0$ for $v \geq v_0$, and the spacetime to the future of v_0 is isometric with a portion of the Schwarzschild spacetime; the marginally trapped tube and the event horizon H meet tangentially at $v = v_0$.*

4.3 Spherically symmetric Einstein-Maxwell-scalar field spacetimes

Spherically symmetric spacetimes in which gravity is coupled with electromagnetism alone are given by a 2-parameter family of metrics known as the *Reisner-Nordström* solutions; in these spacetimes, as in the Schwarzschild solution, the MTTs always coincide with the event horizon. In spherically symmetric Einstein-scalar field spacetimes, the work of Christodoulou shows that the MTT will always be achronal and either coincide with or be asymptotic to the event horizon [8]. A more sophisticated example of MTT behavior arises when one couples gravity with electromagnetism *and* a scalar field. In general, a solution to the Einstein-Maxwell-scalar field equations is a 4-tuple $(M, g, F_{\alpha\beta}, \phi)$, where (M, g) is a Lorentzian 4-manifold, the electromagnetic field $F_{\alpha\beta}$ is an anti-symmetric 2-tensor satisfying the source-free Maxwell equations

$$F_{\alpha\beta}^{;\beta} = 0, \;\; F_{[\alpha\beta,\gamma]} = 0, \tag{4.3}$$

the (massless) scalar field ϕ is a smooth function satisfying

$$g^{\alpha\beta}\phi_{;\alpha\beta} = 0, \tag{4.4}$$

and the Einstein field equations (3.3) are satisfied with stress-energy tensor

$$T_{\alpha\beta} = \phi_{;\alpha}\phi_{;\beta} - \tfrac{1}{2}g_{\alpha\beta}\phi^{;\gamma}\phi_{;\gamma} + F_{\alpha\gamma}F^{\gamma}_{\beta} - \tfrac{1}{4}g_{\alpha\beta}F^{\gamma}_{\delta}F^{\delta}_{\gamma}.$$

To impose spherical symmetry on the system, one requires not only that (M,g) satisfy all the assumptions made in Section 3, but also that F and ϕ be invariant under the $SO(3)$ action.

A careful analysis of the interiors of black holes in Einstein-Maxwell-scalar field spacetimes was undertaken by Mihalis Dafermos in [10] and [11]. In [11], Dafermos solves a spherically symmetric, double characteristic initial value problem for the Einstein-Maxwell-scalar field equations, prescribing the initial data in such a way that, in the maximal development of the data, the "tail" of the massless scalar field ϕ decays at a certain rate with respect to an outgoing null coordinate v along the event horizon, namely

$$|\partial_v\phi| \leq Cv^{-1-\varepsilon} \tag{4.5}$$

for v large, $\varepsilon > 0$ arbitrarily small. Then, as part of the proof of a much broader theorem, he shows that the marginally trapped tube $\mathcal{A}$ is achronal with no ingoing null components and that it terminates at i^+ (the future limit point of $\mathcal{I}^+$); in particular, $\mathcal{A}$ is composed of segments of dynamical and non-expanding horizons and is asymptotic to the event horizon.

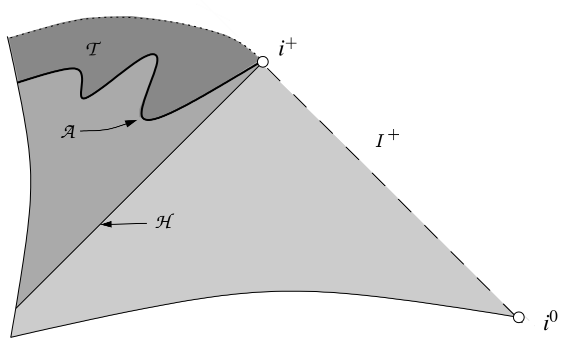

Figure 4.3: *A marginally trapped tube which eventually becomes achronal and is asymptotic to the event horizon* $\mathcal{H}$.

Furthermore, in [16], Dafermos and collaborator Igor Rodnianski show that, given any asymptotically flat, spacelike, spherically symmetric initial data for the Einstein-Maxwell-scalar field equations such that the scalar field ϕ and its gradient have compact support along the initial hypersurface, if the 2-dimensional Lorentzian quotient manifold $\mathcal{Q}^+$ of its future Cauchy development contains at least one trapped surface, then either "the black hole is extremal in the limit" or

$$|\partial_v \phi| \leq C v^{-3-\varepsilon} \tag{4.6}$$

for large v along the event horizon, where again v is a naturally defined outgoing null coordinate and $\varepsilon > 0$ is arbitrarily small. The former statement means that the quantity $m + \frac{e^2}{2r} - |e| \to 0$ as $v \to \infty$ along the event horizon, where the constant e is the charge of the electromagnetic field $F_{\alpha\beta}$. Since it is generally thought that $e << m$ in astrophysically reasonable situations, such a scenario is in some sense unphysical. The decay rate in (4.6) is widely known as *Price's law* [20].

Clearly (4.6) is considerably stronger than (4.5), so we may combine these two results and conclude that in general, an MTT in an Einstein-Maxwell-scalar field spacetime *must* be asymptotic to the event horizon of the black hole, at least as long as the latter is non-extremal as described above.

Chapter 5

ASYMPTOTIC BEHAVIOR OF MTTS WITH ARBITRARY MATTER

In this chapter, we give conditions on a *general* stress-energy tensor $T_{\alpha\beta}$ in a spherically symmetric black hole spacetime which are sufficient to guarantee both that the black hole will contain a (spherically symmetric) marginally trapped tube and that that marginally trapped tube will be achronal, connected, and asymptotic to the event horizon. We then derive some additional results pertaining to the affine lengths of both the black hole event horizon and the marginally trapped tube and show how our main result can be applied in Vaidya spacetimes.

As described in Chapter 3, it is both natural and convenient to formulate and prove these statements about spherically symmetric spacetimes at the level of the 1+1 Lorentzian manifold obtained by taking a quotient of the $SO(3)$-action by isometries. In particular, we restrict ourselves to a characteristic rectangle in Minkowski 2-space with conformal metric and past boundary data constrained in such a way that the rectangle could indeed lie inside the quotient of a spherically symmetric black hole spacetime, with one of its edges coinciding with the event horizon. In order to make this regime both generic and physical, we assume that our spacetime is the maximal future development of some initial data for the metric and the stress-energy tensor prescribed along the two past edges of this characteristic rectangle. We use no explicit evolution equations for $T_{\alpha\beta}$, but we assume that one or more exist and that the resulting $T_{\alpha\beta}$ satisfies the dominant energy condition throughout the maximal development. We also impose a nontrivial extension principle, one which arises in the evolutionary setting for many physically reasonable matter models. Our conditions then take the form of four inequalities which must hold near a point which we call future timelike infinity and denote by i^+. The inequalities relate components of the stress-energy tensor to the conformal factor and radial function for the metric.

It is worth mentioning that the conditions we impose on $T_{\alpha\beta}$ do not directly include or imply Price's law. Originally formulated as an estimate of the decay of radiation tails of massless scalar fields in the exterior of a black hole [20], the appellation "Price's law" is now widely used to refer to inverse power decay of any black hole "hair" along the event horizon itself; see Section 4.3. As noted there, in addressing the double characteristic initial value problem for the Einstein-Maxwell-scalar field equations, Dafermos showed that imposing a weak version of Price law decay on data along an outgoing characteristic yields a maximal future development which does indeed contain an achronal marginally trapped tube asymptotic to the event horizon. Consequently, one might have expected such decay to be central for obtaining the same result in the general setting. In what follows, however, we show that the analogous decay of T_{vv} (v an outgoing null coordinate) is only *a priori* related to the length of the marginally trapped tube, not its terminus. The conditions we use instead to control the tube's asymptotic behavior entail only smallness and monotonicity of certain quantities.

However, it appears that some sort of decay is always necessary in order to retrieve our conditions in practice. Indeed, in the self-gravitating Higgs field setting, our conditions follow rather naturally from the assumption of weak Price-law-like decay on the derivatives of the scalar field and the potential (Theorem 3), exactly analogously to Dafermos' result for Einstein-Maxwell-scalar fields. On the other hand, in Theorem 4, we are able to derive these conditions without making use of an explicit decay rate, instead using only smallness and monotonicity, and indeed one can construct examples which satisfy our conditions but violate even the weak version of Price's law. Still, the specific monotonicity assumptions are themselves quite strong and do imply decay, if not that which is specifically called Price's law.

5.1 First assumptions

5.1.1 Spherical symmetry & the initial value problem

Recall that the study of spherically symmetric 3+1-dimensional spacetimes is essentially equivalent to the study of conformal metrics on subsets of $(\mathbb{R}^2, \eta)$, and the relative simplicity of the latter recommends it as a starting point. Without *a priori* knowledge about an "upstairs" spacetime (M, g) or the embedding $\mathcal{Q} \hookrightarrow \mathbb{R}^2$, it is natural to begin with a generalized initial value problem for the system (3.8)-(3.11).

First, for any values $u, v > 0$, we use $K(u, v)$ to denote the characteristic rectangle given by

$$K(u,v) = [0,u] \times [v, \infty).$$

Next, suppose we choose some values $u_0, v_0 > 0$, *fix* the specific rectangle $K(u_0, v_0)$, and define initial hypersurfaces $\mathcal{C}_{in} = [0, u_0] \times \{v_0\}$ and $\mathcal{C}_{out} = \{u_0\} \times [v_0, \infty)$. These are the initial hypersurfaces along which to prescribe initial data for the metric, namely r and Ω (or m) and their derivatives. In a specific matter model, one would specify exactly how to prescribe these data, but since we are working in the most general case, we simply assume that this has been done in such a way that the four equations (3.8)-(3.11) are satisfied and $r, \Omega > 0$ on $\mathcal{C}_{in} \cup \mathcal{C}_{out}$. Next, specifying just the initial data for $T_{\alpha\beta}$ is not quite sufficient for our purposes, since we are working with a general stress-energy tensor with no evolution equations of its own beyond those imposed by the Bianchi identity, $\mathrm{div}_g T = 0$. We therefore assume not only that the functions T_{uu}, T_{uv}, and T_{vv} have been prescribed along $\mathcal{C}_{in} \cup \mathcal{C}_{out}$, but also that there is some field equation governing their evolution into the interior of $K(u_0, v_0)$. We then assume that we obtain the maximal future development with respect to the system (3.8)-(3.11), $\mathcal{G}(u_0, v_0) \subset K(u_0, v_0)$. (Again, if we were working in the context of a specific matter model, we would cite (or prove!) a global existence result here. But in our situation, we must confine ourselves to assuming one.) The spacetime $(\mathcal{G}(u_0, v_0), -\Omega^2 du dv)$ is the one we are interested in; see Figure 5.1 for a Penrose diagram.

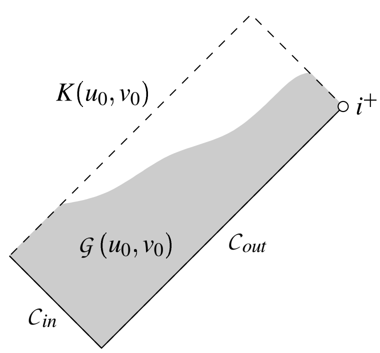

Figure 5.1: *The characteristic rectangle $K(u_0, v_0)$ contains the maximal future development $\mathcal{G}(u_0, v_0)$ of initial data prescribed on $\mathcal{C}_{in} \cup \mathcal{C}_{out}$.*

5.1.2 Black hole & energy assumptions

In this section, we make a number of assumptions to which we will refer later, namely in the statement and proof of our main result. These assumptions are the basic requirements that our stress-energy tensor components and the initial data and their maximal development must satisfy in order to be physically reasonable and relevant to black hole spacetimes. We label them here with upper-case Roman numerals for convenience.

First, on physical grounds, we want the "upstairs" stress-energy tensor $T_{\alpha\beta}$ to satisfy the dominant energy condition. In the 1+1-setting, this condition yields the following pointwise inequalities at the quotient level:

I $$T_{uu} \geq 0, \;\; T_{uv} \geq 0, \;\; \text{and} \;\; T_{vv} \geq 0.$$

Second, because we are not working with a specific global existence result in an evolutionary setting, we explicitly require that the maximal development $\mathcal{G}(u_0, v_0)$ obtained in the previous section be a past subset of $K(u_0, v_0)$, i.e.

II $$J^{-}(\mathcal{G}(u_0, v_0)) \subset \mathcal{G}(u_0, v_0).$$

Next, we assume that along $\mathcal{C}_{out}$ the functions r and m satisfy

III $$r \leq r_{+},$$

and

IV $$0 \leq m \leq m_{+},$$

where the constants r_{+}, $m_{+} < \infty$ are chosen to be the respective suprema of r and m along $\mathcal{C}_{out}$. Since our aim is to say something about the interiors of spherically symmetric black holes, we want to choose the data on the initial hypersurface $\mathcal{C}_{out}$ of our characteristic rectangle $K(u_0, v_0)$ to insure that they would agree with that we would find along (the quotient of) the event horizon in a general spherically symmetric black hole spacetime, and assumption III provides this correspondence. In particular, the boundedness of r along $\mathcal{C}_{out}$ is precisely the requirement that $\mathcal{C}_{out}$ must satisfy in order to lie inside a black hole in (the quotient of) an asymptotically flat spacetime, at least provided the black hole has bounded surface area (or equivalently, bounded entropy); see Section 3.3. Since $\mathcal{C}_{out}$ is necessarily defined for arbitrarily large values of v, while outgoing null rays past the event horizon need not be, it is natural to interpret $\mathcal{C}_{out}$ as lying along the event horizon itself. For this reason we will often refer to its terminal point $(0, \infty)$ — which is not in the spacetime, strictly speaking — as

i^+, future timelike infinity. The first inequality of IV is physically natural, since it just requires that the quasi-local mass m be nonnegative, while the second inequality is actually slightly redundant: given equation (3.2), the boundedness of m along $\mathcal{C}_{out}$ follows immediately from the fact that it is nonnegative and that r is bounded. Indeed, we must have the relation $m_+ \leq \frac{1}{2}r_+$.

We next impose the assumption that no anti-trapped surfaces are present initially (cf. Section 3.3):

V $$\partial_u r < 0 \text{ along } \mathcal{C}_{out}.$$

Proposition 1 then says that $\mathcal{G}(u_0, v_0) = \mathcal{R} \cup \mathcal{T} \cup \mathcal{A}$; see Figure 5.2 for a representative Penrose diagram. Assumption V thus guarantees that a spacelike marginally trapped tube in $\mathcal{G}(u_0, v_0)$ is indeed a dynamical horizon as defined in Section 2.4, since that definition requires both that $\theta_+ = 0$ and $\theta_- < 0$ along $\mathcal{A}$.

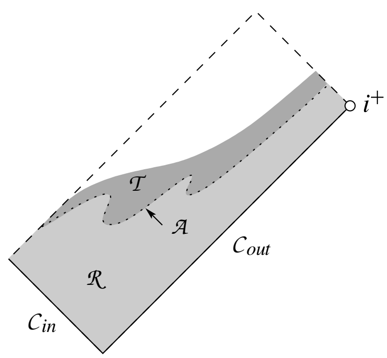

Figure 5.2: *The spacetime $\mathcal{G}(u_0, v_0)$ comprises the regular and trapped regions, $\mathcal{R}$ and $\mathcal{T}$, and a marginally trapped tube $\mathcal{A}$, shown here as a dotted curve. Note that the marginally trapped tube shown is not achronal, but it does comply with Proposition 4.*

Before continuing, we note that a consequence of the dominant energy condition (assumption I), the Einstein equations (3.8)-(3.11), and our definitions of $\mathcal{A}$ and $\mathcal{T}$ is the following proposition due to Christodoulou, which will be of considerable use later on:

Proposition 4. *If $(u, v) \in \mathcal{T} \cup \mathcal{A}$, then $(u, v^*) \in \mathcal{T} \cup \mathcal{A}$ for all $v^* > v$. Similarly, if $(u, v) \in \mathcal{T}$, then $(u, v^*) \in \mathcal{T}$ for all $v^* > v$.*

Proof. Integrating (3.9) along the null ray to the future of (u,v) yields

$$(\Omega^{-2}\partial_v r)(u,v^*) = (\Omega^{-2}\partial_v r)(u,v) - \int_v^{v^*} r\Omega^{-2}T_{vv}(u,V)\,dV$$

for $v^* > v$. Since $T_{vv} \geq 0$ everywhere by assumption I, the righthand side of this equation will be nonpositive if $(\Omega^{-2}\partial_v r)(u,v) \leq 0$, and strictly negative if $(\Omega^{-2}\partial_v r)(u,v) < 0$. Since $\Omega > 0$ everywhere, both statements of the proposition now follow immediately. □

In a black hole spacetime, the trapped region $\mathcal{T}$ must be contained inside the black hole. Since we would like $\mathcal{C}_{out}$ to represent an event horizon, we must therefore require that $\partial_v r \geq 0$ along $\mathcal{C}_{out}$. Combining this inequality with Proposition 4, we see that if $\mathcal{A}$ intersects $\mathcal{C}_{out}$ at a single point, then the two must in fact coincide to the future of that point. This is indeed the case in the Schwarzschild and Reisner-Nordström spacetimes, in which the black hole coincides exactly with the trapped region $\mathcal{T}$. However, we are really only interested in the cases in which the marginally trapped tube $\mathcal{A}$ does *not* coincide with the event horizon, so we will instead assume

VI $$0 < \partial_v r \text{ along } \mathcal{C}_{out}.$$

Note that assumptions II and VI and the fact that $r > 0$ on $\mathcal{C}_{in} \cup \mathcal{C}_{out}$ now together imply that $r > 0$ everywhere in $\mathcal{G}(u_0, v_0)$. Furthermore, it is now clear that the values r_+ and m_+ specified in assumptions III and IV are in fact the asymptotic values of r and m along $\mathcal{C}_{out}$, respectively (the monotonicity of m follows from assumption I and equation (3.11)).

Finally, we will need to assume that $\mathcal{G}(u_0, v_0)$ satisfies the extension principle formulated in [13]. This principle holds for self-gravitating Higgs (scalar) fields and self-gravitating collisionless matter [12, 14], and it is expected to hold for a number of other physically reasonable models [13]. Let Γ denote the center of symmetry of $\mathcal{G}(u_0, v_0)$, the set of points p at which $r(p) = 0$, and regard set closures as being taken with respect to the topology of $K(u_0, v_0)$. Then the extension principle may be formulated as follows:

VII If $p \in \overline{\mathcal{R}} \backslash \overline{\Gamma}$, and $q \in \overline{\mathcal{R}} \cap I^-(p)$ such that $J^-(p) \cap J^+(q) \backslash \{p\} \subset \mathcal{R} \cup \mathcal{A}$, then $p \in \mathcal{R} \cup \mathcal{A}$.

