Math 380 *HOMEWORK #8* *due Tue, November 22, 2016*

Written assignment (4 problems).

W1. Compute the exponential generating function for the sequence satisfying the recurrence $a_0 = a_1 = 1$, $a_n = a_{n-1} + (n-1)a_{n-2}$. Then use the generating function to obtain an expression for a_n (in the form of a sum).

W2. Prove the Vandermonde formula using generating functions:

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}$$

W3.

a) Let $f(x)$ be the ordinary generating function for the sequence $\{a_n\}_{n\geq 0}$. Show that $a_n = f^{(n)}(0)/n!$, where $f^{(n)}$ is the nth formal derivative of f.

b) Fix an integer $k \geq 1$. Let a_n be the number of solutions of the equation $x_1 + x_2 + \ldots + x_k = n$ where $x_1, \ldots, x_k$ are non-negative integers. Find a formula for a_n by using generating functions.

W4. Does there exist a subset S of the non-negative integers so that for any positive integer n the equation

$$x + 2y = n$$

has exactly one solution with $x, y \in S$?

Presentation assignment (4 problems).

P1. Find and prove the closed form for Fibonacci numbers.

P2. Using generating functions, prove that any integer weight between 1 and 100 can be balanced uniquely on a two cup scale using the weights 1, 3, 9, 27 and 81.

P3. Prove that F_{238} is divisible by 239. No computer programs or explicit calculations, please!

P4. Prove that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, for all $n \geq 1$, via ordinary generating functions, as follows. First, compute the ordinary generating function for $a_n = n^2$. Then, multiply it to another well-known ordinary generating function, to obtain the ordinary generating function for $\{\sum_{k=1}^{n} k^2\}_{n\geq 1}$. Finally, use the form of the obtained product to deduce the formula for the sum of squares.