Math 380 **HOMEWORK #8** *due Tue, November 20, 2018*

Written assignment (4 problems).

Writing Problem 1. Compute the exponential generating function for the sequence satisfying the recurrence $a_0 = a_1 = 1$, $a_n = a_{n-1} + (n-1)a_{n-2}$. Then use the generating function to obtain an expression for a_n (in the form of a sum).

Writing Problem 2. Prove Vandermonde formula using generating functions:

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}$$

Writing Problem 3.

a) Let $f(x)$ be the ordinary generating function for the sequence $\{a_n\}_{n\geq 0}$. Show that $a_n = f^{(n)}(0)/n!$, where $f^{(n)}$ is the nth formal derivative of f.

b) Fix an integer $k \geq 1$. Let a_n be the number of solutions of the equation $x_1 + x_2 + \ldots + x_k = n$ where $x_1, \ldots, x_k$ are non-negative integers. Find a formula for a_n by using generating functions.

Writing Problem 4.

1. Show that $\prod_{i\geq 0}(1 + x^{2^i}) = \frac{1}{1-x}$ for $|x| < 1$.

2. Does there exist a subset S of the non-negative integers so that for any positive integer n the equation
$$x + 2y = n$$
has exactly one solution with $x, y \in S$.

Extra Credit Problem 1. Let a_n be the number of partitions of n into distinct parts; and let b_n be the number of partitions of n into odd parts. Show that $a_n = b_n$ establishing an explicit bijection.

Presentation assignment (4 problems).

Presentation Problem 1. Find and prove the closed form for Fibonacci numbers.

Presentation Problem 2. Using generating functions, prove that any integer weight between 1 and 100 can be balanced uniquely on a two cup scale using the weights 1, 3, 9, 27 and 81.

Presentation Problem 3. Prove that F_{238} is divisible by 239. No computer programs or explicit calculations, please!

Presentation Problem 4. Prove that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, for all $n \geq 1$, via ordinary generating functions, as follows. First, compute the ordinary generating function for $a_n = n^2$. Then, multiply it to another well-known ordinary generating function, to obtain the ordinary generating function for $\{\sum_{k=1}^{n} k^2\}_{n\geq 1}$. Finally, use the form of the obtained product to deduce the formula for the sum of squares.