5.2 Achronality & connectedness

The proof of our main result, Theorem 1, which appears in Section 5.3.1, relies on two more general propositions, both of which are of interest in their own right. These propositions require a weak

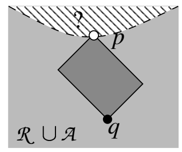

Figure 5.3a: *A priori p could be in* $\overline{\mathcal{G}(u_0,v_0)} \setminus \mathcal{G}(u_0,v_0)\ldots$

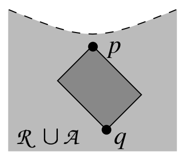

Figure 5.3b: … *but in fact the extension principle says that p must lie in* $\mathcal{G}(u_0,v_0)$.

version of one of the four conditions appearing in Theorem 1, a condition which we now state separately:

A $$T_{uv}\Omega^{-2} < \frac{1}{4r^2}.$$

In practice, we will only require that condition A hold in some small subset of our spacetime $\mathcal{G}(u_0,v_0)$. The expression $T_{uv}\Omega^{-2}$ takes a particularly simple form in many matter models. For a perfect fluid of pressure P and energy density ρ, it is the quantity $\frac{1}{4}(\rho - P)$. For a self-gravitating Higgs field ϕ with potential $V(\phi)$, it is $\frac{1}{2}V(\phi)$. And for an Einstein-Maxwell massless scalar field of charge e, it is $\frac{1}{4}e^2r^{-4}$.

Proposition 5. *Suppose* $(\mathcal{G}(u_0,v_0), -\Omega^2\,du\,dv)$ *is a spacetime obtained as in Section 5.1.1 with radial function r, and suppose it satisfies assumptions* I-VII *of Section 5.1.2. If* $\mathcal{A}$ *is nonempty and condition* A *holds in* $\mathcal{A}$*, then each of its connected components is achronal with no ingoing null segments.*

Remark: In [6], Booth, et al. give a necessary and sufficient condition for a general marginally trapped tube (not necessarily spherically symmetric) to be achronal; that condition is precisely A in our setting. The proof of Proposition 5 essentially duplicates their reasoning, although it is formulated somewhat differently.

Proof. To begin, we must establish that $\mathcal{A}$ is in fact a hypersurface in $\mathcal{G}(u_0, v_0)$. Since $\mathcal{A}$ is defined as a level set, this is equivalent to showing that 0 is a regular value of $\partial_v r$, i.e. that the differential $D(\partial_v r)$ is non-degenerate at points where $\partial_v r = 0$. Since $D(\partial_v r)$ has components $\partial^2_{uv} r$ and $\partial^2_{vv} r$, it suffices to show that $\partial^2_{uv} r < 0$ along $\mathcal{A}$.

Rearranging equation (3.10) and then combining with equations (3.8) and (3.2) yields

$$\begin{aligned} 2r^2\Omega^{-2}T_{uv}\partial_u r &= 2r^2\Omega^{-2}T_{uu}\partial_v r + \partial_u m \\ &= \tfrac{1}{2}(\partial_u r) + 2(\partial_u r)^2(\partial_v r)\Omega^{-2} + 2r(\partial^2_{uv} r)\Omega^{-2}\,\partial_u r, \end{aligned}$$

and solving for $\partial^2_{uv} r$, we obtain

$$\partial^2_{uv} r = -\tfrac{1}{2}\Omega^2 r^{-2}\alpha, \tag{5.1}$$

where α is given by

$$\alpha = m - 2r^3\,\Omega^{-2}T_{uv}. \tag{5.2}$$

Since r and Ω are strictly positive in $\mathcal{G}(u_0, v_0)$, it is enough to show that $\alpha > 0$ along $\mathcal{A}$. Using condition A and the fact that $\partial_v r = 0$ on $\mathcal{A}$, we have

$$\begin{aligned} \alpha &= m - 2r^3\,\Omega^{-2}T_{uv} \\ &= \tfrac{r}{2} - 2r^3\,\Omega^{-2}T_{uv} \\ &= \tfrac{r^3}{2}\left(\tfrac{1}{r^2} - 4\,\Omega^{-2}T_{uv}\right) \\ &> 0, \end{aligned}$$

so $\partial^2_{uv} r < 0$ along $\mathcal{A}$ as desired.

Now, we have $\mathcal{A} \neq \emptyset$, and since we now know it is a 1-dimensional submanifold of the spacetime, we can parameterize some connected component by a curve $\gamma(t) = (u(t), v(t))$. Since $\partial_v r \equiv 0$ along $\mathcal{A}$, at points on $\mathcal{A}$ we have

$$\dot{\gamma}(\partial_v r) = 0 = \frac{du}{dt}(\partial^2_{uv} r) + \frac{dv}{dt}(\partial^2_{vv} r). \tag{5.3}$$

By the result of Proposition 4, we know that we can describe all but the outgoing null segments of $\mathcal{A}$ in terms of a function $v(u)$, defined on some (possibly disconnected) subset of $[0, u_0]$. From equation (5.3) we see that its slope is given by

$$\frac{dv}{du} = \frac{dv/dt}{du/dt} = -\frac{\partial^2_{uv} r}{\partial^2_{vv} r} \tag{5.4}$$

at points where $\partial^2_{vv} r \neq 0$.

Showing that $\mathcal{A}$ is achronal thus amounts to showing that this slope $\frac{dv}{du} \leq 0$ wherever it is defined, since the points at which the slope is not defined correspond to points on outgoing null segments. In fact, we will show that, where it is defined, $\frac{dv}{du} < 0$, thereby excluding the possibility of ingoing null segments.

Expanding the lefthand side of (3.9), we see that along $\mathcal{A}$,

$$\Omega^{-2}\partial^2_{vv} r = -r\Omega^{-2} T_{vv},$$

or rather,

$$\partial^2_{vv} r = -rT_{vv}. \tag{5.5}$$

Substituting equations (5.5) and (5.1) into equation (5.4) then yields

$$\frac{dv}{du} = -\frac{\Omega^2 \alpha}{2r^3 T_{vv}}.$$

Since r, Ω, and α are all positive along $\mathcal{A}$ and T_{vv} is nonnegative by assumption I, the dominant energy condition, we conclude that $\frac{dv}{du} < 0$ at points along $\mathcal{A}$ at which $T_{vv} > 0$, which is exactly what was needed. □

For technical reasons, in the next proposition we use $\overline{K}(u_0, v_0)$ to denote the "compactification" of our initial rectangle $K(u_0, v_0)$, that is, $\overline{K}(u_0, v_0) = [0, u_0] \times [v_0, \infty]$. Set closures are taken with respect to $\overline{K}$ rather than K so as to include points at infinity (the Cauchy horizon).

We also want to confine ourselves to regions of the spacetime in which r is close to r_+, so for any $\delta > 0$, let

$$\mathcal{W} = \mathcal{W}(\delta) = \{(u,v) \in \mathcal{G}(u_0, v_0) : r(u,v) \geq r_+ - \delta\}.$$

Proposition 6. *Suppose $(\mathcal{G}(u_0, v_0), -\Omega^2\, du\, dv)$ is a spacetime obtained as in Section 5.1.1 with radial function r, suppose it satisfies assumptions* I-VII *of Section 5.1.2, and suppose condition* A *is satisfied in $\mathcal{W}$. If $\mathcal{G}(u_0, v_0)$ does not contain a marginally trapped tube which is asymptotic to the event horizon, then $\mathcal{W} \cap \mathcal{R}$ contains a rectangle $K(u_1, v_1)$ for some $u_1 \in (0, u_0]$, $v_1 \in [v_0, \infty)$.*

Proof. The proposition is an immediate consequence of the following two lemmas:

Lemma 1. *Under the hypotheses of Proposition 6, if* $\mathcal{A} \cap \mathcal{W} \neq \emptyset$*, then it is connected and terminates either at* i^+ *(in which case it is asymptotic to the event horizon) or along the Cauchy horizon* $(0,u_0] \times \{\infty\} \subset \overline{K}(u_0,v_0)$*. In the latter case,* $\mathcal{W} \cap \mathcal{R}$ *contains a rectangle* $K(u_1,v_1)$ *for some* $u_1 \in (0,u_0]$*,* $v_1 \in [v_0,\infty)$*.*

Lemma 2. *Under the hypotheses of Proposition 6, if* $\mathcal{A} \cap \mathcal{W} = \emptyset$*, then* $\mathcal{W} \cap \mathcal{R}$ *contains a rectangle* $K(u_1,v_1)$ *for some* $u_1 \in (0,u_0]$*,* $v_1 \in [v_0,\infty)$*.*

Remark. The existence of the rectangle in $\mathcal{W} \cap \mathcal{R}$ is what is required for the proof of the main result, but the first statement of Lemma 1 establishing the connectedness of $\mathcal{A}$ is of independent interest. As we will see in the proof of the lemma, that result hinges on the extension principle (VII) and the achronality of $\mathcal{A}$.

Proof of Lemma 1. First we lay some groundwork. Let $\mathcal{S}$ denote any connected component of $\partial\mathcal{W} \cap \mathcal{G}(u_0,v_0)$, that is, a connected component of the level set $\{r = r_+ - \delta\}$. Since $\partial_u r$ is strictly negative by Proposition 1, the differential Dr is nondegenerate in all of $\mathcal{G}(u_0,v_0)$, and thus $\mathcal{S}$ is a smooth curve segment whose endpoints lie on $\partial\mathcal{G}(u_0,v_0)$. Parameterizing $\mathcal{S}$ by a curve $\gamma(t) = (u(t),v(t))$, we compute that

$$0 = \dot\gamma(r) = (\partial_u r)\tfrac{du}{dt} + (\partial_v r)\tfrac{dv}{dt}. \tag{5.6}$$

Now, $\partial_v r > 0$ in $\mathcal{R}$ and $\partial_u r < 0$ everywhere, so $\frac{du}{dt}$ and $\frac{dv}{dt}$ must have the same sign in $\mathcal{R}$, which in turn implies that

$$\langle\dot\gamma,\dot\gamma\rangle = -\Omega^2\, du\, dv\,(\dot\gamma,\dot\gamma) = -\Omega^2(\tfrac{du}{dt})(\tfrac{dv}{dt}) < 0$$

— that is, $\mathcal{S} \cap \mathcal{R}$ is timelike. Furthermore, equation (5.6) implies that if $\mathcal{S}$ intersects $\mathcal{A}$ and passes into $\mathcal{T}$, it is $\frac{du}{dt}$ that changes sign, while $\frac{dv}{dt}$ does not. Hence $\mathcal{S} \cap (\mathcal{R} \cup \mathcal{A})$ must be causal with no ingoing null segments.

Now let $\mathcal{S}_0$ denote the connected component of $\partial\mathcal{W} \cap \mathcal{G}(u_0,v_0)$ that intersects one of the initial hypersurfaces $\mathcal{C}_{in}$ or $\mathcal{C}_{out}$. (If no component intersects $\mathcal{C}_{in} \cup \mathcal{C}_{out}$, then without loss of generality, we may shrink δ until one does. And by the monotonicity of r along each initial hypersurface, we can be sure that there is at *most* one such component.) Then the above characterization of the causal behavior of $\partial\mathcal{W}$ implies that all of $\mathcal{S}_0$ lies to the future of this endpoint on $\mathcal{C}_{in}$ or $\mathcal{C}_{out}$. This past endpoint lies in $\mathcal{R}$ since both initial hypersurfaces do, so $\mathcal{S}_0 \cap (\mathcal{R} \cup \mathcal{A})$ is nonempty and, by the preceding argument, causal. Let q denote its future endpoint in $\overline{\mathcal{G}(u_0,v_0)}$. Then $J^+(q)\setminus\{q\}\cap\mathcal{W}=\emptyset$. To see this, observe that if $q\in\overline{\mathcal{G}(u_0,v_0)}\setminus\mathcal{G}(u_0,v_0)$, then the fact that $\mathcal{G}(u_0,v_0)$ is a past set (assumption II) in fact implies that $J^+(q)\cap\mathcal{G}(u_0,v_0)=\emptyset$. Otherwise, $q\in\mathcal{G}(u_0,v_0)$ and hence $q\in\mathcal{A}\cap\mathcal{S}_0$, so by the choice of q and Proposition 4, the outgoing null ray to the future of q must lie in the trapped region, i.e.

$$\{u(q)\}\times(v(q),\infty)\cap\mathcal{G}(u_0,v_0)\subset\mathcal{T}.$$

Since $r(q)=r_+-\delta$ and r is strictly decreasing along $\{u(q)\}\times[v(q),\infty)\cap\mathcal{G}(u_0,v_0)$,

$$\{u(q)\}\times(v(q),\infty)\cap\mathcal{G}(u_0,v_0)\cap\mathcal{W}=\emptyset.$$

Then the inequality $\partial_u r<0$ guarantees that $J^+(q)\setminus\{q\}\cap\mathcal{W}=\emptyset$.

Let $\mathcal{A}_0$ denote any connected component of $\mathcal{A}\cap\mathcal{W}$. We will show that it either terminates at i^+ or along the Cauchy horizon, after which the connectedness of $\mathcal{A}\cap\mathcal{W}$ and the statement of the lemma will follow. Since $\mathcal{A}_0$ is a connected curve segment, it has endpoints p_0 and p_1 in $\overline{\mathcal{G}(u_0,v_0)}$. By Proposition 5, $\mathcal{A}_0$ is achronal with no ingoing null segments, so if $p_0\neq p_1$, then one of p_0 and p_1 must have v-coordinate strictly larger than the other — without loss of generality, suppose it's $p_1=(u_*,v_*)$. (See Figure 5.4 for a Penrose diagram depicting $\mathcal{A}_0$ and $\mathcal{S}_0$.) Then for any point $(u,v)\in\mathcal{A}_0$, the achronality of $\mathcal{A}_0$ implies that $u\geq u_*$ and $v\leq v_*$. Our goal is to show that $v_*=\infty$, for then $\mathcal{A}_0$ terminates either at $i^+=(0,\infty)$ or along the Cauchy horizon $[0,u_0]\times\{\infty\}$.

Now, this point p_1 is either contained in the spacetime $\mathcal{G}(u_0,v_0)$ itself or in its boundary, $\overline{\mathcal{G}(u_0,v_0)}\setminus\mathcal{G}(u_0,v_0)$, and we must employ different arguments for each case. In the former case we will deduce that the point p_1 coincides with q, the future endpoint of the curve $\mathcal{S}_0\cap(\mathcal{R}\cup\mathcal{A})$, and from there derive a contradiction to how p_1 and q were chosen. In the latter case, we will use the extension principle, assumption VII, to show that if $v_*<\infty$, then p_1 is in the spacetime, a contradiction. Thus we will conclude that p_1 must indeed lie on the Cauchy horizon.

To begin, suppose that $p_1\in\mathcal{G}(u_0,v_0)$. Then $p_1\in\mathcal{A}_0\cap\partial\mathcal{W}$, and in particular, $r(p_1)=r_+-\delta$. Since $\overline{\mathcal{S}_0}$ is causal with no ingoing null segments, the v-coordinate of its endpoint q must be greater than that of any other point on $\mathcal{S}_0$, i.e. $v(q)>v(\tilde{q})$ for any $\tilde{q}\in\mathcal{S}_0$, $\tilde{q}\neq q$. Now, unless p_1 lies on $\overline{\mathcal{S}_0}$, p_1 must have v-coordinate greater than $v(q)$, i.e. $v_*\geq v(q)$ — otherwise, the fact that $r(p_1)=r_+-\delta$ and $r\equiv r_+-\delta$ along $\mathcal{S}_0$ would contradict assumptions V and VI, the strict monotonicity of r along ingoing null rays and along $\mathcal{C}_{out}$. Furthermore, $p_1\in\partial\mathcal{W}$ implies that $p_1\in\mathcal{W}$, which in turn

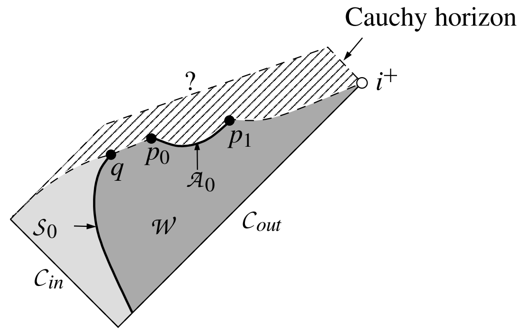

Figure 5.4: *A priori, the curves $\mathcal{S}_0$ and $\mathcal{A}_0$ used in the proof of Lemma 1 may be situated as shown. Their endpoints q, p_0 and p_1 may lie in the spacetime $\mathcal{G}(u_0,v_0)$ or in its boundary, $\overline{\mathcal{G}(u_0,v_0)} \setminus \mathcal{G}(u_0,v_0)$. The dashed curves and diagonal lines indicate boundaries and regions which also may or may not be part of the spacetime. Lemma 1 shows that the point p_1 must in fact lie along the Cauchy horizon.*

yields that if $v_* \geq v(q)$, then either $p_1 = q$ or $u_* < u(q)$ since $J^+(q) \setminus \{q\} \cap \mathcal{W} = \emptyset$. Thus the only possibilities remaining are that either $p_1 \in \overline{\mathcal{S}_0}$, or $u_* < u(q)$ and $v_* \geq v(q)$.

If $u_* < u(q)$ and $v_* \geq v(q)$, then since $p_1 \in \mathcal{A}$, the outgoing null ray to the past of p_1 must lie in $\mathcal{R} \cup \mathcal{A}$ by Proposition 4, but it must also contain some point $\tilde{q} \in \mathrm{int}(\mathcal{W})$. Then $r(\tilde{q}) > r_+ - \delta$, but $r(p_1) = r_+ - \delta$ and $\partial_v r \geq 0$ in $\mathcal{R} \cup \mathcal{A}$, a contradiction.

For the case $p_1 \in \overline{\mathcal{S}_0}$, first note that since $D(\partial_v r)$ is nondegenerate at p_1 (see the proof of Proposition 5), it must be the case that the curve $\mathcal{A}$ leaves $\mathcal{W}$ at p_1 as v increases. Together with the facts that $\mathcal{A}$ is achronal, $\mathcal{S}_0$ is causal, and p_1 was chosen to be the endpoint of $\mathcal{A}_0$ with the largest v-coordinate, this implies that $p_1 = q$.

