Math 380a, Fall 2018 **HOMEWORK #2** *due Tue, October 9*

Homework consists of four parts: the written assignment W1-W4 which counts towards the final grade for the course, the extra credit problems which are for those who are interested in taking the Putnam competition and anyone in search of an opportunity to boost his or her grade, the presentation assignment (which will not be handed in) and (occasionally) a reading assignment.

Please don't get discouraged if you cannot immediately solve all of the problems, especially the presentation problems and the extra credit problems. The instructors are more than happy to discuss any of the problems by e-mail and in person - in particular, during Monday office hours - before the assignment is due. Hints are given upon request.

Written assignment (4 problems).

Writing Problem 1.

1. Prove the following combinatorial identity in two ways:

$$\binom{2n+2}{n+1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}$$

   (a) algebraically (using Pascal's triangle), and
   (b) combinatorially (by counting things in different ways).

2. Prove the following identity combinatorially (by counting something):

$$\sum_{i=1}^{n-1} (i-1)\, i\, (n-i-1) = 2\binom{n}{4}\ .$$

Writing Problem 2. How many ways are there to represent a positive integer n as a sum of

(a) k non-negative integers?

(b) k positive integers?

Note: the order of summation matters. For example, take $n = 3$, $k = 2$. Then the possible sums in (a) are: $3 + 0, 2 + 1, 1 + 2, 0 + 3$.

Writing Problem 3. A permutation of $[n] := \{1, 2, \dots, n\}$ is a bijective function from $[n]$ to $[n]$ (sometimes given as the string $f(1)f(2)f(3)\dots f(n)$, as in 123 representing the identity function on $[3]$. There are $n!$ permutations, for any n.) A fixed point for a permutation f is a value i such that $f(i) = i$. (For example, 231 has no fixed points, while 1243 has two.)

Assume $n \geq 4$. Which set is more numerous: the set of permutations of $[n]$ with no fixed points (also known as *derangements* of $[n]$), or the set of permutations with at least one fixed point?

Writing Problem 4. Let Γ be a connected planar graph with E edges, V vertices and F faces. Assume that V is at least 3. Prove that

$$3V - 6 \geq E$$

Extra Credit Problem 1. Let X be a set of points in an n-dimensional plane such that all point in X have coordinates ± 1. Show that if the cardinality of X is bigger than $2^{n+1}/n$ then there exist three points in X which form an equilateral triangle.

Presentation assignment (5 problems).

Presentation Problem 1. For any $n \geq 4$, calculate the number of derangements of $[n]$ (as before, permutations of $[n]$ with no fixed points). The answer can be in the form of a sum.

Definition. Let Γ be a directed (i.e., oriented) graph. A Hamiltonian path in Γ is a directed path that visits every vertex exactly once.

Presentation Problem 2. Prove that every tournament (complete directed graph with no loops) has a Hamiltonian path.

Definition. A *regular polyhedron* is a polyhedron such that all its vertices have the same degree, and all its faces are congruent regular polygons. A convex regular polyhedron is also called a *Platonic solid*.

Presentation Problem 3. Let $\mathcal{P}$ be a convex regular polyhedron. Show that the degree of any vertex of $\mathcal{P}$ is at most 5.

Presentation Problem 4. An n-domino consists of two squares, each of which is marked with a number of dots from 0 to n. (For any pair of numbers from 0 to n, there is exactly one n-domino corresponding to it.)

1. How many n-dominos are there?
2. For what values of n is it possible to arrange the n-dominos in a circle so that the adjacent halves of neighboring dominos show the same number? (E.g., for $n = 2$ this is possible as follows $00, 01, 11, 12, 22, 20$, and back to 00.)

Presentation Problem 5. You are given a set of n distinct real numbers, $n \geq 2$. What is the minimal cardinality of the set consisting of all distinct averages taken over all pairs of these n numbers?