Homework consists of four parts: the written assignment W1-W4 which counts towards the final grade for the course, the extra credit problems which are for those who are interested in taking the Putnam competition and anyone in search of an opportunity to boost his or her grade, the presentation assignment (which will not be handed in) and (occasionally) a reading assignment.

Please don't get discouraged if you cannot immediately solve all of the problems, especially the presentation problems and the extra credit problems. The instructors are more than happy to discuss any of the problems by e-mail and in person - in particular, during Monday office hours - before the assignment is due. Hints are given upon request.

Written assignment (4 problems).

Writing Problem 1. Let $S(n)$ denote the sum of the digits of n.

1. Prove that for any n, the sequence
$$n, S(n), S(S(n)), \ldots$$
eventually becomes constant. The value of that constant is called the digital sum of n.

2. Prove that for any twin primes, other than 3 and 5, the digital sum of their product is 8 (twin primes are consecutive odd primes, such at 17, 19).

Writing Problem 2.

(a) Show that $9^{121} + 13^{1331}$ is divisible by 22.

(b) Show that $30^{239} + 239^{30}$ is not a prime number.

Writing Problem 3. Let $\{a_n\}_{n\geq 0}$ be a sequence of integers given by the rule
$$a_{n+1} = 2a_n + 1.$$
Does there exist a value for a_0 such that the sequence consists *entirely* of prime numbers?

Writing Problem 4. Let $d_1 = 1 < d_2 < \ldots < d_k = n$ be all the positive divisors of the positive integer n, and let $D = \sum_{i=1}^{k-1} d_i d_{i+1}$. Show that $d < n^2$. (Hint: it must be the case that $d_i \geq i$.)

Extra Credit Problem 1. Let $n > 6$ be an integer with the following property. Let $d_1 = 1 < d_2 < \ldots < d_k < n$ be all the positive integers that are less than n and relatively prime to n; then $d_2 - d_1 = d_3 - d_2 = \ldots = d_k - d_{k-1}$. Prove that either n is prime, or it is a power of 2.

Presentation assignment (5 problems).

Presentation Problem 1. Suppose $n > 1$ is an integer. Show that $n^4 + 4^n$ is not a prime.

Presentation Problem 2. Recall that the sequence of Fibonacci numbers is defined by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ if $n \geq 2$. Show that if F_n is divisible by p for some $n \geq 0$, then there are infinitely many n such that F_n is divisible by p.

Presentation Problem 3. Prove that $\binom{p^k}{n}$ is divisible by p for any n, $1 \leq n \leq p^k - 1$.

Presentation Problem 4. Find all prime numbers in the sequence

$$10001, 100010001, 1000100010001, \ldots$$

Presentation Problem 5. Let $(a_1, \ldots, a_n)$ be positive integers, and let d_k be the Greatest Common Divisor of all products of k integers from the set $(a_1, \ldots, a_n)$, where $2 \leq k \leq n-2$. Prove that $d_k^2 \mid d_{k-1}d_{k+1}$.