We are now in a position to derive the contradiction for this case. First note that both curves $\mathcal{A}$ and $\partial\mathcal{W}$ must extend smoothly through $p_1 = q$. As noted in the preceding paragraph, by definition of $\mathcal{A}_0$ and p_1, the curve $\mathcal{A}$ leaves $\mathcal{W}$ at p_1 as v increases. On the other hand, by definition of $\mathcal{S}_0$ and the characterization of its causal behavior given above, the curve $\partial\mathcal{W}$ must leave $\mathcal{R} \cup \mathcal{A}$ at q as v increases, passing into $\mathcal{T}$; in particular, it must become spacelike for $v > v(q)$. And since $\partial\mathcal{W}$ is leaving $\mathcal{R} \cup \mathcal{A}$, this spacelike curve-continuation past $q = p_1$ must lie to the future of that of $\mathcal{A}$ at least locally. On the other hand, since $\mathcal{A}$ is leaving $\mathcal{R} \cup \mathcal{A}$ at $p_1 = q$, its continuation must lie to the future of that of $\mathcal{A}$, locally. But the two cannot coincide past $p_1 = q$, by the choices of both p_1 and

q, so we have arrived at a contradiction.

Thus p_1 cannot lie in $\mathcal{G}(u_0,v_0)$, so we must have $p_1 \in \overline{\mathcal{G}(u_0,v_0)} \setminus \mathcal{G}(u_0,v_0)$. If $v_* < \infty$, then consider the ingoing null ray to the past of p_1, $[0,u_*] \times \{v_*\}$. Since the point $(0,v_*) \in \mathcal{G}(u_0,v_0)$ and $\mathcal{G}(u_0,v_0)$ is open, there must exist some smallest $\tilde{u} \in (0,u_*]$ such that $(\tilde{u},v_*) \notin \mathcal{G}(u_0,v_0)$. Since p_1 is a limit point of $\mathcal{A}$ but is not in the spacetime, we must have $p_0 \neq p_1$, and so since $\mathcal{A}_0$ is achronal with no ingoing segments, we can parameterize a portion of $\mathcal{A}_0$ in a neighborhood of p_1 by $(u(v),v)$, $v \in (v_* - \varepsilon, v_*]$, some $\varepsilon > 0$. For each value $v \in (v_* - \varepsilon, v_*)$, the ingoing null ray to the past of the point $(u(v),v) \in \mathcal{A} \cap \mathcal{W}$ must be contained in $\mathcal{W} \cap \mathcal{R}$ — the ray must lie in $\mathcal{W}$ since $\partial_u r < 0$ along it, and it must lie in $\mathcal{R}$ since $\partial_u \partial_v r < 0$ in $\mathcal{W}$. So in particular, if we choose some $\tilde{v} \in (v_* - \varepsilon, v_*)$, then $J^-(\tilde{u},v_*) \cap J^+(0,\tilde{v}) \setminus \{(\tilde{u},v_*)\} \subset \mathcal{R} \cup \mathcal{A}$, and hence by the extension principle, we must have $(\tilde{u},v_*) \in \mathcal{R} \cup A$ as well, a contradiction. (See Figure 5.5 for a Penrose diagram of this situation.) Thus we conclude that $v_* = \infty$, and we are done; this is what we wanted to show.

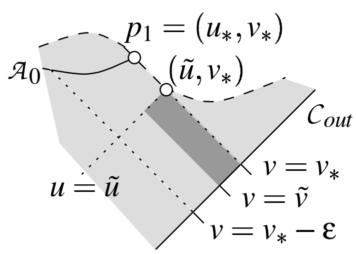

Figure 5.5: *The dashed line indicates the boundary of the spacetime,* $\overline{\mathcal{G}(u_0,v_0)} \setminus \mathcal{G}(u_0,v_0)$. *The points* $p_1 = (u_*,v_*)$ *and* $(\tilde{u},v_*)$ *lie in this boundary, not in* $\mathcal{G}(u_0,v_0)$ *itself. The dark-shaded rectangle is the set* $J^-(\tilde{u},v_*) \cap J^+(0,\tilde{v}) \setminus \{(\tilde{u},v_*)\}$ *to which we apply the extension principle (assumption* VII*) and derive a contradiction.*

Now we have shown that an arbitrary connected component of $\mathcal{A} \cap \mathcal{W}$ must terminate along the Cauchy horizon $[0,u_0] \times \{\infty\}$, so if $\mathcal{A} \cap \mathcal{W}$ is not connected, we can derive a contradiction as follows: Assume that there exist multiple connected components of $\mathcal{A} \cap \mathcal{W}$. Then since they each exist for arbitrarily large v, there must exist some $\tilde{v} > v_0$ such that $[0,u_0] \times \{\tilde{v}\} \cap \mathcal{A} \cap \mathcal{W}$ contains more than one point. If any two of these points are contained in the same component of $\mathcal{A} \cap \mathcal{W}$, we have a contradiction to the fact that that component is achronal with no ingoing null components. Otherwise, we have two or more connected components of $\mathcal{A} \cap \mathcal{W}$ whose other endpoints do not lie on the Cauchy horizon and hence must lie in $\overline{\mathcal{G}(u_0,v_0)} \setminus \mathcal{G}(u_0,v_0)$. Then since $\mathcal{G}(u_0,v_0)$ is a past set, there must also exist some $\tilde{u} > 0$ such that $\{\tilde{u}\} \times [v_0,\infty)$ intersects $\mathcal{A} \cap \mathcal{W}$ at more than one point, say at $(\tilde{u},v_1)$ and $(\tilde{u},v_2)$, which in turn implies that the ray segment between the two points lies entirely in $\mathcal{A}$ by Proposition 4. Furthermore, values of u sufficiently close to $\tilde{u}$ must also have this same property, which implies that $\mathcal{A}$ contains an open subset of $\mathcal{G}(u_0,v_0)$, contradicting the fact that it is codimension 1.

The last statement of the lemma follows immediately. □

Proof of Lemma 2. If $\mathcal{A} \cap \mathcal{W} = \emptyset$, then $\mathcal{W} \subset \mathcal{R}$, so in fact $\mathcal{W} \cap \mathcal{R} = \mathcal{W}$. Fix a reference point $(u_1,v_1) \in \mathrm{int}(\mathcal{W}) \cap \mathcal{R}$ such that $u_1 > 0$. Note that since $\partial_u r < 0$, the past-directed ingoing null ray behind (u_1,v_1) must also be in $\mathrm{int}(\mathcal{W})$, that is, $[0,u_1] \times \{v_1\} \subset \mathrm{int}(\mathcal{W})$.

If $K(u_1,v_1)$ is not wholly contained in $\mathcal{W}$, then there exists some $q \in (\partial\mathcal{W}) \cap K(u_1,v_1) \neq \emptyset$, where the boundary $\partial\mathcal{W}$ is taken here with respect to $K(u_0,v_0)$ (as opposed to $\mathcal{G}(u_0,v_0)$). In particular, q cannot lie on $[0,u_1] \times \{v_1\}$ by choice of (u_1,v_1), nor can it lie elsewhere in $\mathcal{W} \cap K(u_1,v_1)$, since that would imply that $r(q) = r_+ - \delta$, violating the monotonicity of r in $\mathcal{R}$ ($\partial_v r > 0$) and the fact that $r > r_+ - \delta$ on $[0,u_1] \times \{v_1\}$. Hence $q \in \overline{\mathcal{G}(u_0,v_0)} \setminus \mathcal{G}(u_0,v_0)$.

Let v_* be the smallest value in (v_1,∞) such that $[0,u_1] \times \{v_*\} \cap (\overline{\mathcal{G}(u_0,v_0)} \setminus \mathcal{G}(u_0,v_0)) \neq \emptyset$, and set u_* to be the smallest value in $(0,u_1]$ such that $(u_*,v_*) \in \overline{\mathcal{G}(u_0,v_0)} \setminus \mathcal{G}(u_0,v_0)$. Then by construction, the rectangle $J^-(u_*,v_*) \cap J^+(0,v_1) \setminus \{(u_*,v_*)\} \subset \mathcal{R}$, and so since r is bounded below by $r_+ - \delta$ near (u_*,v_*), the extension principle implies that $(u_*,v_*) \in \mathcal{G}(u_0,v_0)$, a contradiction.

Thus we must in fact have $(\partial\mathcal{W}) \cap K(u_1,v_1) = \emptyset$, which implies that $K(u_1,v_1) \subset \mathcal{W} \cap \mathcal{R}$. □

5.3 Main result

5.3.1 Asymptotic behavior

We are now ready to state and prove our main result characterizing the asymptotic behavior of certain marginally trapped tubes. The theorem has the immediate corollary that the event horizon of the given spacetime must be future geodesically complete. Afterward we present a second theorem relating the lengths of such tubes to Price law decay and indicate how both theorems apply to the ingoing Vaidya spacetime.

Theorem 1. *Suppose $(\mathcal{G}(u_0,v_0), -\Omega^2\,du\,dv)$ is a spacetime obtained as in Section 5.1.1 with radial function r, and suppose it satisfies the assumptions* I-VII *of Section 5.1.2. Define*

$$\mathcal{W}(\delta) = \{(u,v) \in \mathcal{G}(u_0,v_0) : r(u,v) \geq r_+ - \delta\}$$

and assume that there exist a constant $0 < c_0 < \frac{1}{4r_+^2}$, constants c_1, $c_2 > 0$, constants $0 < \varepsilon < \frac{1}{4r_+^2} - c_0$ and $v' \geq v_0$, and some small $\delta > 0$ such that for $\mathcal{W} = \mathcal{W}(\delta)$ the following conditions hold:

A′ $T_{uv}\Omega^{-2} \leq c_0$ *in* $\mathcal{W}$;

B1 $T_{uu}/(\partial_u r)^2 \leq c_1$ *in* $\mathcal{W} \cap \mathcal{R}$;

B2 $\partial_v(\Omega^{-2}T_{uv})(u,\cdot) \in L^1([v_0,\infty))$ *for all* $u \in [0,u_0]$, *and*

$\int_{v'}^{v} \partial_v(\Omega^{-2}T_{uv})(u,\tilde{v})\,d\tilde{v} < \varepsilon$ *for all* $(u,v) \in \mathcal{W} \cap \mathcal{R}$ *with* $v \geq v'$;

C $(-\partial_u r)\Omega^{-2} \leq c_2$ *along* $\mathcal{C}_{out} \cap \mathcal{W}$.

Then the spacetime $\mathcal{G}(u_0,v_0)$ contains a marginally trapped tube $\mathcal{A}$ which is asymptotic to the event horizon, i.e. for every small $u > 0$, there exists some $v > v_0$ such that $(u,v) \in \mathcal{A}$. Furthermore, for large v, $\mathcal{A}$ is connected and achronal with no ingoing null segments.

See Figure 5.6 for a representative Penrose diagram.

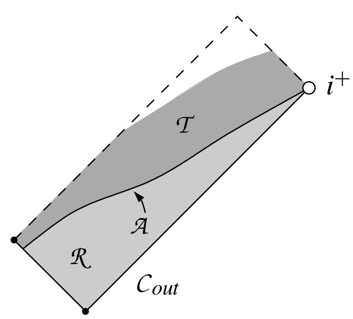

Figure 5.6: *Theorem 1 says that the spacetime must contain a (small) characteristic rectangle whose Penrose diagram looks like this – in particular, the marginally trapped tube $\mathcal{A}$ is achronal and terminates at i^+.*

Remarks: The physical meaning of condition A′, which is somewhat stronger than condition A, is readily apparent from Proposition 5: it controls the causal behavior of the marginally trapped tube, if one exists. Conditions B1 and B2 have no obvious general physical meaning. (But note that B2 is automatically satisfied if $\partial_v(\Omega^{-2}T_{uv}) \leq 0$ in $\mathcal{W} \cap \mathcal{R}$.) Condition C may alternately be expressed as saying that the quantities $(1-\frac{2m}{r})$ and $\partial_v r$ approach zero at proportional rates as v tends to infinity along $\mathcal{C}_{out}$. It also implies that $m \to \frac{1}{2}r$ along $\mathcal{C}_{out}$, i.e. that $2m_+ = r_+$.

These four conditions, as well as the proof of the theorem, were obtained by extrapolating portions of the bootstrap argument in Section 7 of [11] for Einstein-Maxwell scalar fields. Conditions A$'$, B2, and C are all satisfied for sufficiently small δ in *any* Einstein-Maxwell-scalar field black hole, provided $e < 2m_+$. (The particular choice of the upper bound for c_0 in A$'$ is analogous to the condition in [11] that the black hole not be "extremal in the limit," i.e. $e < 2m_+$; see Section 4.3.) Condition B1 also holds in all the spacetimes considered in [11], but there the proof hinges on the Price law decay imposed on T_{vv}; one integrates the scalar field equation by parts and uses the polynomial decay of T_{vv} in the v-direction to obtain the bound on $T_{uu}/(\partial_u r)^2$. (In fact one obtains something stronger than B1 this way, that $T_{uu}/(\partial_u r)^2$ decays polynomially with v.)

Proof. Our first step is to shrink δ in order to align with the choice of ε. For this, consider the quantity

$$\Lambda(\varepsilon^*,\delta^*) := \left(\frac{r_+-\delta^*}{r_+}\right)^3\left(\frac{1}{4r_+^2}\left(1-2\varepsilon^* c_2 e^{c_1 r_+ \delta^*}\right)-c_0\right)-\frac{\delta^*}{r_+}(2c_0+3M),$$

where

$$M := \sup_{(u,v)\in\mathcal{G}(u_0,v_0)} \int_{v_0}^{v} |\partial_v(\Omega^{-2}T_{uv})(u,\tilde{v})|\, d\tilde{v}.$$

Clearly Λ is positive for ε^* and δ^* sufficiently small and $\Lambda \nearrow \frac{1}{4r_+^2}-c_0$ as $\varepsilon^*,\delta^* \searrow 0$. Then since $\varepsilon < \frac{1}{4r_+^2}-c_0$, there exist ε_1, $\delta_1 > 0$ such that $\Lambda(\varepsilon_1,\delta_1) > \varepsilon$. Without loss of generality, we may assume that $\delta_1 \leq \delta$, and henceforth we restrict our attention to the (possibly) smaller region $\mathcal{W}(\delta_1)$, using $\mathcal{W}$ to denote it rather than $\mathcal{W}(\delta)$. We make use of ε_1 in what follows.

Now, by Proposition 6, either $\mathcal{W} \cap \mathcal{R}$ contains a rectangle $K(u_1,v_1)$ for some $u_1 \in (0,u_0]$, $v_1 \in [v_0,\infty)$, or the spacetime contains a marginally trapped tube $\mathcal{A}$ which is asymptotic to the event horizon. We will show that the existence of the rectangle in the former case leads to a contradiction and thus conclude that the latter statement is true. Furthermore, given how Proposition 6 was proved, we will then know that in fact it is $\mathcal{A} \cap \mathcal{W}$ which is asymptotic to the event horizon, so in particular $\mathcal{A}$

must lie in $\mathcal{W}$ for large v and hence must be achronal with no ingoing null segments (by Proposition 5, since condition A$'$ holds in $\mathcal{W}$) and connected (by Lemma 1), proving the theorem.

For the remainder of the proof, we restrict our attention to the region $K(u_1, v_1)$. (We may assume without loss of generality that $v_1 \geq v'$.) Define $\kappa = -\frac{1}{4}\Omega^2(\partial_u r)^{-1}$. Then $-\Omega^{-2}\partial_u r = \frac{1}{4\kappa}$. Now, using equation (3.8) and condition B1, we have

$$\begin{aligned} \partial_u \log(-\Omega^{-2}\partial_u r) &= -r(\partial_u r)^{-1} T_{uu} \\ &\leq -c_1 r(\partial_u r), \end{aligned}$$

so integrating along an ingoing null ray, we have

$$\begin{aligned} \log\left(\frac{\kappa(0,v)}{\kappa(u,v)}\right) &= \int_0^u \partial_u \log(-\Omega^{-2}\partial_u r)(\tilde{u}, v)d\tilde{u} \\ &\leq -\int_0^u c_1 r(\partial_u r)(\tilde{u}, v)d\tilde{u} \\ &= -\tfrac{c_1}{2}(r^2(u,v) - r^2(0,v)) \\ &\leq -\tfrac{c_1}{2}((r_+ - \delta_1)^2 - r_+^2) \\ &\leq c_1 r_+ \delta_1, \end{aligned}$$

which yields

$$\kappa(u,v) \geq \kappa(0,v)e^{-c_1 r_+ \delta_1}.$$

Then since condition C implies that $\kappa(0,v) \geq \frac{1}{4c_2}$ for all $v \geq v_1$, we have a lower bound

$$\kappa \geq \kappa_0 := \tfrac{1}{4c_2}e^{-c_1 r_+ \delta_1} > 0$$

in all of $K(u_1, v_1)$.

