Math 308O **Name (Print):** ____________________
Winter 2015
Final Exam
March 16, 2015
Time Limit: 1 Hour 50 Minutes Instructor ____________________

This exam contains 10 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

For question 1 (to be completed separately in the first ten minutes and handed in) you may not use any notes or calculators. For questions 2-7, you may use a scientific calculator and a sheet of notes, 8.5×11 and handwritten by you, but no other devices, books, or notes are permitted.
You are required to show your work on each problem on this exam, unless otherwise specified. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	15	
2	15	
3	15	
4	10	
5	15	
6	21	
7	9	
Total:	100	

1. (15 points) Suppose S is a subspace of $\mathbb{R}^n$. Prove that the orthogonal complement of S, $S^{\perp}$, is also a subspace of $\mathbb{R}^n$.

2. Suppose we have

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

(a) (3 points) Is $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ an orthogonal set? Justify your answer.

(b) (4 points) Is $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ a basis for $\mathbb{R}^3$? Justify your answer.

(c) (5 points) Use projections to find an orthogonal basis for $\mathbb{R}^3$ that includes the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$. That is, you must find a vector $\mathbf{w}$ so that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{w}\}$ is an orthogonal basis. (Remember to check your work.)

(d) (3 points) What is $\text{proj}_S(\mathbf{w})$, where $S = \text{span}\{\mathbf{v_1}, \mathbf{v_2}\}$?

3. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) (5 points) Find the characteristic polynomial and all eigenvalues of the matrix A.

(b) (7 points) Choose an eigenvalue λ of the matrix A. (It doesn't matter which one you pick.) For this eigenvalue, find a basis for the corresponding eigenspace.

(c) (3 points) For the eigenvalue you chose in part (b), does the multiplicity of the eigenvalue equal the dimension of the corresponding eigenspace?

4. (a) (5 points) Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Is A^3 invertible? Explain. If it is, find the determinant of the inverse of A^3. In other words, find $\det((A^3)^{-1})$.

(b) (5 points) Suppose A, B, and $A+B$ are invertible matrices. Simplify the following expression as much as possible.

$$(A+B)^T(A^T+B^T)^{-1}B(A+B)^{-1}B^{-1}.$$

5. (a) (5 points) Let $\mathcal{S}$ be the standard basis, and suppose $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. If $x_{\mathcal{B}_1} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}_{\mathcal{B}_1}$, find $x_{\mathcal{S}}$.

(b) (5 points) Now consider the basis $\mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. If $x_{\mathcal{S}} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}_{\mathcal{S}}$, find $x_{\mathcal{B}_2}$.

(c) (5 points) If $x_{\mathcal{B}_1} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{B}_1}$, compute $x_{\mathcal{B}_2}$.

6. Read each of the following statements carefully, and decide whether it is true or false. You are not required to justify your answers, but I recommend justifying them to yourself.

   (a) (3 points) If the columns of A form an orthonormal basis for $\mathbb{R}^n$, then $A^T = A^{-1}$.

   (b) (3 points) Suppose $\mathbf{x_1}$ and $\mathbf{x_2}$ are vectors in $\mathbb{R}^m$, and A is an $n \times m$ matrix. If $\mathbf{x_1}$ and $\mathbf{x_2}$ are both solutions to the equation $A\mathbf{x} = \mathbf{0}$, then so is the vector $\mathbf{x_1} + \mathbf{x_2}$.

   (c) (3 points) The set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 15 \\ 0 \\ 0 \end{bmatrix} \right\}$ spans all of $\mathbb{R}^4$.

   (d) (3 points) Suppose $\hat{\mathbf{x}}$ in $\mathbb{R}^m$ is a least squares solution to the system $A\mathbf{x} = \mathbf{y}$. Then $\|A\hat{\mathbf{x}} - \mathbf{y}\| \leq \|\mathrm{A}\mathbf{x} - \mathbf{y}\|$ for any $\mathbf{x}$ in $\mathbb{R}^m$.

(e) (3 points) If λ is an eigenvalue for A, then λ^2 is an eigenvalue of A^2.

(f) (3 points) The function $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 + 1 \end{bmatrix}$ is a linear transformation.

(g) (3 points) Suppose $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly independent set of vectors, and we run the Gram-Schmidt process on them to obtain an orthogonal set of vectors $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$. Then $\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \text{span}\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$.

7. (a) (3 points) Give an example of an orthogonal set of 5 vectors in $\mathbb{R}^5$ that does not span $\mathbb{R}^5$.

(b) (6 points) Find a value for k so that the set $\left\{ \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix} \right\}$ is linearly dependent.