Math 380a, Fall 2018 ***HOMEWORK #5*** ***due Tue, October 30***

Homework consists of four parts: the written assignment W1-W4 which counts towards the final grade for the course, the extra credit problems which are for those who are interested in taking the Putnam competition and anyone in search of an opportunity to boost his or her grade, the presentation assignment (which will not be handed in) and (occasionally) a reading assignment.

Please don't get discouraged if you cannot immediately solve all of the problems, especially the presentation problems and the extra credit problems. I am happy to discuss any of the problems by e-mail and in person - in particular, during Monday office hours - before the assignment is due. Hints are given upon request.

This homework has no reading assignment.

Written assignment (4 problems).

Writing Problem 1. Prove that consecutive Fibonacci numbers are always relatively prime.

Writing Problem 2. Suppose $p(x) = \sum_{i=0}^{r} a_i x^i$ is a polynomial with all coefficients a_i integers. Assume that a_0, a_r, and $p(1)$ are all odd. Show that $p(x)$ cannot have rational roots.

Writing Problem 3. The nth Fermat number is defined as $\Phi_n = 2^{2^n} + 1$.

1. Prove that Fermat numbers satisfy the following relation:

$$\Phi_n = \Phi_0 \Phi_1 \dots \Phi_{n-1} + 2.$$

2. Prove **"Goldbach's theorem"**: any two Fermat numbers are co-prime.

Writing Problem 4. Let f be a function defined on positive integers such that $f(1) = 1$, $f(2n) = f(n)$, and $f(2n+1) = f(n) + 1$. Find an expression for f.

Extra Credit Problem 1. Let n be a positive integer such that $\sigma(n) = 2n + 1$ (recall the definition of σ from class). Show that n is the square of an odd number.

Presentation assignment (4 problems).

Presentation Problem 1. What is the minimal number of weights needed to weigh any integer number of grams from 1 to 100 on a standard, two-cup weighing scale, if you are allowed to put weights on *both* sides of the scale?

Presentation Problem 2. We call an integer *square-free* if it is the product of single powers of primes (i.e., no square power of a prime divides it). For example, 6 is square-free, while 12 is not.

Given $n \in \mathbb{N}$, show that one can find a large enough integer A such that *none* of the numbers A, $A+1$, $A+2$, $\ldots$, $A+n-1$ is square-free.

Presentation Problem 3. A polynomial with integer coefficients is called *primitive* if its coefficients are relatively prime.

1. Show that a product of two primitive polynomials is primitive.

2. Prove **Gauss' Lemma**: If a polynomial with integer coefficients can be factored as a product of two non-constant polynomials with rational coefficients then it can also be factored as a product of two non-constant polynomials with integer coefficients.

Presentation Problem 4. A positive integer n is called *perfect* if $\sigma(n) = 2n$.

1. Show that if $2^k - 1$ is a prime then $2^{k-1}(2^k - 1)$ is perfect.

2. Prove the partial converse: every even perfect number must be of the form $2^{k-1}(2^k - 1)$, where $2^k - 1$ is a prime.