MATH 380 **FINAL EXAM** **Autumn 2016**

December 8, 2016

NAME: SIGNATURE:

Instructions: there are six problems on the final. You need to solve (completely and impeccably) four of them to get 100%. Partial credit will be given on all six but sparingly. One complete solution is worth more than several partial ones. Please check your solutions very carefully and justify all your claims.

Problem	Number of points	Points obtained
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total		

You can use handouts from the class' webpage but no other materials. Please turn off and put away your cell phone and all other electronic devices.

<u>Problem 1.</u> Let p be an odd prime and a, b, c, d be integers, not all of them 0, such that the polynomial $p(x) = ax^3 + bx^2 + cx + d$ is divisible by p for any integer x. Find all possibilities for the set $\{p, a, b, c, d\}$.

Problem 2. For what values of x does $\sum_{n=1}^{\infty} \sin(nx)$ converge?

Problem 3. Let C be a circle and P a given point on the plane. Each line through P that intersects C determines a chord of C. Show that the midpoints of those chords lie on a circle.

Problem 4.

a) [5pt] Let $a_0 = A > 0$, $a_{n+1} = a_n^2 + \frac{1}{4}$. Analyze the convergence/divergence of the sequence, depending on A.

b) [5pt] Let $\lambda > 0$, and let $\{b_n\}_{n\geq 0}$ be the sequence defined by $b_0 = B > 0$, $b_{n+1} = b_n^2 + \lambda$. For what values of λ there exists a unique number $L_\lambda \in \mathbb{R}$ (not depending on B) such that, **if** $\{b_n\}_{n\geq 0}$ converges, its limit is L_λ?

Problem 5. Let H_n be the n-cube in n-space, given by all 2^n vectors $(a_1, a_2, \ldots, a_n)$ such that $a_i \in \{0,1\}$, for all $i \in \{1, 2, \ldots, n\}$. An edge in H_n joins two vertices whose coordinates differ in exactly one place. (See pictures below for H_2, H_3.)

a) [2pt] Show that H_2 and H_3 have Hamiltonian cycles.

b) [8pt] Show that H_n has a Hamiltonian cycle, for any $n \geq 4$.

A Hamiltonian cycle is a cycle which passes through each vertex exactly once.

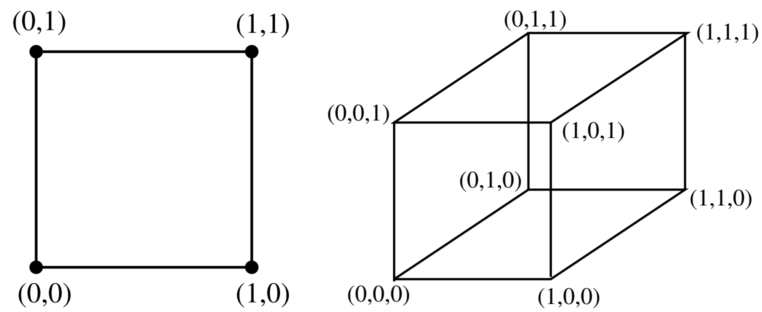

Figure 1: The graphs H_2 (left) and H_3 (right).

<u>Problem 6.</u> A student writes a sequence of N 0's and 1's on the board, as follows: the first two digits are 1 and 0, and thereafter the $n+1$st digit is 1 with probability i/n, where i is the number of 1's among the first n digits. For example, the third digit is 1 with probability $1/2$; if the third digit is 1, then the fourth digit is 1 with probability $2/3$, whereas if the third digit is 0, the fourth digit is 1 with probability $1/3$., etc.

Show that for any $k > 0$ the probability of having exactly k 1's in the sequence of length N is independent of k.