Math 308H
Spring 2016
Final Exam
June 9, 2016
Time Limit: 110 Minutes

Name (Print): ____________________

Instructor ____________________

This exam contains 10 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a sheet of notes, 8.5×11 and handwritten by you, but no other devices, books, or notes are permitted.

Unless otherwise stated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	10	
3	10	
4	10	
5	15	
6	20	
7	15	
Total:	100	

1. Suppose we have

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}.$$

(a) (5 points) Is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ a basis for $\mathbb{R}^3$? Justify your answer.

Yes. We can check that the matrix $[u_1 u_2 u_3]$ has determinant $6 \neq 0$.

(b) (7 points) Run the Gram-Schmidt procedure on $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

We get

$$\begin{aligned}
v_1 &= u_1 \\
v_2 &= u_2 - \frac{u_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\
v_3 &= u_3 - \frac{u_3 \cdot v_1}{\|v_1\|^2} v_1 - \frac{u_3 \cdot v_2}{\|v_2\|^2} v_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} - 0 - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}
\end{aligned}$$

(c) (4 points) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ denote the basis found by Gram-Schmidt in part (b).

Let $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$. Find coefficients c_1, c_2 and c_3 so that $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$.

$$\begin{aligned} c_1 &= \frac{x \cdot v_1}{\|v_1\|^2} = \frac{3}{3} = 1. \\ c_2 &= \frac{x \cdot v_2}{\|v_2\|^2} = \frac{2}{2} = 1. \\ c_3 &= \frac{x \cdot v_3}{\|v_3\|^2} = \frac{6}{6} = 1. \end{aligned}$$

(d) (4 points) Let $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, and $S = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ where $\mathbf{u}_1$, $\mathbf{u}_2$ are as in part (a).

Find $\operatorname{proj}_S \mathbf{x}$.

An orthogonal basis for S is $\{v_1, v_2\}$, so we get that the projection is $c_1v_1 + c_2v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

2. (a) (5 points) Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Is A^3 invertible? Explain. If it is, find the determinant of the inverse of A^3. In other words, find $\det((A^3)^{-1})$.

$\det(A) = 4$, so A is invertible, and hence A^3 is invertible (with inverse $(A^{-1})^3$, in case you're wondering.) $\det((A^3)^{-1}) = \frac{1}{\det(A)^3} = \frac{1}{64}$.

(b) (5 points) Suppose $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is an orthonormal basis for $\mathbb{R}^4$, and let $U = [\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_4]$ be the corresponding 4×4 matrix. What is U^TU? Explain.

U^TU is the 4×4 identity matrix. We see this because the i, j entry of U^TU is exactly $u_i \cdot u_j$. If $i = j$, this is $\|u_i\|^2 = 1$, and if $i \neq j$ then this is zero since they're orthogonal.

3. Given two sets A and B, we will use $A \cup B$ to denote their union - that is, $A \cup B$ is the set of all vectors that are in at least one of A, B. We will use $A \cap B$ to denote their intersection - that is, $A \cap B$ is the set of all vectors that are in both A and B.

   (a) (5 points) Suppose S_1 and S_2 are subspaces of $\mathbb{R}^n$ and $S = S_1 \cup S_2$. For each of the three subspace conditions, decide whether that condition is always satisfied by S. If it is not, provide a counter-example.

   Contains zero. Closed under scalar multiplication. Not closed under addition - consider for example the union of the x-axis and the y-axis in $\mathbb{R}^2$.

   (b) (5 points) Suppose S_1 and S_2 are subspaces of $\mathbb{R}^n$ and $S = S_1 \cap S_2$. For each of the three subspace conditions, decide whether that condition is always satisfied by S. If it is not, provide a counter-example.

   All three conditions are satisfied.

4. Let $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$.

   (a) (5 points) Compute the characteristic polynomial of A and find the eigenvalues λ_1 and λ_2.

   The characteristic polynomial is $p(\lambda) = (\lambda + 3)(\lambda - 1)$, so the eigenvalues are $-3, 1$.

   (b) (5 points) For one of the eigenvalues in part (a) (it doesn't matter which you pick), find a basis for the corresponding eigenspace.

   For $\lambda = -3$, get basis of $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$. For $\lambda = 1$, get basis of $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

5. Decide whether the following statements are true or false, and circle the explanation that best explains why.

(a) (3 points) 5 vectors in $\mathbb{R}^4$ cannot form an orthogonal set.