Next, since $r(0,v) \to r_+$ as $v \to \infty$, $\partial_v r$ cannot have a positive lower bound along $\mathcal{C}_{out}$; thus there must exist some $V \geq v_1$ such that $\partial_v r(0,V) < \varepsilon_1$. By continuity, there is a neighborhood of $(0,V)$ in $\mathcal{G}(u_0, v_0)$ in which this inequality holds, and in particular, there exists some $0 < U \leq u_1$ such that

$$\partial_v r(u,V) < \varepsilon_1 \text{ for all } 0 \leq u \leq U. \tag{5.7}$$

Now, since $\Lambda(\varepsilon_1, \delta_1) > \varepsilon$, we have

$$\frac{\delta_1}{r_+}(2c_0 + 3M) + \varepsilon < \left(\frac{r_+ - \delta_1}{r_+}\right)^3 \left(\frac{1}{4r_+^2}\left(1 - 2\varepsilon_1 c_2 e^{c_1 r_+ \delta_1}\right) - c_0\right),$$

or equivalently, setting $r_0 = r_+ - \delta_1$,

$$2\delta_1 r_+^2(2c_0+3M)+2r_+^3\varepsilon < \frac{r_0^3}{2}\left(\frac{1}{r_+^2}\left(1-\frac{\varepsilon_1}{2\kappa_0}\right)-4c_0\right).$$

We can thus fix constants α_0 and α_1 such that

$$2\delta_1 r_+^2(2c_0+3M)+2r_+^3\varepsilon < \alpha_0 < \frac{r_0^3}{2}\left(\frac{1}{r_+^2}\left(1-\frac{\varepsilon_1}{2\kappa_0}\right)-4c_0\right) \tag{5.8}$$

and

$$\alpha_1 = \alpha_0 - 2\delta_1 r_+^2(2c_0+3M) - 2r_+^3\varepsilon > 0. \tag{5.9}$$

As in the proof of Proposition 5, define a function α on $\mathcal{G}(u_0,v_0)$:

$$\alpha(u,v) = m - 2r^3\,\Omega^{-2}T_{uv}\,(u,v).$$

Using our lower bound for κ, (5.7), (5.8), and condition A$'$, we see that $\alpha > \alpha_0$ on $[0,U]\times\{V\}$:

$$\begin{aligned}
\alpha &= m - 2r^3\,\Omega^{-2}T_{uv} \\
&= \frac{r}{2}\left(1+4\Omega^{-2}\partial_u r\partial_v r\right) - 2r^3\,\Omega^{-2}T_{uv} \\
&= \frac{r^3}{2}\left(\frac{1}{r^2}\left(1-\frac{1}{2\kappa}\partial_v r\right)-4\,\Omega^{-2}T_{uv}\right) \\
&\geq \frac{r_0^3}{2}\left(\frac{1}{r_+^2}\left(1-\frac{\varepsilon_1}{2\kappa_0}\right)-4c_0\right) \\
&> \alpha_0.
\end{aligned}$$

Our goal now is to deduce that $\alpha > \alpha_1$ in $K(U,V)$, using B2. First we compute:

$$\begin{aligned}
\partial_v\alpha &= \partial_v(m-2r^3\,\Omega^{-2}T_{uv}) \\
&= \partial_v m - \partial_v(2r^3\,\Omega^{-2}T_{uv}) \\
&= 2r^2\Omega^{-2}(T_{uv}\partial_v r - T_{vv}\partial_u r) - 2r^3\partial_v(\Omega^{-2}T_{uv}) - 6r^2(\partial_v r)\Omega^{-2}T_{uv} \\
&= -4r^2\Omega^{-2}T_{uv}\partial_v r - 2r^2\Omega^{-2}T_{vv}\partial_u r - 2r^3\partial_v(\Omega^{-2}T_{uv}). \\
&\geq -4r^2\Omega^{-2}T_{uv}\partial_v r - 2r^3\partial_v(\Omega^{-2}T_{uv}),
\end{aligned} \tag{5.10}$$

where the last inequality follows from assumptions I and V. The next step is to integrate (5.10) along an outgoing null ray $\{u\}\times[V,v)$, but first let us consider the two summands on the right hand side separately. First,

$$\begin{aligned}\int_V^v -4r^2\Omega^{-2}T_{uv}\partial_v r &= \int_V^v -\tfrac{4}{3}(\partial_v r^3)(\Omega^{-2}T_{uv})\\ &\geq \int_V^v -\tfrac{4}{3}c_0(\partial_v r^3)\\ &> -\tfrac{4}{3}c_0(r_+^3-(r_+-\delta_1)^3)\\ &> -4c_0 r_+^2\delta_1.\end{aligned}$$

For the second summand of (5.10), we use the following notation: given a function f, $f^+ = \max\{f,0\}$ and $f^- = \max\{-f,0\}$, so that $f = f^+ - f^-$. Then:

$$\begin{aligned}\int_V^v 2r^3\partial_v(\Omega^{-2}T_{uv}) &= \int_V^v 2r^3[\partial_v(\Omega^{-2}T_{uv})]^+ - 2r^3[\partial_v(\Omega^{-2}T_{uv})]^-\\ &\leq \int_V^v 2r_+^3[\partial_v(\Omega^{-2}T_{uv})]^+ - 2(r_+-\delta_1)^3[\partial_v(\Omega^{-2}T_{uv})]^-\\ &= \int_V^v 2r_+^3(\partial_v(\Omega^{-2}T_{uv}))\\ &\quad + \int_V^v 2(3r_+^2\delta_1 - 3r_+\delta_1^2+\delta_1^3)[\partial_v(\Omega^{-2}T_{uv})]^-\\ &\leq 2r_+^3\varepsilon + 6r_+^2\delta_1\int_V^v[\partial_v(\Omega^{-2}T_{uv})]^-\\ &\leq 2r_+^3\varepsilon + 6r_+^2\delta_1\int_V^v|\partial_v(\Omega^{-2}T_{uv})|\\ &\leq 2r_+^3\varepsilon + 6r_+^2\delta_1 M.\end{aligned}$$

Integrating (5.10) now yields

$$\begin{aligned}\alpha(u,v) &> \alpha(u,V) - 4c_0r_+^2\delta_1 - (2r_+^3\varepsilon + 6r_+^2\delta_1 M)\\ &> \alpha_0 - 2r_+^3\varepsilon - 2r_+^2\delta_1(2c_0+3M)\\ &= \alpha_1.\end{aligned}$$

Thus we conclude that $\alpha > \alpha_1$ in all of $K(U,V)$.

Finally, recall from equation (5.1) that

$$\partial_{uv}^2 r = -\tfrac{1}{2}\Omega^2 r^{-2}\alpha,$$

or, using the definition of κ,

$$\partial_{uv}^2 r = 2\kappa r^{-2}(\partial_u r)\alpha.$$

Rearranging and applying our bounds in our region of interest, we have

$$\begin{aligned}\partial_v \log(-\partial_u r) &= 2\kappa r^{-2}\alpha \\ &> 2\kappa_0 r_+^{-2}\alpha_1,\end{aligned}$$

and so integrating along an outgoing ray yields

$$\frac{\partial_u r(u,v)}{\partial_u r(u,V)} > e^{2\kappa_0 r_+^{-2}\alpha_1(v-V)},$$

and hence

$$-\partial_u r(u,v) > -\partial_u r(u,V)\, e^{2\kappa_0 r_+^{-2}\alpha_1(v-V)}. \tag{5.11}$$

Assume $\partial_u r(u,V) \leq -b_0 < 0$ for all $0 \leq u \leq U$, let

$$b_1 = 2\kappa_0 r_+^{-2}\alpha_1,$$

and set

$$b_2 = b_0 e^{-b_1 V};$$

then

$$-\partial_u r(u,v) > b_2 e^{b_1 v}$$

and so integrating along an ingoing null ray, we get

$$r(0,v) - r(u,v) > b_2 e^{b_1 v} u,$$

i.e.

$$r(u,v) < r(0,v) - b_2 e^{b_1 v} u.$$

But for any $u > 0$, the right-hand side tends to $-\infty$ as $v \to \infty$, while the left-hand side is positive. Thus we have arrived at a contradiction, so no such rectangle $K(U,V)$ can be contained in $\mathcal{R}$, and the statement of the theorem follows. □

Corollary. *Under the hypotheses of Theorem 1, the event horizon of the black hole is future geodesically complete, i.e., $\mathcal{C}_{out}$ has infinite affine length.*

Proof. Suppose s is an affine parameter for $\mathcal{C}_{out} = \{0\} \times [v_0, \infty) = \{(0, v(s))\}$ which increases to the future. Then the vector field $X = \frac{dv}{ds}\frac{\partial}{\partial v}$ satisfies $\nabla_X X = 0$, which in this setting becomes

$$\frac{dv}{ds}\left[\partial_v\left(\frac{dv}{ds}\right) + \left(\frac{dv}{ds}\right)\left(\partial_v(\log\Omega^2)\right)\right] = 0,$$

or equivalently, since $\frac{dv}{ds} > 0$,

$$\partial_v\left(\log(\Omega^2\frac{dv}{ds})\right) = 0.$$

Integrating, we have

$$\frac{dv}{ds} = a_0\Omega^{-2}$$

for some $a_0 > 0$, and so now taking s as a function of the outgoing null coordinate v, we have

$$s(v) = a_1 + a_2\int_{v_0}^{v}\Omega^2(0,\bar{v})d\bar{v},$$

some $a_1, a_2 \in \mathbb{R}$, $a_2 > 0$. Thus s has infinite range if and only if $\Omega \notin L^2([v_0,\infty))$.

Now, one of the hypotheses of Theorem 1 was that $(-\partial_u r)\Omega^{-2} \leq c_2$ along $\mathcal{C}_{out} \cap \mathcal{W}$ for some constant $c_2 > 0$ (condition C), and in the proof of the theorem, we found that for some $U > 0$, $V \geq v_0$ and some $b_1, b_2 > 0$,

$$-\partial_u r(u,v) > b_2 e^{b_1 v}$$

for all $0 \leq u \leq U$, $V \leq v$ (equation (5.11)). Putting these inequalities together and evaluating along $\mathcal{C}_{out}$, we have

$$b_2 e^{b_1 v} < -\partial_u r(0,v) \leq c_2\Omega^2(0,v)$$

for all $V \leq v$, so in fact $\Omega \notin L^2([v_0,\infty))$ and $\mathcal{C}_{out}$ is future complete. □

5.4 Immediate applications

5.4.1 Price law decay & length

Theorem 2. *Suppose* $(\mathcal{G}(u_0,v_0), -\Omega^2\,du\,dv)$ *is a spacetime obtained as in Section 5.1.1 with radial function* r*, and suppose it satisfies assumptions* I-VII *of Section 5.1.2. Suppose* $\mathcal{A}_0$ *is a connected component of* $\mathcal{A}$ *along which condition* A *is satisfied, and suppose that in addition,*

$$T_{vv} \leq c_3 v^{-2-\varepsilon} \text{ along } \mathcal{A}_0$$

for some $c_3 \geq 0$, $\varepsilon > 0$. *Then* $\mathcal{A}_0$ *has finite length with respect to the induced metric.*

Remark: The rate of decay of T_{vv} given corresponds to that of Price's law; cf. [20, 11]. Theorem 2 applies in particular to the case of the marginally trapped tube $\mathcal{A} \cap \mathcal{W}$ obtained in Theorem 1, but it does not require that the tube terminate at i^+ in order to be valid. In the context of Theorem 1, this decay rate gives a direct measure of how quickly the tube approaches the event horizon.

Proof. Since $\mathcal{A}_0$ is connected and achronal with no ingoing null segments, we may parameterize it by its v coordinate, i.e. $\gamma(v) = (u(v), v) \in \mathcal{A}_0$. If the domain of the v coordinate is bounded, then the result is trivially true, so suppose v has domain $[V, \infty)$ for some $V \geq v_0$. Then we have

$$|\dot{\gamma}(v)|^2 = \langle \dot{\gamma}(v), \dot{\gamma}(v) \rangle = -\Omega^2 (\tfrac{du}{dv}).$$

Using the relation $\dot{\gamma}(\partial_v r) = 0$, we readily compute that

$$\frac{du}{dv} = -\frac{\partial^2_{vv} r}{\partial^2_{uv} r} = -\frac{2r^3 T_{vv}}{\Omega^2 \alpha},$$

so

$$|\dot{\gamma}(v)|^2 = \frac{2r^3 T_{vv}}{\alpha}.$$

As in the proof of Proposition 5, we compute that $\alpha > \alpha_0$ along $\mathcal{A}$, so setting $b_3 = \sqrt{2r_+^3 \alpha_0^{-1}}$, we have

$$\int_V^\infty |\dot{\gamma}(v)| dv \leq \int_V^\infty b_3 (T_{vv}(\gamma(v)))^{\frac{1}{2}} dv \leq \int_V^\infty b_3 c_3 v^{-1-\varepsilon/2} dv < \infty,$$

i.e., the length of $\mathcal{A}_0$ is finite. □

5.4.2 Vaidya spacetimes

As noted in Section 4.2, perhaps the simplest example at hand when one is working with dynamical horizons is that of the (ingoing) Vaidya spacetime, the spherically symmetric solution to Einstein's equations with an ingoing null fluid as source. It is widely accepted that the Vaidya marginally trapped tube is asymptotic to the event horizon, but the literature seems to be lacking an analytical proof of this behavior for an arbitrary mass function, so it is worth seeing how our results apply to this case.

Recall that the ingoing Vaidya metric is given in terms of the ingoing Eddington-Finkelstein coordinates (v,r,θ,ϕ) by

$$g = -\left(1-\frac{2M(v)}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

with stress-energy tensor

$$T = \frac{\dot{M}(v)}{r^2}dv^2,$$

where and $M(v)$ is any smooth function of v. One can show directly that the marginally trapped tube is the hypersurface at which $r = 2M(v)$; it is spacelike where $\dot{M}(v) > 0$, null where $\dot{M}(v) = 0$, and timelike where $\dot{M}(v) < 0$.

By inspection T satisfies the dominant energy condition, assumption I, if and only if $M(v)$ is nondecreasing. We restrict our attention to a characteristic rectangle in which M is strictly positive and indicate how the remaining assumptions II-VII are satisfied: the metric is regular everywhere in the rectangle except at the singularity at $r = 0$ (albeit not in the coordinates given above), so it follows that assumption VII is satisfied, and furthermore, that singularity is evidently spacelike, so II is satisfied as well. The inner expansion of each round 2-sphere is (some positive multiple of) $-\frac{2}{r}$, i.e. it is strictly negative, so assumption V holds. Finally, assuming $M(v) < M_0$ for some limiting value $M_0 < \infty$, assumptions III, IV, and VI are all satisfied as well (the strict inequality is what yields VI).

Now, in order to check the hypotheses of Theorem 1, it seems we ought to convert from Eddington-Finkelstein to double-null coordinates. Unfortunately, to make such a conversion analytically is impossible in general; see [23]. However, we can still compute the relevant quantities from first principles. Following the treatment given in [23], we find that in double-null coordinates (u,v,θ,ϕ) with v scaled such that its domain is $[v_0,\infty)$, the only nonzero component of $T_{\alpha\beta}$ is T_{vv}. Thus conditions A$'$, B1, and B2 are all trivially satisfied. We can also deduce that the term $-(\partial_u r)\Omega^{-2} = \frac{1}{2}$ everywhere, so C is satisfied as well. Thus we conclude from Theorem 1 that the marginally trapped tube is asymptotic to the event horizon. Furthermore, as one might expect, Theorem 2 tells us that the length of the tube depends on the rate of decay of $\dot{M}(v)$ as $v \to \infty$. In particular, if $\dot{M}(v) = O(v^{-2-\varepsilon})$ for some $\varepsilon > 0$, or more generally, if $(\dot{M}(v))^{1/2} \in L^1([V,\infty))$ for some V, then the tube has finite length.

Chapter 6

ASYMPTOTIC BEHAVIOR OF MTTS WITH A SELF-GRAVITATING HIGGS FIELD MATTER MODEL

We now turn our attention from spacetimes with arbitrary matter to those containing a self-gravitating Higgs field with non-zero potential. This matter model consists of a scalar function ϕ on the spacetime and a potential function $V(\phi)$ such that

$$\Box\phi = g^{\alpha\beta}\phi_{;\alpha\beta} = V'(\phi). \tag{6.1}$$

The stress-energy tensor then takes the form

$$T_{\alpha\beta} = \phi_{;\alpha}\phi_{;\beta} - \left(\tfrac{1}{2}\phi_{;\gamma}\phi^{;\gamma} + V(\phi)\right) g_{\alpha\beta}. \tag{6.2}$$

In our spherically symmetric setting, $\phi = \phi(u,v)$, so the evolution equation (6.1) becomes

$$V'(\phi) = -4\Omega^{-2}(\partial^2_{uv}\phi + \partial_u\phi\,(\partial_v \log r) + \partial_v\phi\,(\partial_u \log r)) \tag{6.3}$$

and in double-null coordinates, (6.2) yields

$$T_{uu} = (\partial_u\phi)^2$$

$$T_{vv} = (\partial_v\phi)^2$$

and

$$T_{uv} = \tfrac{1}{2}\Omega^2 V(\phi).$$

Note that the dominant energy condition (I) is satisfied if and only if $V(\phi)$ is nonnegative. The extension principle (VII) is known to hold for self-gravitating Higgs fields if $V \geq -C$ for any finite C [12].

We give two applications of Theorem 1 to self-gravitating Higgs field black hole spacetimes. In Theorem 3, we assume that the scalar field and the potential both satisfy weak Price-law-like decay conditions on the event horizon, namely that $|\partial_v\phi|$ and $|V'(\phi)| \in O(v^{-p})$ for some constant $p > \frac{1}{2}$.

For Theorem 4, we make only certain monotonicity and smallness assumptions, including that V is convex. In both cases, we extract the hypotheses of Theorem 1 and conclude that the respective black holes each contain a marginally trapped tube asymptotic to the event horizon, which is additionally achronal with no ingoing null segments, and connected for large v.

The advantage of Theorem 4 over Theorem 3 is that it does not require an explicit rate of decay. However, the assumptions which we must make in the absence of such a decay rate are highly nontrivial. In particular, in the case of a Klein-Gordon potential of mass $\overline{m}$, i.e. $V(\phi) = \frac{1}{2}\overline{m}^2\phi^2$, the hypotheses of Theorem 4 are satisfied only if ϕ decays exponentially along the event horizon. In the case of an exponential potential $V(\phi) = ce^{k\phi}$, the hypotheses cannot even be satisfied simultaneously. By contrast, Theorem 3 can be applied in both such settings. On the other hand, for any potential of the form $V(\phi) = c\phi^{k+2}$, $k > 0$, the hypotheses of Theorem 4 may be readily satisfied; in this case they imply that $\phi \in O(v^{-\frac{1}{k}})$ and $V'(\phi) \in O(v^{-1-\frac{1}{k}})$ along the event horizon but make no *a priori* restriction on $|\partial_v\phi|$. Indeed, it is possibly to construct an admissible ϕ along the event horizon in this setting such that $\limsup_{v\to\infty} |\partial_v\phi| v^p = \infty$ for any $p > 0$, which then implies that Price law decay per se does not hold.

For both of the following theorems, we assume we have initial data r, Ω, ϕ for a self-gravitating Higgs field along the null hypersurfaces $\mathcal{C}_{in} \cup \mathcal{C}_{out} = [0,u_0] \times \{v_0\} \cup \{0\} \times [v_0,\infty)$ with nonnegative potential function $V \in C^2(\mathbb{R})$, and suppose the data satisfy assumptions III-VI, namely: $r \leq r_+$ and $0 \leq m \leq m_+$ along $\mathcal{C}_{in} \cup \mathcal{C}_{out}$, and $\partial_u r < 0$, $\partial_v r > 0$ along $\mathcal{C}_{out}$.

