Homework consists of four parts: the written assignment W1-W4 which counts towards the final grade for the course, the extra credit problems which are for those who are interested in taking the Putnam competition and anyone in search of an opportunity to boost his or her grade, the presentation assignment (which will not be handed in) and (occasionally) a reading assignment.

Please don’t get discouraged if you cannot immediately solve all of the problems, especially the presentation problems and the extra credit problems. The instructors are more than happy to discuss any of the problems by e-mail and in person - in particular, during Monday office hours - before the assignment is due. Hints are given upon request.

This homework has no reading assignment.

Written assignment (4 problems).

Writing Problem 1. There are 2015 distinct positive integers placed in a circle. Is it possible that the ratio of any two consecutively placed numbers (the largest to smallest) is a prime? What if we replace 2015 with 2014?

Writing Problem 2. For which positive integers \( N \) there exist \( N \) distinct positive integers such that the sum of all of them is divisible by each one of them?

Writing Problem 3. Suppose \( x \) is a positive real number such that \( x + \frac{1}{x} \) is an integer. Prove that \( x^{2014} + \frac{1}{x^{2014}} \) is an integer.

Writing Problem 4. Integers from 1 to 2014 are written on the board. At any point you can erase any two numbers \( a, b \) from the board and replace them with \( a + b - 1, a - b - 1 \). Is it possible to get all 0’s?

Extra Credit Problem 1. Define

\[
q(n) = \left\lfloor \frac{n}{\sqrt{n}} \right\rfloor.
\]

Determine (with proof) all positive integers \( n \) such that

(i) \( q(n) < q(n + 1), \)
(ii) \( q(n) = q(n + 1), \)
(iii) \( q(n) > q(n + 1). \)
Presentation assignment (4 problems).

Presentation Problem 1. The sequence of Fibonacci numbers is defined by

\[ F_1 = F_2 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad \text{if} \quad n \geq 2. \]

(a) Prove that any positive integer can be represented as a sum of several different Fibonacci numbers.
(b) Prove that \( F_n \) is divisible by 3 if and only if \( n \) is divisible by 4.

Presentation Problem 2. The \( n \)-dimensional space is colored with \( n \) colors such that every point in the space is assigned a color. Show that there exist two points of the same color exactly a mile from each other.

Presentation Problem 3. Three frogs are placed on three vertices of a square. Every minute, one frog leaps over another frog, in such a way that the “leapee” is at the midpoint of the line segment whose endpoints are the starting and ending positions of the “leaper”. Will a frog ever occupy the vertex of the square which was originally unoccupied?

Presentation Problem 4. Let \( \overline{abc} \) be a three digit integer. Show that there exist a concatenation \( \overline{abcabc} \ldots \overline{abc} \) which is divisible by 101.