LECTURE 8, GENERATING FUNCTIONS AND RECURRENCES

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Topics (4.3)

- Generating functions
- Recurrences
- Partitions

1. Generating Functions

Consider a sequence of numbers $\{a_n\}_{n\in\mathbb{N}}$, and define two kinds of functions encoding this sequence:

a) *ordinary* generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and
b) *exponential* generating function $g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$.

We are not in the least concerned with whether f and g make sense as actual functions, rather than formal power series. They are only used for computation, and convergence of the RHSs is not explored at all at this time.

Multiplication for sequences a_n, b_n with generating functions $f_1(x)$, $f_2(x)$:

- ordinary: $f_1(x)f_2(x)$ corresponds to $c_n = \sum_{k=0}^{n} a_k b_{n-k}$
- exponential: $f_1(x)f_2(x)$ corresponds to $d_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$

Addition, in both cases, corresponds to addition of sequences.

Example 1.1. Find the ogf and egf corresponding to the sequence of all 1.

Proof.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \; ; \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x \; .$$

□

Example 1.2. Find the ogf and egf corresponding to the sequence $\{2^n\}_{n\in\mathbb{N}}$.

Proof.

$$\sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x} \; ; \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n = e^{2x} \; .$$

□

Example 1.3. Find the ogf, resp egf corresponding to $a_n = n$.

Proof.

$$\sum_{n=0}^{\infty} n x^n = x \sum_{n=0}^{\infty} n x^{n-1} = x \left(\frac{1}{1-x} \right)' = \frac{x}{(1-x)^2} \; .$$

On the other hand the egf is

$$\sum_{n=0}^{\infty} \frac{n}{n!} x^n = x \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} = x(e^x)' = xe^x \; .$$

□

Example 1.4. Find the unique sequence satisfying $\sum_{k=0}^{n} a_k a_{n-k} = 1$, $\forall n \geq 0$.

Proof. Look above; the sum is the coefficient of x^n in $(f(x))^2$, where $f(x)$ is the ogf for a_n. Thus, $(f(x))^2 = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

Thus $f(x) = \frac{1}{(1-x)^{1/2}} = (1-x)^{-1/2} = \sum_{n\geq 0} (-1)^n \binom{-1/2}{n} x^n$. Hence

$$\begin{aligned} a_n &= (-1)^n \binom{-1/2}{n} = (-1)^n \frac{(-1/2)(-1/2-1)\dots(-1/2-(n-1))}{n!} \\ &= \frac{(1/2)(3/2)\dots((2n-1)/2)}{n!} = \frac{(2n-1)!!}{2^n n!} . \end{aligned}$$

□

2. Recurrences

One can use gfs to solve recurrences, e.g., $a_{n+1} = 3a_n - 2a_{n-1}$, $a_0 = 1$, $a_1 = 3$.

Proof. Multiply whole thing by x^{n+1}, sum over $n \geq 1$, and use ogf:

$$\begin{aligned} \sum_{n=1}^{\infty} a_{n+1} x^{n+1} &= 3x \sum_{n=1}^{\infty} a_n x^n - 2x^2 \sum_{n=1}^{\infty} a_{n-1} x^{n-1} \\ f(x) - a_1 x - a_0 &= 3x(f(x) - a_0) - 2x^2 f(x) , \end{aligned}$$

which yields

$$f(x)(1 - 3x + 2x^2) = x(a_1 - 3a_0) + a_0 ,$$

i.e. $f(x) = \frac{1}{(1-2x)(1-x)}$. Write

$$f(x) = \frac{1}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x} ,$$

solve to get $A = -1$, $B = 2$. Hence $a_n = 2^{n+1} - 1$. □

What happened? It follows that for a recurrence of the type $a_{n+1} = \alpha a_n + \beta a_{n-1}$, one should solve $1 - \alpha x - \beta x^2 = 0$, get the solutions x_1 and x_2, and then find $a_n = A x_1^n + B x_2^n$, and find A, B by solving $a_0 = A + B$, $a_1 = A x_1 + B x_2$. Note this is how one gets for example the Fibonacci numbers!

This works wonderfully for constant-coefficient recurrences. What about more complicated stuff?

Example 2.1. Find the general term in the sequence satisfying $a_0 = 1$,

$$2a_{n+1} = \sum_{i=0}^{n} \binom{n}{i} a_i a_{n-i} , \ n \geq 0.$$

Proof. This suggest egf, see format of RHS! Let

$$g(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n$$

Then

$$2g'(x) = 2 \sum_{n\geq 0} \frac{a_{n+1}}{n!} x^n$$

and $g(x)^2 = \sum \frac{b_n}{n!} x^n$ with $b_n = \sum\limits_{i=0}^{n} \binom{n}{i} a_i a_{n-i}$.

We need to solve:

$$2g'(x) = (g(x))^2\ ,\ g(0) = 1\ .$$

We have $-\frac{g'(x)}{(g(x))^2} = -\frac{1}{2}$, so $\frac{1}{g(x)} = C - \frac{1}{2}x$. Since $g(0) = 1$, $C = 1$. Thus $g(x) = \frac{1}{1-\frac{1}{2}x}$, and $a_n = 2^{-n}n!$. □

3. Partitions

Another interesting application is in computing/proving things about partitions of a given integer. Partition is a way of writing n as a sum of numbers without ordering.

Example 3.1. Suppose a_n is the number of partitions of n into distinct parts, and b_n is the number of partitions of n into odd parts. Show that $a_n = b_n$. (E.g., $n = 4$; $\{4\}$ and $\{(1, 3)\}$, resp., $\{1, 3\}$ and $\{1, 1, 1, 1\}$).

Proof. For $\{a_n\}$, can think of a_n as the coefficient of x^n in $\prod_{i=1}^{\infty}(1 + x^i)$. On the other hand, can think of b_n as the coefficient of x^n in $\prod_{i \text{ odd}}(1 + x^i + x^{2i} + ...) = \prod_{i \text{ odd}} \frac{1}{1-x^i}$.

Note that

$$\prod_{i=1}^{\infty}(1 + x^i) = \prod_{i=1}^{\infty} \frac{1 - x^{2i}}{1 - x^i}\ ;$$

note that in the above, every factor $(1 - x^{2i})$ in the numerator gets cancelled eventually, so you are left with $\prod_{i \text{ odd}} \frac{1}{1-x^i}$, thus the two generating functions are equal, hence the sequences are equal. □

References

Z [Z] P. Zeitz, *The art and craft of problem solving*, Second edition, John Wiley and Sons, (2007).