6.1 Application of Theorem 1 assuming explicit decay rates

Theorem 3. *Fix a constant $p > \frac{1}{2}$ and a function $\eta(v) > 0$ such that $\eta(v)$ decreases monotonically to 0 as v tends to infinity. Suppose the second derivative of the potential V is bounded, i.e. there exists a constant B such that*

$$|V''(x)| \leq B$$

on the interval $(\phi_0 - \delta_0, \phi_1 + \delta_0)$ for some $\delta_0 > 0$, where ϕ_0 and ϕ_1 are the (possibly infinite) inf *and* sup *of ϕ along $\mathcal{C}_{out}$, respectively. If along $\mathcal{C}_{out}$ the initial data satisfy*

$$\partial_v r < \eta(v),$$

$$|\partial_v \phi| < \tfrac{1}{2} b_1 v^{-p},$$

$$|V'(\phi)| < \tfrac{1}{2} b_2 v^{-p},$$

$$c_3 \le -(\partial_u r)\Omega^{-2} \le \tfrac{1}{2} c_2, \tag{6.4}$$

and

$$\liminf_{v\to\infty} V(\phi) < \frac{1}{4r_+^2}, \tag{6.5}$$

for some positive constants b_1, b_2, c_2, and c_3, then the result of Theorem 1 holds for maximal development of these initial data.

Remark: Note that if the constant $p > 1$, then $\partial_v \phi$ is integrable along the event horizon, which in turn implies that the domain of ϕ is compact and hence that V'' is *a priori* bounded on the relevant domain. Thus the hypothesis that V'' be bounded is only necessary for $\frac{1}{2} < p \le 1$.

Proof. Let $\mathcal{G}(u_0, v_0)$ denote the maximal development of the given initial data. By (6.5), we may choose a positive constant c_0 such that

$$\liminf_{v\to\infty} V(\phi(0,v)) < c_0 < \tfrac{1}{4r_+^2}.$$

Thus there exist some small $0 < \varepsilon < 2c_0$ and a sequence of values $\{\overline{v}_k\} \to \infty$ such that $V(\phi(0,\overline{v}_k)) < c_0 - \frac{1}{2}\varepsilon$ for all k. Choose $\varepsilon' > 0$ such that

$$\varepsilon' < \min\{c_0 - \tfrac{1}{2}\varepsilon, \tfrac{1}{4r_+^2} - c_0\}.$$

Since $\overline{v}_k \to \infty$, we can find K sufficiently large that for $v \ge \overline{v}_K$,

$$\frac{b_1 b_2}{2p-1} v^{1-2p} < \varepsilon',$$

$$2c_2\eta(v) < r_+^2\varepsilon,$$

and

$$2c_2 r_+ p\left(\frac{\log v}{v}\right) < r_+^2\varepsilon - 2c_2\eta(v). \tag{6.6}$$

Set $v_1 = \overline{v}_K$ and note that by construction $V(\phi(0,v_1)) < c_0 - \frac{1}{2}\varepsilon$.

Next, set $b_0 = \eta(v_1)$, let

$$\begin{aligned}\overline{b_3} &= 2\left(\frac{b_2}{4c_3}+\frac{2b_1}{r_+}\right)\left(\frac{2c_2r_+}{r_+^2\varepsilon-2c_2b_0}\right)\\ &\quad\cdot\left(v_1^{-p}+\left(1-\left(\frac{2c_2r_+p}{r_+^2\varepsilon-2c_2b_0}\right)\frac{\log v_1}{v_1}\right)^{-p}\right),\end{aligned}$$

and fix b_3 such that

$$b_3 > \max\left\{2\left|\frac{\partial_u\phi}{\partial_u r}(0,v_1)\right|v_1^p,\ \overline{b_3}\right\}.$$

Now, continuity at the point $(0,v_1)$ and our initial conditions along $\mathcal{C}_{out}$ imply that there exists $u_1 > 0$ sufficiently small that

$$\partial_v r(u,v_1) < b_0,$$

$$|\partial_v\phi(u,v_1)| < \tfrac{1}{2}b_1v_1^{-p},$$

$$|V'(\phi(u,v_1))| < \tfrac{1}{2}b_2v_1^{-p},$$

$$\left|\frac{\partial_u\phi}{\partial_u r}(u,v_1)\right| < \tfrac{1}{2}b_3v_1^{-p},$$

and

$$V(\phi(u,v_1)) < c_0 - \tfrac{1}{2}\varepsilon$$

for all $u \in [0,u_1]$. Set $\mathcal{C}'_{in} = [0,u_1]\times\{v_1\}$ and $\mathcal{C}'_{out} = \{0\}\times[v_1,\infty)$. Henceforth we will consider the subregion of $\mathcal{G}(u_0,v_0)$ given by

$$\mathcal{G}(u_1,v_1) := K(u_1,v_1)\cap\mathcal{G}(u_0,v_0),$$

i.e. the maximal development of the induced initial data on $\mathcal{C}'_{in}\cup\mathcal{C}'_{out}$.

Finally, we choose

$$0 < \delta < \min\left\{\frac{r_+}{2},\ \frac{b_1}{2}\left(\frac{b_2}{4c_3}+\frac{2b_0b_3+2b_1}{r_+}\right)^{-1},\ \frac{b_2}{2Bb_3},\ \frac{\delta_0v_1^p}{b_3},\ \frac{v_1^{2p}}{2b_3^2r_+}\right\},$$

and as usual, define

$$\mathcal{W} = \{(u,v)\in\mathcal{G}(u_1,v_1)\mid r(u,v)\geq r_+-\delta\}.$$

Define a region $\mathcal{V}$ as the set of points $(u,v) \in \mathcal{G}(u_1,v_1)$ such that the following seven inequalities hold for all $(\tilde{u},\tilde{v}) \in J^-(u,v) \cap \mathcal{G}(u_1,v_1)$:

$$\partial_v r < b_0 \tag{6.7}$$

$$|\partial_v \phi| < b_1 v^{-p} \tag{6.8}$$

$$|V'(\phi)| < b_2 v^{-p} \tag{6.9}$$

$$\left|\frac{\partial_u \phi}{\partial_u r}\right| < b_3 v^{-p} \tag{6.10}$$

$$-(\partial_u r)\Omega^{-2} < c_2 \tag{6.11}$$

$$V(\phi) < 2c_0 - \varepsilon \tag{6.12}$$

$$\partial_v r > 0. \tag{6.13}$$

Note that (6.13) implies that $\mathcal{V} \subset \mathcal{R}$, so $\mathcal{W} \cap \mathcal{V} \subset \mathcal{W} \cap \mathcal{R}$. Since we can easily extract the hypotheses of Theorem 1 from these inequalities, our goal is to prove that in fact $\mathcal{W} \cap \mathcal{R} \subset \mathcal{V}$. We accomplish this by means of a bootstrap argument showing that $\mathcal{W} \cap \mathcal{R} = \mathcal{W} \cap \mathcal{V}$. Before proceeding, however, let us show that the hypotheses of Theorem 1 are satisfied.

First we check that conditions A$'$-C hold in $\mathcal{V}$. Note that $\mathcal{V}$ is a past set by definition, and our choices of u_1 and v_1 imply that $\mathcal{C}'_{in} \subset \mathcal{V}$. Thus for $(u,v) \in \mathcal{V}$, we have

$$\int_{v_1}^{v} |\partial_v V(\phi(u,\tilde{v})|\, d\tilde{v} \leq b_1 b_2 \int_{v_1}^{v} \tilde{v}^{-2p}\, d\tilde{v} < \frac{b_1 b_2}{2p-1} v_1^{1-2p} < \varepsilon'. \tag{6.14}$$

Since $\varepsilon' < \frac{1}{4r_+^2} - c_0$, (6.14) implies that B2 is satisfied, and since $\varepsilon' < c_0 - \frac{1}{2}\varepsilon$, it also implies that

$$\tfrac{1}{2}V(\phi(u,v)) < \tfrac{1}{2}V(\phi(u,v_1)) + \tfrac{1}{2}\varepsilon' < c_0 - \tfrac{1}{2}\varepsilon < c_0 < \tfrac{1}{4r_+^2},$$

so A$'$ is satisfied as well. Condition B1 follows immediately from (6.10), and C follows from the hypothesis (6.4).

We must also verify that assumptions I-VII hold in $\mathcal{G}(u_1,v_1)$. The requirement that V be nonnegative implies I, while II follows by construction, since $\mathcal{G}(u_1,v_1)$ is the maximal development of initial data on $\mathcal{C}'_{in} \cup \mathcal{C}'_{out}$. Assumptions III and IV hold on $\mathcal{C}'_{in} \cup \mathcal{C}'_{out}$ by monotonicities of r and m in $\mathcal{R}$, respectively, while V and VI were among the hypotheses of the theorem. Finally, assumption VII holds by [12] since V is nonnegative.

We now turn to the bootstrap argument, which we carry out as follows: we first retrieve (strict) inequalities (6.10) and (6.12) in $\overline{\mathcal{V}}$, where the set closure is taken with respect to $\mathcal{G}(u_1,v_1)$, i.e.

$\overline{\mathcal{V}} = \overline{\mathcal{V}} \cap \mathcal{G}(u_1, v_1)$. Since inequalities (6.7)-(6.9), (6.11) and (6.13) hold along $\mathcal{C}'_{out}$ by hypothesis, a continuity argument then implies that $\mathcal{C}'_{out} \subset \mathcal{V}$, i.e. that both (6.10) and (6.12) hold along all of $\mathcal{C}'_{out}$. Thus $\mathcal{W} \cap \mathcal{V} \neq \emptyset$, since $\mathcal{W}$ must contain a neighborhood of $i^+ = (0, \infty)$. We then retrieve inequalities (6.7)-(6.9) and (6.11) in $\overline{\mathcal{W} \cap \mathcal{V}}$ and conclude, again by continuity, that in fact $\mathcal{W} \cap \mathcal{V} = \mathcal{W} \cap \mathcal{R}$.

It is again convenient to make use of the quantity κ introduced in the proof of Theorem 1,

$$\kappa = -\tfrac{1}{4}\Omega^2 (\partial_u r)^{-1} = \frac{\partial_v r}{1 - \frac{2m}{r}}. \tag{6.15}$$

Equation (3.8) implies that $\partial_u \kappa \leq 0$, and combining this fact with (6.4), we have $\kappa \leq \frac{1}{4c_3}$ in all of $\mathcal{G}(u_1, v_1)$. The bootstrap inequality (6.11) implies that $\kappa \geq \frac{1}{4c_2}$ in all of $\overline{\mathcal{V}}$ as well, so by (6.7),

$$\left(1 - \tfrac{2m}{r}\right)(u,v) = (\partial_v r)\,\kappa^{-1}(u,v) \leq 4c_2 b_0 \tag{6.16}$$

in $\overline{\mathcal{V}}$. Also, note that combining equations (3.8) and (3.10) (or alternately (3.9) and (3.11)) yields

$$\partial^2_{uv} r = \tfrac{1}{4}\Omega^2 r^{-1}\left(2r^2 V(\phi) + (1 - \tfrac{2m}{r}) - 1\right). \tag{6.17}$$

Let us now retrieve inequality (6.10) in $\mathcal{V}$. First we observe that (6.3) may be rearranged as

$$\partial^2_{uv}\phi = -\tfrac{1}{4}\Omega^2 V'(\phi) - (\partial_u\phi)(\partial_v \log r) - (\partial_v\phi)(\partial_u \log r). \tag{6.18}$$

Set $r_1 = r(0, v_1)$ and note that $\partial_v r > 0$ implies that $r \geq r_1$ on all of $\mathcal{C}_{out}$. We compute:

$$\begin{aligned}
\partial_v\left(\frac{\partial_u\phi}{\partial_u r}\right) &= \frac{\partial^2_{uv}\phi}{\partial_u r} - \left(\frac{\partial_u\phi}{\partial_u r}\right)\left(\frac{\partial^2_{uv} r}{\partial_u r}\right)\\
&= \left(-\tfrac{1}{4}\Omega^2 V'(\phi) - (\partial_u\phi)(\partial_v \log r) - (\partial_v\phi)(\partial_u \log r)\right)(\partial_u r)^{-1}\\
&\quad - \left(\frac{\partial_u\phi}{\partial_u r}\right)\left(\tfrac{1}{4}\Omega^2 r^{-1}\left[2r^2 V(\phi) - (1 - \tfrac{2m}{r}) - 1\right]\right)(\partial_u r)^{-1}\\
&= \kappa V'(\phi) - \left(\frac{\partial_u\phi}{\partial_u r}\right)(\partial_v \log r) - \frac{\partial_v\phi}{r}\\
&\quad - \left(\frac{\partial_u\phi}{\partial_u r}\right)\kappa r^{-1}\left[1 - 2r^2 V(\phi) - (1 - \tfrac{2m}{r})\right]\\
&= \kappa V'(\phi) - \frac{\partial_v\phi}{r}\\
&\quad - \left(\frac{\partial_u\phi}{\partial_u r}\right)\left(\kappa r^{-1}\left[1 - 2r^2 V(\phi) - (1 - \tfrac{2m}{r})\right] + (\partial_v \log r)\right).
\end{aligned}$$

Let

$$A := \kappa r^{-1}\left[1 - 2r^2 V(\phi) - (1 - \tfrac{2m}{r})\right] + (\partial_v \log r),$$

so that we may write

$$\partial_v\left(\frac{\partial_u\phi}{\partial_u r}\right) = -A\left(\frac{\partial_u\phi}{\partial_u r}\right) + \left(\kappa V'(\phi) - \frac{\partial_v\phi}{r}\right). \tag{6.19}$$

Using (6.12), (6.16) and the fact that $c_0 < \frac{1}{4r_+^2}$, we estimate

$$A \geq \frac{1-2r_+^2(2c_0-\varepsilon)-4c_2b_0}{4c_2r_1} > \frac{r_+^2\varepsilon - 2c_2b_0}{2c_2r_1} =: a_0.$$

The constant a_0 is positive by our choice of b_0. Also, (6.8) and (6.9) imply that

$$\left|\kappa V'(\phi) - \frac{\partial_v\phi}{r}\right| \leq \left(\frac{b_2}{4c_3} + b_1r_1^{-1}\right)v^{-p} =: a_1v^{-p}$$

Then for $(u,v) \in \overline{\mathcal{V}}$, integrating (6.19) along the outgoing null ray $\{u\}\times[v_1,v]$ yields

$$\begin{aligned}\left(\frac{\partial_u\phi}{\partial_u r}\right)(u,v) \;&=\; e^{-\int_{v_1}^{v}A(u,\tilde v)\,d\tilde v}\left(\frac{\partial_u\phi}{\partial_u r}\right)(u,v_1)\\ &\quad + \int_{v_1}^{v} e^{-\int_{\tilde v}^{v}A(u,\tilde v)\,d\tilde v}\left(\kappa V'(\phi) - \frac{\partial_v\phi}{r}\right)(u,\tilde v)\,d\tilde v,\end{aligned} \tag{6.20}$$

so

$$\begin{aligned}\left|\frac{\partial_u\phi}{\partial_u r}\right|(u,v) \;&\leq\; e^{-\int_{v_1}^{v}A(u,\tilde v)\,d\tilde v}\left|\frac{\partial_u\phi}{\partial_u r}\right|(u,v_1)\\ &\quad + \int_{v_1}^{v} e^{-\int_{\tilde v}^{v}A(u,\tilde v)\,d\tilde v}\left|\kappa V'(\phi) - \frac{\partial_v\phi}{r}\right|(u,\tilde v)\,d\tilde v\\ &\leq\; \tfrac{1}{2}b_3v_1^{-p}e^{-\int_{v_1}^{v}a_0\,d\tilde v} + \int_{v_1}^{v} a_1\tilde v^{-p}e^{-\int_{\tilde v}^{v}a_0\,d\tilde v}\,d\tilde v\\ &\leq\; \tfrac{1}{2}b_3v_1^{-p}e^{a_0(v_1-v)} + a_1e^{-a_0v}\int_{v_1}^{v}\tilde v^{-p}e^{a_0\tilde v}\,d\tilde v.\end{aligned}$$

Integrating the second term by parts, we have:

$$\begin{aligned}a_1e^{-a_0v}\int_{v_1}^{v}\tilde v^{-p}e^{a_0\tilde v}\,d\tilde v \;&=\; a_1e^{-a_0v}\Big(a_0^{-1}v^{-p}e^{a_0v} - a_0^{-1}v_1^{-p}e^{a_0v_1}\\ &\quad + \int_{v_1}^{v}a_0^{-1}p\tilde v^{-p-1}e^{a_0\tilde v}\,d\tilde v\Big)\\ &\leq\; a_1a_0^{-1}v^{-p} + pa_1a_0^{-1}e^{-a_0v}\int_{v_1}^{v}\tilde v^{-p-1}e^{a_0\tilde v}\,d\tilde v.\end{aligned}$$

Furthermore, we estimate that

$$\begin{aligned}
\int_{v_1}^{v} \tilde{v}^{-p-1} e^{a_0\tilde{v}}\, d\tilde{v} &= \int_{v-\frac{p}{a_0}\log v}^{v} \tilde{v}^{-p-1} e^{a_0\tilde{v}}\, d\tilde{v} + \int_{v_1}^{v-\frac{p}{a_0}\log v} \tilde{v}^{-p-1} e^{a_0\tilde{v}}\, d\tilde{v} \\
&\leq e^{a_0 v} \int_{v-\frac{p}{a_0}\log v}^{v} \tilde{v}^{-p-1}\, d\tilde{v} + \\
&\quad + e^{a_0 v - p\log v} \int_{v_1}^{v-\frac{p}{a_0}\log v} \tilde{v}^{-p-1}\, d\tilde{v} \\
&= e^{a_0 v}\left(\frac{-v^{-p}}{p} + \frac{(v-\frac{p}{a_0}\log v)^{-p}}{p}\right) + \\
&\quad + e^{a_0 v} v^{-p}\left(\frac{-(v-\frac{p}{a_0}\log v)^{-p}}{p} + \frac{{v_1}^{-p}}{p}\right) \\
&\leq p^{-1} e^{a_0 v} v^{-p}\left[\left(-1 + (1-\tfrac{p}{a_0}v^{-1}\log v)^{-p}\right) + {v_1}^{-p}\right].
\end{aligned}$$

Putting it all back together yields

$$\begin{aligned}
\left|\frac{\partial_u \phi}{\partial_u r}\right|(u,v) &\leq \tfrac{1}{2} b_3 v_1^{-p} e^{a_0(v_1-v)} + a_1 a_0^{-1} v^{-p} + (p a_1 a_0^{-1} e^{-a_0 v}) \cdot \\
&\quad \cdot \left(p^{-1} e^{a_0 v} v^{-p}\left[\left(-1 + (1-\tfrac{p}{a_0}v^{-1}\log v)^{-p}\right) + {v_1}^{-p}\right]\right) \\
&\leq \tfrac{1}{2} b_3 v_1^{-p} e^{a_0(v_1-v)} + a_1 a_0^{-1} v^{-p}\left((1-\tfrac{p}{a_0}v^{-1}\log v)^{-p} + {v_1}^{-p}\right) \\
&< \left(\tfrac{1}{2} b_3 v^p {v_1}^{-p} e^{a_0(v_1-v)} + \tfrac{1}{2} b_3\right) v^{-p} \\
&< b_3 v^{-p},
\end{aligned}$$

where in the second to last line we have used the monotonicity of $v^{-1}\log v$ together with the definition of b_3 and the fact that $r_1 < r_+$, and in the last line we have used the fact that $v^p e^{-a_0 v}$ decreases monotonically for $v > p a_0^{-1}$, a lower bound which is guaranteed by (6.6) and our choice of v_1 (assuming without loss of generality that $v_1 \geq e$). Thus we have retrieved (6.10) in $\overline{\mathcal{V}}$.

