LECTURE I FOR MATH 380A

INTRODUCTION: METHODS OF ARGUMENT

Topics:

- Induction
- Pigeon Hole
- Contradiction
- Test Cases
- Random ideas (coloring, ...)

Sections 2.3, 3.3, 3.4 in [Z]

1. Induction

Introduce (or recall) basic principles of induction (see, e.g., [Z], p.45).

Standard Induction.
We want to prove some assertion $P(n)$ for all $n \geq n_0$.

(1) Base: Establish that $P(n_0)$ is true
(2) Induction step: Assume $P(n)$ is true and prove $P(n+1)$.

Strong Induction.

(1) Base: Establish that $P(n_0)$ is true (sometimes, need to check the first several values n_0, n_0+1, ...
(2) Induction step: Assume $P(n_0), P(n_0+1), \ldots, P(n)$ are all true and prove $P(n+1)$.

Often we need just $P(n)$ and $P(n-1)$.

Example 1.1. Sum formulas, such as

(a) $1 + 3 + \ldots + (2k-1) = k^2$
(b) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdot + \frac{1}{n-1\cdot n} = \frac{n-1}{n}$

We introduce formulas one by one without the answers and ask students to figure them out and then prove.

Example 1.2. ([Z, 2.2.16].) For each positive integer n, find positive integer solutions $x_1, \ldots, x_n$ to the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \ldots x_n} = 1$$

Solution: $x_1 = 2$; from $\frac{1}{x_n} + \frac{1}{x_1 \ldots x_n} = \frac{1}{x_1 \ldots x_{n-1}}$ solve to get $x_n = x_1 \ldots x_{n-1} + 1$.

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2. Pigeon Hole

Other names: Dirichlet principle, rabbits and cages.

Example 2.1 (Z, 3.3.1)**.** Every point on the plane is colored either red or blue. Show that there exist two points of the same color which are exactly on mile apart.

Example 2.2.

(1) You are given $n + 1$ numbers. Show that you can choose two of them such that their difference is divisible by n.
(2) You are given $n + 2$ numbers. Prove that there are two numbers among them such that either their sum or difference is divisible by $2n$.

3. Contradiction

Show example with $\sqrt{2}$ being irrational.
See homework for more examples, such as $\sqrt{p} + \sqrt{q}$ is an irrational problem.

4. Coloring/Parity/Invariants

Example 4.1.

(1) A grasshoper jumps along the line. He can go exactly 1cm in either direction. Can he get back to his initial position after 2011 jumps?
(2) Now the grasshopper changes his pattern. He jumps on a rectangular grid; he is allowed to go 2cm left, right, up, or down. He is also allowed to along the diagonal of any unit square of the grid (e.g., from (0,0) to (1,1)). He starts at $(0, 1)$. Can there be a sequence of steps which takes him to the origin?
(3) Now back on the line. His first jump is 1 cm in length, the second - 2cm, the third - 3cm and so on. Each jump can take him either right or left. Can the grasshoper return to its initial position after 2011 jumps? (2009 jumps?)

(1), (2) is parity. For (3) consider the positions mod 4. Then the pattern has period 7: 1, -1, 2, 2, -1, 1, 0 (Or check that $\frac{n(n+1)}{2}$ is divisible by 4 for the first time when $n = 7$). So to get back the number of jumps should be at least divisible by 7. Hence, the answer is NO for 2011. For 2009 there is an explicit construction (done in fragments of length 7). Discuss the difference between necessary and sufficient conditions.

References

[Z] P. Zeitz, *The art and craft of problem solving*, Second edition, John Wiley and Sons, (2007).