Homework consists of four parts: the written assignment W1-W4 which counts towards the final grade for the course, the extra credit problems which are for those who are interested in taking the Putnam competition and anyone in search of an opportunity to boost his or her grade, the presentation assignment (which will not be handed in) and (occasionally) a reading assignment.

Written assignment (4 +1 problems).

Writing Problem 1. Let n be a positive integer, and denote by $S(n)$ the sum of digits of n.

1. Prove that for any n, the sequence

$$n, S(n), S(S(n)), \dots$$

eventually becomes constant. The value of that constant is called the digital sum of n.

2. Prove that for any twin primes, other than 3 and 5, the digital sum of their product is 8 (twin primes are consecutive odd primes, such at 17, 19).

Writing Problem 2.

1. We say that $x \in \{0, 1, \dots, (n-1)\}$ is a square modulo n if there exists a remainder $y \in \{0, 1, \dots, (n-1)\}$ such that $y^2 = x \pmod{n}$. Show that if $n = m^2 + 1$ and x is a square modulo n, so is $n - x$.
2. Show that if $m^2 + n^2 + p^2 = 0 \pmod 5$, then at least one of them is divisible by 5.

Writing Problem 3. Let $\{a_n\}_{n \geq 0}$ be a sequence of integers given by the rule

$$a_{n+1} = 2a_n + 1.$$

Does there exist a value for a_0 such that the sequence consists *entirely* of prime numbers?

Writing Problem 4. Prove that $\binom{p^k}{n}$ is divisible by p for any n, $1 \leq n \leq p^k - 1$, k a positive integer.

Extra Credit Problem 1. Let $f(x) = x^n + 5x^{n-1} + 3$ where $n > 1$ is an integer. Prove that $f(x)$ does not decompose as a product of two polynomials with integer coefficients of degree at least 1.

Presentation assignment (2 + 4 problems).

Left-over from HW 2:

Presentation Problem 1. Let $\mathcal{P}$ be a convex regular polyhedron. Show that the degree of any vertex of $\mathcal{P}$ is at most 5.

Presentation Problem 2. You are given a set of n distinct real numbers, $n \geq 2$. What is the minimal cardinality of the set consisting of all distinct averages taken over all pairs of these n numbers?

New:

Presentation Problem 3. Suppose $n > 1$ is an integer. Show that $n^4 + 4^n$ is not a prime.

Presentation Problem 4. Let $S(n)$ be the sum of digits of n. Find $S(S(S(n)))$ for $n = 4444^{4444}$.

Presentation Problem 5. Recall that the sequence of Fibonacci numbers is defined by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ if $n \geq 2$. Show that if F_n is divisible by p for some $n \geq 0$, then there are infinitely many n such that F_n is divisible by p.

Presentation Problem 6. Let $f(n) = \sum_{i=1}^{p-1} in^{i-1}$. Prove that if $f(m) = f(n) \pmod p$ then $m = n \pmod p$.