For (6.12), we compute as in (6.14) that for $(u,v) \in \overline{\mathcal{V}}$,

$$V(\phi(u,v)) \leq V(\phi(u,v_1)) + \int_{v_1}^{v} |\partial_v(V(\phi))(u,\tilde{v})|\, d\tilde{v} < c_0 - \tfrac{1}{2}\varepsilon + \varepsilon' < 2c_0 - \varepsilon,$$

where the last inequality follows from our choice of ε'. Thus we have retrieved (6.12) in $\overline{\mathcal{V}}$.

As discussed above, we can now conclude that $\mathcal{C}'_{out} \subset \mathcal{V}$ and hence that $\mathcal{W} \cap \mathcal{V} \neq \emptyset$. We now turn to improving inequalities (6.7)-(6.9) and (6.11) in $\overline{\mathcal{W} \cap \mathcal{V}}$.

For (6.7), note that equation (6.17) and inequalities (6.12) and (6.7) imply that

$$\begin{aligned}\partial^2_{uv} r &\leq \tfrac{1}{4}\Omega^2 r^{-1}\left(2r_+^2(2c_0-\varepsilon)+4c_2b_0-1\right)\\ &\leq \tfrac{1}{4}\Omega^2 r^{-1}\left(4c_2b_0-2r_+^2\varepsilon\right)\\ &\leq 0\end{aligned}$$

in $\overline{\mathcal{V}}$. Since $\partial_v r(0,v) < \eta(v) \leq \eta(v_1) = b_0$ for all $v \geq v_1$, this yields (6.7).

Next we turn to (6.8). Rearranging (6.18), we have

$$\partial^2_{uv}\phi = (\partial_u r)\left(\kappa V'(\phi) - \frac{\partial_u \phi}{\partial_u r}(\partial_v \log r) - \frac{\partial_v \phi}{r}\right),$$

so for $(u,v) \in \overline{\mathcal{W} \cap \mathcal{V}}$, using inequalities (6.7)-(6.10) and the fact that $r \geq r_+ - \delta$ implies

$$|\partial^2_{uv}\phi(u,v)| \leq -(\partial_u r(u,v))\left(\frac{b_2}{4c_3} + \frac{b_0b_3+b_1}{r_+-\delta}\right)v^{-p}.$$

Thus

$$\begin{aligned}|\partial_v\phi(u,v)| &\leq |\partial_v\phi(0,v)| + \int_0^u |\partial^2_{uv}\phi(\tilde{u},v)|\,d\tilde{u}\\ &\leq |\partial_v\phi(0,v)| - \left(\frac{b_2}{4c_3} + \frac{b_0b_3+b_1}{r_+-\delta}\right)v^{-p}\int_0^u \partial_u r(\tilde{u},v)\,d\tilde{u}\\ &< \tfrac{1}{2}b_1 v^{-p} + \delta\left(\frac{b_2}{4c_3} + \frac{2b_0b_3+2b_1}{r_+}\right)v^{-p}\\ &< b_1 v^{-p},\end{aligned}$$

where in the second-to-last and last lines we have used our choice of δ. Thus we have obtained (6.8) in $\overline{\mathcal{W} \cap \mathcal{V}}$.

Next we retrieve (6.9). First observe that for $(u,v) \in \overline{\mathcal{W} \cap \mathcal{V}}$,

$$\left|\int_0^u (\partial_u\phi)(\tilde{u},v)\,d\tilde{u}\right| \leq -b_3 v^{-p}\int_0^u \partial_u r(\tilde{u},v)\,d\tilde{u} \leq b_3 v_1^{-p}\delta < \delta_0,$$

where we have used (6.10) and our choice of δ. Thus $\phi(u,v) \in (\phi_0 - \delta_0, \phi_1 + \delta_0)$, so in particular, $|V''(\phi(u,v))| \leq B$ for all $(u,v) \in \overline{\mathcal{W} \cap \mathcal{V}}$. Using (6.10) once more, we have:

$$\begin{aligned}|V'(\phi(u,v))| &\leq |V'(\phi(0,v))| + \int_0^u |V''(\phi)||\partial_u\phi|(\tilde{u},v)\,d\tilde{u}\\ &\leq \tfrac{1}{2}b_2 v^{-p} - Bb_3 v^{-p}\int_0^u \partial_u r(\tilde{u},v)\,d\tilde{u}\\ &\leq \tfrac{1}{2}b_2 v^{-p} + Bb_3\delta v^{-p}\\ &< b_2 v^{-p},\end{aligned}$$

where in the last line we have again used our choice of δ.

It remains only to retrieve (6.11). Note that from (3.8) we have

$$\partial_u(-\Omega^{-2}\partial_u r) = r\Omega^{-2}(\partial_u\phi)^2,$$

and combining (6.11) and (6.10) yields

$$\Omega^{-2}(\partial_u\phi)^2 \le b_3^2 v^{-2p}\,\Omega^{-2}(\partial_u r)^2 < -b_3^2 c_2 v^{-2p}(\partial_u r).$$

Integrating along an ingoing null ray and using (6.4), we have

$$\begin{aligned}(-\Omega^{-2}\partial_u r)(u,v) &\le (-\Omega^{-2}\partial_u r)(0,v) - \int_0^u b_3^2 c_2 v^{-2p} r(\partial_u r)\,d\tilde{u} \\ &\le \tfrac{1}{2}c_2 + b_3^2 c_2 v^{-2p} r_+ \delta \\ &< c_2,\end{aligned}$$

once more using the choice of δ.

Thus inequalities (6.7)-(6.12) hold in all of $\overline{\mathcal{W}\cap\mathcal{V}}$, which implies that the boundary of $\mathcal{W}\cap\mathcal{V}$ relative to $\mathcal{W}\cap\mathcal{R}$ is empty. Thus $\mathcal{W}\cap\mathcal{V} = \mathcal{W}\cap\mathcal{R}$, so in particular, $\mathcal{W}\cap\mathcal{R}\subset\mathcal{V}$ and the theorem follows. □

6.2 Application of Theorem 1 assuming smallness and monotonicity

Theorem 4. *Suppose there exist positive constants c_0, c_1, c_2, c_3, and c_4 such that*

$$c_0 < \frac{1}{4r_+^2}; \tag{6.21}$$

along $\mathcal{C}_{in}$

$$\left(\frac{\partial_u\phi}{\partial_u r}\right)^2 < c_1; \tag{6.22}$$

and along $\mathcal{C}_{out}$

$$c_3 \le -(\partial_u r)\Omega^{-2} \le \tfrac{1}{2}c_2 \tag{6.23}$$

and

$$V'(\phi) < c_4|\partial_v\phi|. \tag{6.24}$$

Suppose also that the potential V satisfies

$$0 \leq V''(x) \leq B \tag{6.25}$$

on the interval $(\phi_0 - \delta_0, \phi_1)$ *for some* $\delta_0 > 0$*, where* ϕ_0 *and* ϕ_1 *are the (possibly infinite)* inf *and* sup *of* ϕ *along* $\mathcal{C}_{out}$*, respectively, and B is a constant satisfying* $B < r_+^{-2}$.

If along $\mathcal{C}_{out}$ *the initial data satisfy*

$$\partial_v r < \varepsilon$$

$$V(\phi) < \tfrac{1}{2}\varepsilon'$$

$$|\partial_v \phi| < \tfrac{1}{2}\varepsilon''$$

for sufficiently small ε, ε'*, and* $\varepsilon'' > 0$*, as well as*

$$\begin{aligned} \partial_v \phi &< 0 \\ \partial_u \phi &< 0 \\ V'(\phi) - 4\sqrt{c_1}c_2(\partial_v \log r) - 4c_3 r_+^{-1}(\partial_v \phi) &> 0, \end{aligned}$$

and either

$$V'(\phi) \leq 0$$

or

$$|\phi_0| < \infty,$$

then the result of Theorem 1 holds for maximal development of these initial data.

Remarks: Stated more precisely, the requirement that the constants ε, ε', and ε'' be sufficiently small is the following:

$$2\varepsilon' + 8c_2 r_+^{-2}\varepsilon < r_+^{-2} - B,$$

$$\varepsilon' < \min\left\{\frac{1}{2r_+^2} - 2c_0, 2c_0\right\},$$

and

$$\varepsilon'' < \frac{\sqrt{c_1}c_3(1 - 2r_+^2\varepsilon' - 4c_2\varepsilon)}{c_2(c_4 r_+ + 4c_3)}.$$

The existence of constants c_0 and c_1 satisfying (6.21) and (6.22) is not restrictive, but that of constants c_2 and c_3 in (6.23) is. Inequalities (6.26) and (6.27) together imply that the scalar field ϕ has a timelike gradient. Note that if $V'(\phi) \leq 0$ along $\mathcal{C}_{out}$, then (6.24) is trivially true for any c_4.

Proof. Consider the maximal development $\mathcal{G}(u_0, v_0)$ of the given initial data, and define a region $\mathcal{V}_0$ as the set of all $(u,v) \in \mathcal{G}(u_0,v_0)$ such that the following ten inequalities hold for all $(\tilde{u},\tilde{v}) \in J^-(u,v)$:

$$\partial_v \phi < 0 \tag{6.26}$$

$$\partial_u \phi < 0 \tag{6.27}$$

$$V'(\phi) < c_4 |\partial_v \phi| \tag{6.28}$$

$$V'(\phi) - 4\sqrt{c_1} c_2 (\partial_v \log r) - 4c_3 r_+^{-1} (\partial_v \phi) > 0, \tag{6.29}$$

$$\partial_v r < \varepsilon \tag{6.30}$$

$$V(\phi) < \varepsilon' \tag{6.31}$$

$$|\partial_v \phi| < \varepsilon'' \tag{6.32}$$

$$|\partial_u \phi| < \sqrt{c_1} |\partial_u r| \tag{6.33}$$

$$-(\partial_u r)\Omega^{-2} < c_2 \tag{6.34}$$

$$\partial_v r > 0. \tag{6.35}$$

Clearly $\mathcal{V}_0$ is open in $\mathcal{G}(u_0,v_0)$. Consequently, our assumptions on the initial data imply that $\mathcal{V}_0$ must contain some neighborhood of $(0,v_0)$ in $\mathcal{G}(u_0,v_0)$, so by shrinking u_0 as necessary, we may in fact assume that they all hold along $\mathcal{C}_{in}$. Since all of the inequalities except (6.33) are known to hold on $\mathcal{C}_{out}$, our first step will be to retrieve (6.33) in $\overline{\mathcal{V}_0}$, where the set closure is taken relative to $\mathcal{G}(u_0,v_0)$, i.e. $\overline{\mathcal{V}_0} = \overline{\mathcal{V}_0} \cap \mathcal{G}(u_0,v_0)$. Then by a continuity argument, we can conclude that $\mathcal{C}_{in} \cup \mathcal{C}_{out} \subset \mathcal{V}_0$.

As in the proof of Theorem 3, recall that quantity κ is given by

$$\kappa = -\tfrac{1}{4}\Omega^2 (\partial_u r)^{-1} = \frac{(\partial_v r)}{1 - \frac{2m}{r}}.$$

From (6.34), we have $\kappa \geq \frac{1}{4c_2}$ in $\overline{\mathcal{V}_0}$, and since $\partial_u \kappa \leq 0$ by equation (3.8), (6.23) implies $\kappa \leq \frac{1}{4c_3}$ in all of $\mathcal{G}(u_0,v_0)$. Also, from (6.30) we have that

$$(1 - \tfrac{2m}{r}) = (\partial_v r)\kappa^{-1} \leq 4c_2 \varepsilon \tag{6.36}$$

in $\overline{\mathcal{V}_0}$. Let $r_0 = r(0,v_0)$ and observe that $\partial_v r > 0$ implies that $r \geq r_0$ on all of $\mathcal{C}_{out}$.

Now, equation (6.20) may be derived exactly as in the proof of Theorem 3, namely

$$\begin{aligned}\left(\frac{\partial_u\phi}{\partial_u r}\right)(u,v) &= e^{-\int_{v_0}^{v}A(u,\tilde v)\,d\tilde v}\left(\frac{\partial_u\phi}{\partial_u r}\right)(u,v_0)\\ &\quad+\int_{v_0}^{v}e^{-\int_{v'}^{v}A(u,\tilde v)\,d\tilde v}\left(\kappa V'(\phi)-\frac{\partial_v\phi}{r}\right)(u,v')\,dv',\end{aligned}$$

where

$$A := \kappa r^{-1}\left[(1-2r^2V(\phi))-(1-\tfrac{2m}{r})\right]+(\partial_v\log r).$$

Using the above bounds, (6.31), and (6.35), in $\overline{\mathcal{V}_0}$ we estimate

$$A \geq \frac{1-2r_+^2\varepsilon'-4c_2\varepsilon}{4c_2r_0} =: a_0.$$

The constant a_0 is positive by our choices of ε and ε'. Also, (6.32) and (6.28) imply that

$$\kappa V'(\phi)-\frac{\partial_v\phi}{r} \leq \left(\frac{c_4}{4c_3}+\frac{1}{r_0}\right)\varepsilon'' =: a_1.$$

Then applying these bounds and using (6.22), for $(u,v)\in\overline{\mathcal{V}_0}$ we have

$$\begin{aligned}\left(\frac{\partial_u\phi}{\partial_u r}\right)(u,v) &< \sqrt{c_1}\,e^{-a_0(v-v_0)}+a_1\int_{v_0}^{v}e^{-a_0(v-v')}\,dv'\\ &= \sqrt{c_1}\,e^{-a_0(v-v_0)}+a_1a_0^{-1}\left(1-e^{-a_0(v-v_0)}\right)\\ &= e^{-a_0(v-v_0)}\left(\sqrt{c_1}-a_1a_0^{-1}\right)+a_1a_0^{-1}.\end{aligned}$$

Our choices of ε, ε' and ε'' imply that $\sqrt{c_1}-a_1a_0^{-1}>0$, so for $v\geq v_0$, we have

$$\left(\frac{\partial_u\phi}{\partial_u r}\right)(u,v) < \left(\sqrt{c_1}-a_1a_0^{-1}\right)+a_1a_0^{-1}=\sqrt{c_1}.$$

Thus (6.33) holds in all of $\overline{\mathcal{V}_0}$, so in particular, $\mathcal{C}_{out}\subset\mathcal{V}_0$.

Our next step is to choose a suitably small $\delta>0$ to use in defining $\mathcal{W}$: we let

$$\delta < \min\left\{\frac{\delta_0}{\sqrt{c_1}},\ \frac{\varepsilon' r_+}{8\sqrt{c_1}c_3\varepsilon''},\ r_+\left(1-2^{-\left(\frac{c_4r_+}{4c_3}+1\right)^{-1}}\right),\ \frac{1}{2c_1r_+}\right\}$$

and set $\mathcal{W}=\mathcal{W}(\delta)=\{(u,v)\in\mathcal{G}(u_0,v_0)\,|\,r(u,v)\geq r_+-\delta\}$. Now, $r\nearrow r_+$ along $\mathcal{C}_{out}$, so there must exist some $v_1\geq v_0$ such that $\mathcal{W}$ contains an open neighborhood of the ray $\{0\}\times[v_1,\infty)$. Also, since $\mathcal{V}_0$ and $\mathcal{W}$ each contain some neighborhood of the point $(0,v_1)$, we can find $0<u_1\leq u_0$ such that $[0,u_1]\times\{v_1\}\subset\mathcal{V}_0\cap\mathcal{W}$. Set $\mathcal{C}'_{in}=[0,u_1]\times\{v_1\}$ and $\mathcal{C}'_{out}=\{0\}\times[v_1,\infty)$. Henceforth we restrict

our attention to the subset $\mathcal{G}(u_1,v_1) = K(u_1,v_1) \cap \mathcal{G}(u_0,v_0)$, that is, the maximal development of the induced data on $\mathcal{C}_{in}' \cup \mathcal{C}_{out}'$.

The proof now proceeds by a bootstrap argument. Let $\mathcal{V}$ be the set of all points $(u,v) \in \mathcal{G}(u_1,v_1)$ such that $(\tilde{u},\tilde{v}) \in \mathcal{V}_0 \cap \mathcal{W}$ for all $(\tilde{u},\tilde{v}) \in J^-(u,v) \cap \mathcal{G}(u_1,v_1)$. Clearly $\mathcal{V} \subset \mathcal{W} \cap \mathcal{R}$. By construction, we have $\mathcal{C}_{in}' \cup \mathcal{C}_{out}' \subset \mathcal{V}$. We will retrieve inequalities (6.26)-(6.34) in $\overline{\mathcal{V}}$ and consequently conclude that $\mathcal{V} = \mathcal{W} \cap \mathcal{R} \cap \mathcal{G}(u_1,v_1)$. At that point we can easily extract the hypotheses of Theorem 1.

We proceed through the ten inequalities in order, beginning with (6.26) and (6.27). Rearranging equation (6.3) and applying (6.33) and our bounds for κ yields

$$\begin{aligned}\partial_{uv}^2\phi &= -\tfrac{1}{4}\Omega^2\left(V'(\phi) - \frac{\partial_u\phi}{\partial_u r}(\partial_v \log r)\kappa^{-1} - \frac{\partial_v\phi}{r\kappa}\right)\\ &\leq -\tfrac{1}{4}\Omega^2\left(V'(\phi) - 4\sqrt{c_1}c_2(\partial_v \log r) - 4c_3 r_+^{-1}(\partial_v\phi)\right),\end{aligned}$$

so (6.29) now implies that

$$\partial_{uv}^2\phi \leq 0 \tag{6.37}$$

in $\overline{\mathcal{V}}$. Thus since (6.26) and (6.27) hold along $\mathcal{C}_{out}'$ and $\mathcal{C}_{in}'$, respectively, they must hold in $\overline{\mathcal{V}}$ as well.

