LECTURE II, MATH 380A

BASIC COMBINATORICS

Topics:

- Counting and Binomial formula
- Graph theory
- Exclusion/inclusion

Sections in the book: 6.1, 6.2 and 6.3.

1. Counting and binomial formula

Recall binomial coefficients and formula, notation $\binom{n}{k} = C_n^k$, Pascal triangle.

Example 1.1. Show in two ways, algebraic and enumerative, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Example 1.2. Balls in Urns formula. The number of different ways we can place n *indistinguishable* balls into k *distinguishable* urns is

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

How to see that? Write down $n+k-1$ dots in a row, and choose $k-1$ dots. The dots between ith and $(i+1)$st dot go into the $(i+1)$st urn.

Example 1.3. Vandermonde convolution formula

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}$$

<u>Algebraic proof.</u> Compare the coefficient of x^k in $(1+x)^{n+m}$ and in $(1+x)^n(1+x)^m$.

<u>Enumerative (combinatorial, counting) proof.</u> Plot $n+m$ dots, and divide into 2 groups of n and m dots respectively. To choose k dots we need to choose i dots from the first group and $k-i$ from the second, for all $0 \le i \le k$.

2. Inclusion/Exclusion formula

$$|\bigcup A_i| = \sum |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| - \dots$$

In particular, we can get better and better estimations of $|\bigcup A_i|$ by taking the first few terms; lower bounds with even numbers of terms and upper bounds with odd number of terms.

$$\sum |A_i| \ge |\bigcup A_i| \ge \sum |A_i| - \sum_{i,j} |A_i \cap A_j|$$

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$$\sum |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \geq |\bigcup A_i| \geq$$
$$\sum |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| - \sum_{i,j,k,\ell} |A_i \cap A_j \cap A_k \cap A_\ell|$$

etc.

3. Graphs

Vertices, edges, degrees, trees, completed graphs, directed graphs, tournaments, Euler formula.

Example 3.1. Question: is it possible that in a class of 77 people each student is friends with exactly 13 other students.

Theorem. The sum of degrees is always even.

Observation. In a tree, the number of vertices exceeds the numbers of edges by 1.

Example 3.2. Prove the converse.

Definition 3.3. Euler cycle = cycle that traverses every edge once; Euler path = path that traverses every edge once.

Example 3.4. In every connected graph with all vertices of even degrees there exists an Euler cycle.

Definition 3.5. (Homework related). A Hamiltonian path is a path that travels exactly once through every vertex of the graph. A Hamiltonian cycle is a Hamiltonian path which is also a cycle.

Example 3.6. Show that the Euler formula holds for any planar graph:

$$V - E + F = 2$$

Prove by induction on E: delete an edge until we get a tree, then use te "observation" for a tree.

Note: Also mention polygons; and emphasize that $V - E + F$ is an invariant in this problem.