1. True. Orthogonal sets are always linearly independent, and 5 vectors cannot be linearly independent in $\mathbb{R}^4$.
2. True. This follows from the big theorem.
3. **False**. Consider for example $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{0}\}$, where $\mathbf{e}_i \in \mathbb{R}^4$ is the vector with a one in the ith position, zeros elsewhere.
4. False. In 4 dimensions we can space out 5 unit vectors so they are all orthogonal.

(b) (3 points) For any square matrix A, $\det(5A) = 5\det(A)$.

1. True. $\det(5A) = \det((5I)A) = \det(5I) \cdot \det(A) = 5\det(A)$.
2. True. The determinant is a linear map.
3. False. Consider for example A being the zero matrix.
4. **False**. Consider for example $A = I$.

(c) (3 points) If A and B are invertible $n \times n$ matrices, then $AB = BA$.

1. True. We know $ABA = ABA$, and then we can multiply both sides of the equation by A^{-1} to get $AB = BA$.
2. True. Invertible matrices are always diagonalizable, and we can compute that $AB = BA$ for diagonal matrices.
3. **False**. Consider the matrices $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.
4. False. Consider the matrices $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Notice that the multiplication BA is not even defined.

(d) (3 points) If A is an $n \times m$ matrix with linearly independent columns, then $\text{row}(A) = \mathbb{R}^m$.

1. True. The number of rows is at least m, and so they must span $\mathbb{R}^m$.
2. **True**. The dimension of $\text{col}(A)$ is m, and so the dimension of $\text{row}(A)$ is also m.
3. False. We might have $n \neq m$, and so the big theorem does not apply.
4. False. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. We have $m = 3$ but there are only two row vectors, so their span can't be all of $\mathbb{R}^3$.

(e) (3 points) If A is an $n \times m$ matrix with orthogonal columns, then the rows of A are also orthogonal.

1. True. The dimensions of the row space and column space are the same, so if the columns are orthogonal then so are the rows.
2. True. Notice that the columns of A are precisely the rows of A^T.
3. False. Consider for example the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.
4. **False**. Consider for example the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

6. Read each of the following statements carefully, and decide whether it is true or false. You are not required to justify your answers, but I recommend justifying them to yourself.

   (a) (2 points) If $\mathbf{y} \in \text{col}(A)$, then any least-squares solution to $A\mathbf{x} = \mathbf{y}$ is in fact a solution to $A\mathbf{x} = \mathbf{y}$.

   True

   (b) (2 points) Suppose a linear system has 5 variables and 3 equations, and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ is a solution to the system. Then the system has infinitely-many solutions.

   True

   (c) (2 points) Suppose a linear system has 3 variables and 5 equations, and $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a solution to the system. Then the system has only the trivial solution.

   False

   (d) (2 points) If a diagonal matrix has all nonzero entries along the diagonal, then it is invertible.

   True

   (e) (2 points) If A is an $n \times n$ matrix, then $\text{row}(A) = \text{col}(A)$.

   False

(f) (2 points) If T_1 and T_2 are linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$ and we define $T(x) = T_1(x) + T_2(x)$, then T is also a linear transformation.

True

(g) (2 points) Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are vectors in $\mathbb{R}^m$, and A is an $n \times m$ matrix. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions to the equation $A\mathbf{x} = \mathbf{0}$, then so is the vector $\mathbf{x}_1 + \mathbf{x}_2$.

True

(h) (2 points) The set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 15 \\ 0 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

False

(i) (2 points) The set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ is linearly independent.

False

(j) (2 points) If $T_1 : \mathbb{R}^m \to \mathbb{R}^k$ and $T_2 : \mathbb{R}^k \to \mathbb{R}^n$, define $T : \mathbb{R}^m \to \mathbb{R}^n$ by $T(\mathbf{x}) = T_2(T_1(\mathbf{x}))$. If T_1 and T_2 are onto linear transformations, then T is also an onto linear transformation.

True

7. Give an example of each of the following:

(a) (5 points) A 2×2 matrix so that $A^2 = 0$ but $A \neq 0$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) (5 points) A 3×3 matrix A so that $A^3 = 0$ but $A^2 \neq 0$. (Hint: Think about how your matrix in (a) acts on the standard basis vectors)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) (5 points) A 2×3 matrix A and a 3×2 matrix B so that $AB = I_2$ but $BA \neq I_3$.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$