Now we turn to (6.28) and compute that

$$\partial_u\left(\frac{V'(\phi)}{\partial_v\phi}\right) = \frac{V''(\phi)(\partial_u\phi)(\partial_v\phi) - V'(\phi)(\partial_{uv}^2\phi)}{(\partial_v\phi)^2} \geq -\frac{V'(\phi)(\partial_{uv}^2\phi)}{(\partial_v\phi)^2}.$$

Suppose $(u,v) \in \overline{\mathcal{V}}$. If $V'(\phi(u,v)) \leq 0$, then clearly (6.28) holds at (u,v), since $|\partial_v\phi| > 0$. If $V'(\phi(u,v)) > 0$, let u_* be the smallest value such that $V'(\phi) \geq 0$ along $[u_*,u) \times \{v\}$ and integrate the above inequality, noting that the righthand side is positive along this ray by (6.37). If $u_* = 0$, then by our hypotheses on $\mathcal{C}_{out}$, we have $V'(\phi(u_*,v))(\partial_v\phi(u_*,v))^{-1} > -c_4$. On the other hand, if $u_* > 0$, then by choice of u_*, $V'(\phi(u_*,v))(\partial_v\phi(u_*,v))^{-1} = 0$. Thus in either case we have

$$\left(\frac{V'(\phi)}{\partial_v\phi}\right)(u,v) \geq \left(\frac{V'(\phi)}{\partial_v\phi}\right)(u_*,v) > -c_4,$$

thereby obtaining (6.28) in $\overline{\mathcal{V}}$.

Next, before proceeding to (6.29), let us first show that our initial bounds for $V''(\phi)$ continue to hold in $\overline{\mathcal{V}}$. On one hand, integrating (6.33) yields

$$\phi(0,v) - \phi(u,v) \leq \sqrt{c_1}(r(0,v) - r(u,v)) < \sqrt{c_1}\delta < \delta_0.$$

On the other hand, (6.27) implies

$$\phi(u,v) \le \phi(0,v).$$

Thus for $(u,v) \in \overline{\mathcal{V}}$ we have $\phi(u,v) \in (\phi_0 - \delta_0, \phi_1)$, and hence

$$0 \le V''(\phi(u,v)) \le B.$$

For (6.29), we first derive an upper estimate for $\partial^2_{uv} r$ in $\overline{\mathcal{V}}$. Recalling equation (6.17), we have

$$\begin{aligned}
\partial^2_{uv} r &= \tfrac{1}{4}\Omega^2 r^{-1}\left[2r^2 V(\phi) + (1 - \tfrac{2m}{r}) - 1\right] \\
&\le -\kappa(\partial_u r) r^{-1}\left[2r_+^2 \varepsilon' + 4c_2\varepsilon - 1\right] \\
&\le \frac{\partial_u r}{4c_2 r_+}\left[1 - 2r_+^2\varepsilon' - 4c_2\varepsilon\right],
\end{aligned}$$

where we have used (6.31), (6.36), and $\kappa \ge \frac{1}{4c_2}$. Setting

$$a_2 := \frac{1}{4c_2 r_+}\left[1 - 2r_+^2 \varepsilon' - 4c_2\varepsilon\right] > 0,$$

we thus have

$$\partial^2_{uv} r \le a_2(\partial_u r). \tag{6.38}$$

Consequently, differentiating the lefthand side of (6.29) and using inequalities (6.33), (6.27), (6.37), (6.30), and (6.38) yields

$$\begin{aligned}
&\partial_u\big(V'(\phi) - 4\sqrt{c_1}c_2(\partial_v \log r) - 4c_3 r_+^{-1}(\partial_v\phi)\big) \\
&\quad= V''(\phi)(\partial_u\phi) - 4\sqrt{c_1}c_2(\partial^2_{uv}\log r) - 4c_3 r_+^{-1}(\partial^2_{uv}\phi) \\
&\quad\ge \sqrt{c_1}B(\partial_u r) - 4\sqrt{c_1}c_2 r^{-1}\partial^2_{uv} r + 4\sqrt{c_1}c_2 r^{-2}(\partial_u r)(\partial_v r) \\
&\quad\ge \sqrt{c_1}B(\partial_u r) - 4\sqrt{c_1}c_2 a_2 r_+^{-1}(\partial_u r) + 4\sqrt{c_1}c_2 r_+^{-2}\varepsilon(\partial_u r) \\
&\quad= \sqrt{c_1}(\partial_u r)\left(B - 4c_2 a_2 r_+^{-1} + 4c_2 r_+^{-2}\varepsilon\right) \\
&\quad= \sqrt{c_1}(\partial_u r)\left(B - r_+^{-2} + 2\varepsilon' + 8c_2 r_+^{-2}\varepsilon\right) \\
&\quad> 0,
\end{aligned}$$

where the last line follows from the choices of ε and ε'. Thus we have retrieved (6.29) in $\overline{\mathcal{V}}$.

That inequality (6.30) holds follows immediately from (6.38), since the latter implies that $\partial^2_{uv} r < 0$.

For (6.31), observe that from (6.29), we have

$$V'(\phi) \geq 4\sqrt{c_1}c_2(\partial_v \log r) + 4c_3 r_+^{-1}(\partial_v\phi) \geq 4c_3 r_+^{-1}(\partial_v\phi).$$

Multiplying through by $\partial_u\phi$ and using (6.27), (6.33), and (6.32) yields

$$\begin{aligned} \partial_u(V(\phi)) &\leq 4c_3 r_+^{-1}|\partial_v\phi||\partial_u\phi| \\ &\leq -4\sqrt{c_1}c_3 r_+^{-1}\varepsilon''(\partial_u r). \end{aligned}$$

Integrating and using the assumption that $V(\phi) < \frac{1}{2}\varepsilon'$ on $\mathcal{C}_{out}$ then gives

$$\begin{aligned} V(\phi)(u,v) &\leq \tfrac{1}{2}\varepsilon' + 4\sqrt{c_1}c_3 r_+^{-1}\varepsilon''\delta \\ &< \varepsilon' \end{aligned}$$

by our choice of δ.

Next we turn to (6.32). Using equation (6.3) and inequalities (6.26)-(6.28), we have

$$\begin{aligned} \partial_u \log|\partial_v\phi| &= (\partial_{uv}^2\phi)(\partial_v\phi)^{-1} \\ &= -\tfrac{1}{4}\Omega^2 V'(\phi)(\partial_v\phi)^{-1} - (\partial_u\phi)(\partial_v\phi)^{-1}(\partial_v \log r) - (\partial_u \log r) \\ &\leq -c_4\kappa(\partial_u r) - \partial_u \log r \\ &\leq -\partial_u \log r \left(\frac{c_4 r_+}{4c_3} + 1\right), \end{aligned}$$

so integrating yields

$$\begin{aligned} |\partial_v\phi(u,v)| &\leq |\partial_v\phi(0,v)| \left(\frac{r_+}{r_+ - \delta}\right)^{\frac{c_4 r_+}{4c_3}+1} \\ &< \tfrac{1}{2}\varepsilon'' \left(\frac{r_+}{r_+ - \delta}\right)^{\frac{c_4 r_+}{4c_3}+1} \\ &< \varepsilon'', \end{aligned}$$

where in the last line we have again used our choice of δ. Thus (6.32) holds in $\overline{\mathcal{V}}$.

We have already shown that (6.33) holds in $\overline{\mathcal{V}_0}$, so naturally it holds in $\overline{\mathcal{V}}$ as well.

Lastly we retrieve (6.34). Note that from (3.8) we have

$$\partial_u(-\Omega^{-2}\partial_u r) = r\Omega^{-2}(\partial_u\phi)^2,$$

and combining (6.34) and (6.33) yields

$$\Omega^{-2}(\partial_u\phi)^2 \le c_1\Omega^{-2}(\partial_u r)^2 < -c_1c_2(\partial_u r).$$

Integrating along an ingoing null ray and using (6.34) again, we have

$$\begin{aligned}(-\Omega^{-2}\partial_u r)(u,v) &\le (-\Omega^{-2}\partial_u r)(0,v) - \int_0^u c_1c_2 r(\partial_u r)\,d\tilde{u}\\ &\le \tfrac{1}{2}c_2 + c_1c_2r_+\delta\\ &< c_2,\end{aligned}$$

once more using our choice of δ in the last line.

The bootstrap is now completed; we have shown that inequalities (6.26)-(6.34) hold in all of $\mathcal{W}\cap\mathcal{R}$, and hence that $\mathcal{W}\cap\mathcal{R}\subset\mathcal{V}$. It remains to show that the hypotheses of Theorem 1 hold in this region.

For A$'$, we note that by (6.31), $\Omega^{-2}T_{uv} = \frac{1}{2}V(\phi) < \frac{1}{2}\varepsilon' < c_0 < \frac{1}{4}(r_+)^{-2}$. Conditions B1 and C are immediate by (6.33) and (6.34), respectively. For the second part of condition B2, we use (6.31) and the nonnegativity of $V(\phi)$ to estimate that

$$\int_{v_0}^{v}\partial_v(\Omega^{-2}T_{uv})\,d\tilde{v} = \int_{v_0}^{v}\partial_v(\tfrac{1}{2}V(\phi))\,d\tilde{v} = \tfrac{1}{2}\left[V(\phi(u,v)) - V(\phi(u,v_0))\right] < \tfrac{1}{2}\varepsilon'$$

and then observe that $\frac{1}{2}\varepsilon' < \frac{1}{4r_+^2} - c_0$. For the first part of B2, recall that one of our hypotheses was that either $V'(\phi)\le 0$ along $\mathcal{C}_{out}$ or $|\phi_0| < \infty$. In the former case, we may differentiate $V'(\phi)$ and use (6.27) and the nonnegativity of V'' to obtain

$$\partial_u(V'(\phi)) = V''(\phi)(\partial_u\phi) \le 0.$$

Thus $V'(\phi)\le 0$ in all of $\overline{\mathcal{V}}$, so combining this with inequality (6.26), we have that $|\partial_v(\frac{1}{2}V(\phi)| = \partial_v(\frac{1}{2}V(\phi))$, and the first part of B2 now follows from the second. In the latter case, we have $\phi_0 - \delta_0 < \phi(u,v) < \phi_1$ for all $(u,v)\in\mathcal{V}$, where we are now assuming that $|\phi_0| < \infty$. Since ϕ decreases along

c_{out}, $|\phi_1| < \infty$ as well. Thus, using (6.28), (6.26), and (6.32), we have

$$\begin{aligned}\int_{v_0}^{v} |\partial_v(\tfrac{1}{2}V(\phi))|\, d\tilde{v} &= \int_{v_0}^{v} \tfrac{1}{2}|V'(\phi)||\partial_v\phi|\, d\tilde{v} \\ &< -\int_{v_0}^{v} \tfrac{1}{2}c_4\varepsilon''(\partial_v\phi)\, d\tilde{v} \\ &< \tfrac{1}{2}c_4\varepsilon''(\phi(u,v_0)-\phi(u,v)) \\ &< \tfrac{1}{2}c_4\varepsilon''(\phi_1-\phi_0+\delta_0) \\ &< \infty.\end{aligned}$$

Thus condition B2 holds in either case. The verification of assumptions I-VII is identical to that in the proof of Theorem 3. □

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Appendix A

CHRISTOFFEL SYMBOLS FOR A SPHERICALLY SYMMETRIC METRIC

Here we explicitly compute the Christoffel symbols for the metric

$$g = -\Omega^2 du\,dv + r^2 g_{S^2}.$$

Recall that the functions r and Ω depend only on the coordinates u and v. Assign the coordinates u, v, θ, ϕ labels 1, 2, 3, 4 respectively. Then we have

$$\begin{aligned} g_{12} = g_{21} &= -\tfrac{1}{2}\Omega^2 & \text{and} \quad g^{12} = g^{21} &= -2\Omega^{-2} \\ g_{33} &= r^2 & g^{33} &= r^{-2} \\ g_{44} &= r^2 \sin^2\theta, & g^{44} &= r^{-2}(\sin\theta)^{-2}, \quad \text{all others } 0. \end{aligned}$$

In order to compute this metric's Christoffel symbols, we first write down all of the partial derivatives of its components that are non-zero. These are:

$$\begin{aligned} g_{12,1} = g_{21,1} &= -\Omega(\partial_u \Omega) & g_{44,1} &= 2r(\partial_u r)\sin^2\theta \\ g_{12,2} = g_{21,2} &= -\Omega(\partial_v \Omega) & g_{44,2} &= 2r(\partial_v r)\sin^2\theta \\ g_{33,1} &= 2r(\partial_u r) & g_{44,3} &= 2r^2 \sin\theta\cos\theta \\ g_{33,2} &= 2r(\partial_v r). \end{aligned}$$

Now, we have the coordinate formula for the Christoffel symbols

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}).$$

So we can compute:

$$\begin{aligned} \Gamma^1_{ij} &= \tfrac{1}{2} g^{1l}(g_{il,j} + g_{jl,i} - g_{ij,l}) \\ &= \tfrac{1}{2} g^{12}(g_{i2,j} + g_{j2,i} - g_{ij,2}) \\ &= -\Omega^{-2}(g_{i2,j} + g_{j2,i} - g_{ij,2}); \end{aligned}$$

thus

$$
\begin{aligned}
\Gamma^1_{11} &= -\Omega^{-2}(g_{12,1}+g_{12,1}-g_{11,2}) = -\Omega^{-2}(2g_{12,1}) = 2\Omega^{-1}(\partial_u\Omega),\\
\Gamma^1_{12} = \Gamma^1_{21} &= -\Omega^{-2}(g_{12,2}+g_{22,1}-g_{12,2}) = 0,\\
\Gamma^1_{33} &= -\Omega^{-2}(g_{32,3}+g_{32,3}-g_{33,2}) = \Omega^{-2}g_{33,2} = 2\Omega^{-2}r(\partial_v r)\\
\Gamma^1_{44} &= -\Omega^{-2}(g_{42,4}+g_{42,4}-g_{44,2}) = \Omega^{-2}g_{44,2} = 2\Omega^{-2}r(\partial_v r)\sin^2\theta.
\end{aligned}
$$

An identical procedure gives us the Christoffel symbols of the form Γ^2_{ij}, and we have

$$
\begin{aligned}
\Gamma^2_{12} = \Gamma^2_{21} &= -\Omega^{-2}(g_{11,2}+g_{21,1}-g_{12,1}) = 0\\
\Gamma^2_{22} &= -\Omega^{-2}(g_{21,2}+g_{21,2}-g_{22,1}) = -\Omega^{-2}(2g_{21,2}) = 2\Omega^{-1}(\partial_v\Omega)\\
\Gamma^2_{33} &= -\Omega^{-2}(g_{31,3}+g_{31,3}-g_{33,1}) = \Omega^{-2}g_{33,1} = 2\Omega^{-2}r(\partial_u r)\\
\Gamma^2_{44} &= -\Omega^{-2}(g_{41,4}+g_{41,4}-g_{44,1}) = \Omega^{-2}g_{44,1} = 2\Omega^{-2}r(\partial_u r)\sin^2\theta.
\end{aligned}
$$

And

$$
\begin{aligned}
\Gamma^3_{ij} &= \tfrac{1}{2}g^{3l}(g_{il,j}+g_{jl,i}-g_{ij,l})\\
&= \tfrac{1}{2}g^{33}(g_{i3,j}+g_{j3,i}-g_{ij,3})\\
&= \tfrac{1}{2}r^{-2}(g_{i3,j}+g_{j3,i}-g_{ij,3});
\end{aligned}
$$

thus

$$
\begin{aligned}
\Gamma^3_{13} = \Gamma^3_{31} &= \tfrac{1}{2}r^{-2}(g_{13,3}+g_{33,1}-g_{13,3}) = \tfrac{1}{2}r^{-2}g_{33,1} = \tfrac{1}{2}r^{-2}(2r(\partial_u r)) = r^{-1}(\partial_u r)\\
\Gamma^3_{23} = \Gamma^3_{32} &= \tfrac{1}{2}r^{-2}(g_{23,3}+g_{33,2}-g_{23,3}) = \tfrac{1}{2}r^{-2}g_{33,2} = \tfrac{1}{2}r^{-2}(2r(\partial_v r)) = r^{-1}(\partial_v r)\\
\Gamma^3_{44} &= \tfrac{1}{2}r^{-2}(g_{43,4}+g_{43,4}-g_{44,3}) = -\tfrac{1}{2}r^{-2}g_{44,3} = -\sin\theta\cos\theta.
\end{aligned}
$$

Likewise,

$$
\Gamma^4_{ij} = \tfrac{1}{2}r^{-2}(\sin\theta)^{-2}(g_{i4,j}+g_{j4,i}-g_{ij,4}),
$$

so

$$
\begin{aligned}
\Gamma^4_{14} = \Gamma^4_{41} &= \tfrac{1}{2}r^{-2}(\sin\theta)^{-2}(g_{14,4}+g_{44,1}-g_{14,4}) = \tfrac{1}{2}r^{-2}(\sin\theta)^{-2}g_{44,1} = r^{-1}(\partial_u r)\\
\Gamma^4_{24} = \Gamma^4_{42} &= \tfrac{1}{2}r^{-2}(\sin\theta)^{-2}(g_{24,4}+g_{44,2}-g_{24,4}) = \tfrac{1}{2}r^{-2}(\sin\theta)^{-2}g_{44,2} = r^{-1}(\partial_v r)\\
\Gamma^4_{34} = \Gamma^4_{43} &= \tfrac{1}{2}r^{-2}(\sin\theta)^{-2}(g_{34,4}+g_{44,3}-g_{34,4}) = \tfrac{1}{2}r^{-2}(\sin\theta)^{-2}g_{44,3} = \cot\theta.
\end{aligned}
$$

To summarize: all Christoffel symbols are zero except

$$\begin{aligned} \Gamma^1_{11} &= 2\,(\partial_u\Omega)\Omega^{-1} & \Gamma^2_{22} &= 2\,(\partial_v\Omega)\Omega^{-1} \\ \Gamma^1_{33} &= 2r(\partial_v r)\Omega^{-2} & \Gamma^2_{33} &= 2r(\partial_u r)\Omega^{-2} \\ \Gamma^1_{44} &= 2r(\partial_v r)\sin^2\theta\,\Omega^{-2} & \Gamma^2_{44} &= 2r(\partial_u r)\sin^2\theta\,\Omega^{-2} \end{aligned}$$

$$\begin{aligned} \Gamma^3_{13} = \Gamma^3_{31} &= (\partial_u r)r^{-1} & \Gamma^4_{14} = \Gamma^4_{41} &= (\partial_u r)r^{-1} \\ \Gamma^3_{23} = \Gamma^3_{32} &= (\partial_v r)r^{-1} & \Gamma^4_{24} = \Gamma^4_{42} &= (\partial_v r)r^{-1} \\ \Gamma^3_{44} &= -\sin\theta\cos\theta & \Gamma^4_{34} = \Gamma^4_{43} &= \cot\theta. \end{aligned}$$

Appendix B

DERIVING THE EINSTEIN FIELD EQUATIONS

In order to write down the field equations, we must first find the Ricci and scalar curvatures for the metric $g = \Omega^2 du\,dv + r^2 g_{S^2}$. Now, the components of the curvature tensor are

$$R_{ijkl} = g_{ml}(\Gamma^m_{jk,i} - \Gamma^m_{ik,j}) + g_{pl}(\Gamma^m_{jk}\Gamma^p_{im} - \Gamma^m_{ik}\Gamma^p_{jm}),$$

and so the Ricci tensor is given by

$$R_{jk} = \Gamma^i_{jk,i} - \Gamma^i_{ik,j} + \Gamma^m_{jk}\Gamma^i_{im} - \Gamma^m_{ik}\Gamma^i_{jm}.$$

Thus, using our coordinates (u, v, θ, ϕ) (labeled 1, 2, 3, and 4 respectively) and using the results of Appendix A, we have

$$\begin{aligned}
R_{11} &= \Gamma^i_{11,i} - \Gamma^i_{i1,1} + \Gamma^m_{11}\Gamma^i_{im} - \Gamma^m_{i1}\Gamma^i_{1m} \\
&= \Gamma^1_{11,1} - (\Gamma^1_{11,1} + \Gamma^3_{31,1} + \Gamma^4_{41,1}) + \Gamma^1_{11}(\Gamma^1_{11} + \Gamma^3_{31} + \Gamma^4_{41}) \\
&\quad -(\Gamma^1_{11}\Gamma^1_{11} + \Gamma^3_{31}\Gamma^3_{13} + \Gamma^4_{41}\Gamma^4_{14}) \\
&= -(\Gamma^3_{31,1} + \Gamma^4_{41,1}) + \Gamma^1_{11}(\Gamma^3_{31} + \Gamma^4_{41}) - (\Gamma^3_{31}\Gamma^3_{13} + \Gamma^4_{41}\Gamma^4_{14}) \\
&= -2\partial_u\left((\partial_u r)r^{-1}\right) + 2(\partial_u\Omega)\Omega^{-1}(2(\partial_u r)r^{-1}) - 2\left((\partial_u r)r^{-1}\right)^2 \\
&= -2(\partial^2_{uu}r)r^{-1} + 2(\partial_u r)^2 r^{-2} + 2\,(\partial_u\Omega)\Omega^{-1}\left(2(\partial_u r)r^{-1}\right) - 2\left((\partial_u r)r^{-1}\right)^2 \\
&= -2(\partial^2_{uu}r)r^{-1} + 4\,(\partial_u\Omega)(\partial_u r)(\Omega r)^{-1};
\end{aligned}$$

$$\begin{aligned}
R_{12} = R_{21} &= \Gamma^i_{12,i} - \Gamma^i_{i2,1} + \Gamma^m_{12}\Gamma^i_{im} - \Gamma^m_{i2}\Gamma^i_{1m} \\
&= -(\Gamma^2_{22,1} + \Gamma^3_{32,1} + \Gamma^4_{42,1}) - (\Gamma^3_{32}\Gamma^3_{13} + \Gamma^4_{42}\Gamma^4_{14}) \\
&= -\partial_u\left(2(\partial_v\Omega)\Omega^{-1} + 2(\partial_v r)r^{-1}\right) - 2\left((\partial_v r)r^{-1}\right)\left((\partial_u r)r^{-1}\right) \\
&= -2(\partial^2_{uv}\Omega)\Omega^{-1} + 2\,(\partial_u\Omega)(\partial_v\Omega)\Omega^{-2} - 2(\partial^2_{uv}r)r^{-1} + 2(\partial_u r)(\partial_v r)r^{-2} \\
&\quad -2(\partial_u r)(\partial_v r)r^{-2} \\
&= -2(\partial^2_{uv}\Omega)\Omega^{-1} + 2\,(\partial_u\Omega)(\partial_v\Omega)\Omega^{-2} - 2(\partial^2_{uv}r)r^{-1};
\end{aligned}$$

by the symmetry between u and v,

$$R_{22} = -2(\partial_{vv}^2 r)r^{-1} + 4(\partial_v \Omega)(\partial_v r)(\Omega r)^{-1};$$

$$\begin{aligned} R_{13} &= \Gamma^i_{13,i} - \Gamma^i_{i3,1} + \Gamma^m_{13}\Gamma^i_{im} - \Gamma^m_{i3}\Gamma^i_{1m} \\ &= \Gamma^3_{13,3} - \Gamma^4_{43,1} + \Gamma^3_{13}\Gamma^4_{43} - \Gamma^4_{43}\Gamma^4_{14} \\ &= 0 - 0 + \left((\partial_u r)r^{-1}\right)\cot\theta - \cot\theta\left((\partial_u r)r^{-1}\right) \\ &= 0, \end{aligned}$$

and similarly $R_{23} = 0$;

$$\begin{aligned} R_{14} &= \Gamma^i_{14,i} - \Gamma^i_{i4,1} + \Gamma^m_{14}\Gamma^i_{im} - \Gamma^m_{i4}\Gamma^i_{1m} \\ &= \Gamma^4_{14,4} \\ &= 0, \end{aligned}$$

and similarly $R_{24} = 0$;

$$\begin{aligned} R_{33} &= \Gamma^i_{33,i} - \Gamma^i_{i3,3} + \Gamma^m_{33}\Gamma^i_{im} - \Gamma^m_{i3}\Gamma^i_{3m} \\ &= \Gamma^1_{33,1} + \Gamma^2_{33,2} - \Gamma^4_{43,3} + \Gamma^1_{33}(\Gamma^1_{11} + \Gamma^3_{31} + \Gamma^4_{41}) + \Gamma^2_{33}(\Gamma^2_{22} + \Gamma^3_{32} + \Gamma^4_{42}) \\ &\quad -(2\Gamma^3_{13}\Gamma^1_{33} + 2\Gamma^2_{33}\Gamma^3_{32} + \Gamma^4_{43}\Gamma^4_{34}) \\ &= \Gamma^1_{33,1} + \Gamma^2_{33,2} - \Gamma^4_{43,3} + \Gamma^1_{33}\Gamma^1_{11} + \Gamma^2_{33}\Gamma^2_{22} - \Gamma^4_{43}\Gamma^4_{34} \\ &= \partial_u\left(2r(\partial_v r)\Omega^{-2}\right) + \partial_v\left(2r(\partial_u r)\Omega^{-2}\right) - \partial_\theta(\cot\theta) + \left(2r(\partial_v r)\Omega^{-2}\right)\left(2(\partial_u\Omega)\Omega^{-1}\right) \\ &\quad + \left(2r(\partial_u r)\Omega^{-2}\right)\left(2(\partial_v\Omega)\Omega^{-1}\right) - (\cot\theta)^2 \\ &= 2(\partial_u r)(\partial_v r)\Omega^{-2} + 2r(\partial_{uv}^2 r)\Omega^{-2} - 4r(\partial_v r)(\partial_u\Omega)\Omega^{-3} \\ &\quad +2(\partial_v r)(\partial_u r)\Omega^{-2} + 2r(\partial_{uv}^2 r)\Omega^{-2} - 4r(\partial_u r)(\partial_v\Omega)\Omega^{-3} \\ &\quad +1 + 4r(\partial_v r)(\partial_u\Omega)\Omega^{-3} + 4r(\partial_u r)(\partial_v\Omega)\Omega^{-3} \\ &= 4(\partial_u r)(\partial_v r)\Omega^{-2} + 4r(\partial_{uv}^2 r)\Omega^{-2} + 1; \end{aligned}$$

$$\begin{aligned} R_{34} = R_{43} &= \Gamma^i_{34,i} - \Gamma^i_{i4,3} + \Gamma^m_{34}\Gamma^i_{im} - \Gamma^m_{i4}\Gamma^i_{3m} \\ &= 0; \end{aligned}$$

and finally,

$$
\begin{aligned}
R_{44} &= \Gamma^i_{44,i} - \Gamma^i_{i4,4} + \Gamma^m_{44}\Gamma^i_{im} - \Gamma^m_{i4}\Gamma^i_{4m} \\
&= \Gamma^1_{44,1} + \Gamma^2_{44,2} + \Gamma^3_{44,3} + \Gamma^1_{44}(\Gamma^1_{11} + \Gamma^3_{31} + \Gamma^4_{41}) + \Gamma^2_{44}(\Gamma^2_{22} + \Gamma^3_{32} + \Gamma^4_{42}) \\
&\quad + \Gamma^3_{44}\Gamma^4_{43} - (2\Gamma^1_{44}\Gamma^4_{41} + 2\Gamma^2_{44}\Gamma^4_{42} + 2\Gamma^3_{44}\Gamma^4_{43}) \\
&= \Gamma^1_{44,1} + \Gamma^2_{44,2} + \Gamma^3_{44,3} + \Gamma^1_{44}\Gamma^1_{11} + \Gamma^2_{44}\Gamma^2_{22} - \Gamma^3_{44}\Gamma^4_{43} \\
&= \partial_u\left(2r(\partial_v r)\sin^2\theta\,\Omega^{-2}\right) + \partial_v\left(2r(\partial_u r)\sin^2\theta\,\Omega^{-2}\right) + \partial_\theta(-\sin\theta\cos\theta) \\
&\quad + \left(2r(\partial_v r)\sin^2\theta\,\Omega^{-2}\right)\left(2(\partial_u\Omega)\Omega^{-1}\right) + \left(2r(\partial_u r)\sin^2\theta\,\Omega^{-2}\right)\left(2(\partial_v\Omega)\Omega^{-1}\right) \\
&\quad - (-\sin\theta\cos\theta)\cot\theta \\
&= 2(\sin^2\theta)(\partial_u r)(\partial_v r)\Omega^{-2} + 2(\sin^2\theta)r(\partial^2_{uv} r)\Omega^{-2} - 4(\sin^2\theta)r(\partial_v r)(\partial_u\Omega)\Omega^{-3} \\
&\quad + 2(\sin^2\theta)(\partial_v r)(\partial_u r)\Omega^{-2} + 2(\sin^2\theta)r(\partial^2_{uv} r)\Omega^{-2} - 4(\sin^2\theta)r(\partial_u r)(\partial_v\Omega)\Omega^{-3} \\
&\quad + \sin^2\theta + 4(\sin^2\theta)r(\partial_v r)(\partial_u\Omega)\Omega^{-3} + 4(\sin^2\theta)r(\partial_u r)(\partial_v\Omega)\Omega^{-3} \\
&= 4(\sin^2\theta)(\partial_u r)(\partial_v r)\Omega^{-2} + 4(\sin^2\theta)r(\partial^2_{uv} r)\Omega^{-2} + \sin^2\theta.
\end{aligned}
$$

To summarize: all components of the Ricci tensor are zero except

$$
\begin{aligned}
R_{11} &= -2(\partial^2_{uu} r)r^{-1} + 4(\partial_u\Omega)(\partial_u r)(\Omega r)^{-1} \\
R_{12} = R_{21} &= -2(\partial^2_{uv}\Omega)\Omega^{-1} + 2(\partial_u\Omega)(\partial_v\Omega)\Omega^{-2} - 2(\partial^2_{uv} r)r^{-1} \\
R_{22} &= -2(\partial^2_{vv} r)r^{-1} + 4(\partial_v\Omega)(\partial_v r)(\Omega r)^{-1} \\
R_{33} &= 4(\partial_u r)(\partial_v r)\Omega^{-2} + 4r(\partial^2_{uv} r)\Omega^{-2} + 1 \\
R_{44} &= (\sin^2\theta)\left(4(\partial_u r)(\partial_v r)\Omega^{-2} + 4r(\partial^2_{uv} r)\Omega^{-2} + 1\right).
\end{aligned}
$$

Now the scalar curvature $R = g^{jk}R_{jk}$ is just

$$
\begin{aligned}
R &= 2g^{12}R_{12} + g^{33}R_{33} + g^{44}R_{44} \\
&= -4\Omega^{-2}\left(-2(\partial^2_{uv}\Omega)\Omega^{-1} + 2(\partial_u\Omega)(\partial_v\Omega)\Omega^{-2} - 2(\partial^2_{uv} r)r^{-1}\right) \\
&\quad + r^{-2}\left(4(\partial_u r)(\partial_v r)\Omega^{-2} + 4r(\partial^2_{uv} r)\Omega^{-2} + 1\right) \\
&\quad + (r\sin\theta)^{-2}\left((\sin^2\theta)\left(4(\partial_u r)(\partial_v r)\Omega^{-2} + 4r(\partial^2_{uv} r)\Omega^{-2} + 1\right)\right) \\
&= 8(\partial^2_{uv}\Omega)\Omega^{-3} - 8(\partial_u\Omega)(\partial_v\Omega)\Omega^{-4} + 16(\partial^2_{uv} r)r^{-1}\Omega^{-2} + 8(\partial_u r)(\partial_v r)r^{-2}\Omega^{-2} + 2r^{-2},
\end{aligned}
$$

and so the Einstein tensor $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ has the following non-zero components:

$$G_{11} = R_{11} = -2(\partial^2_{uu}r)r^{-1} + 4\,(\partial_u\Omega)(\partial_u r)(\Omega r)^{-1};$$

$$G_{22} = R_{22} = -2(\partial^2_{vv}r)r^{-1} + 4\,(\partial_v\Omega)(\partial_v r)(\Omega r)^{-1};$$

$$\begin{aligned} G_{12} &= R_{12} - \tfrac{1}{2}R\,g_{12} \\ &= -2(\partial^2_{uv}\Omega)\Omega^{-1} + 2(\partial_u\Omega)(\partial_v\Omega)\Omega^{-2} - 2(\partial^2_{uv}r)r^{-1} \\ &\quad + \tfrac{1}{4}\Omega^2\left(8\,(\partial^2_{uv}\Omega)\Omega^{-3} - 8\,(\partial_u\Omega)(\partial_v\Omega)\Omega^{-4} + 16(\partial^2_{uv}r)r^{-1}\Omega^{-2}\right. \\ &\qquad \left. +8(\partial_u r)(\partial_v r)r^{-2}\Omega^{-2} + 2r^{-2}\right) \\ &= 2(\partial^2_{uv}r)r^{-1} + 2(\partial_u r)(\partial_v r)r^{-2} + \tfrac{1}{2}r^{-2}\Omega^2; \end{aligned}$$

$$\begin{aligned} G_{33} &= R_{33} - \tfrac{1}{2}R\,g_{33} \\ &= 4(\partial_u r)(\partial_v r)\Omega^{-2} + 4r(\partial^2_{uv}r)\Omega^{-2} + 1 \\ &\quad -\tfrac{1}{2}r^2\left(8(\partial^2_{uv}\Omega)\Omega^{-3} - 8(\partial_u\Omega)(\partial_v\Omega)\Omega^{-4} + 16(\partial^2_{uv}r)r^{-1}\Omega^{-2}\right. \\ &\qquad \left. +8(\partial_u r)(\partial_v r)r^{-2}\Omega^{-2} + 2r^{-2}\right) \\ &= -4r(\partial^2_{uv}r)\Omega^{-2} - 4r^2(\partial^2_{uv}\Omega)\Omega^{-3} + 4r^2(\partial_u\Omega)(\partial_v\Omega)\Omega^{-4}; \end{aligned}$$

and by inspection,

$$G_{44} = (\sin^2\theta)\left(-4r(\partial^2_{uv}r)\Omega^{-2} - 4r^2\,(\partial^2_{uv}\Omega)\Omega^{-3} + 4r^2\,(\partial_u\Omega)(\partial_v\Omega)\Omega^{-4}\right).$$

Plugging these components into the Einstein equation $G = 2T$ now clearly yields equations (3.4) through (3.7).

Appendix C

COMPUTING $\theta_{\pm}$ IN SPHERICAL SYMMETRY

Fix coordinates $u = u_0$ and $v = v_0$ and consider the two-sphere of radius $r = r(u_0, v_0)$ comprising the points $\{(u_0, v_0, \theta, \phi)\} \subset M$. We previously defined ∂_u as the "ingoing" direction and ∂_v as the "outgoing" one; denoting these vectors by n^a and ℓ^a, respectively, we then have

$$\theta_+ = \theta_{(\ell)} = h^{ab}\nabla_b \ell_a = h^{ab}(\ell_{a,b} - \ell_c \Gamma^c_{ab}),$$

where $h_{ab} = r^2(g_{S^2})_{ab}$ is the induced metric on the given two-sphere and h^{ab} is its inverse. Using the coordinate labeling as in Appendices A and B, we then have that $h^{33} = r^{-2}$, $h^{44} = r^{-2}(\sin\theta)^{-2}$, and all other components are zero. Also,

$$\ell_a = g_{ab}\ell^b = g_{a2},$$

so $\ell_1 = -\frac{1}{2}\Omega^2$ and the other three components are zero. In particular, it is now clear that $h^{ab}\ell_{a,b} = 0$ for all $a, b = 1, 2, 3, 4$, and we are left with

$$\begin{aligned}
\theta_+ &= -h^{ab}\ell_c\Gamma^c_{ab} \\
&= -h^{33}\ell_1\Gamma^1_{33} - h^{44}\ell_1\Gamma^1_{44} \\
&= -r^{-2}\cdot\left(-\tfrac{1}{2}\Omega^2\right)\cdot(2r(\partial_v r)\Omega^{-2}) - r^{-2}(\sin\theta)^{-2}\cdot\left(-\tfrac{1}{2}\Omega^2\right)\cdot\left(2r(\partial_v r)(\sin^2\theta)\Omega^{-2}\right) \\
&= 2(\partial_v r)r^{-1}.
\end{aligned}$$

Similarly, one computes that $\theta_- = \theta_{(n)} = 2(\partial_u r)r^{-1}$.

VITA

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