Math 308H
Spring 2016
Midterm 2
May 13, 2016
Time Limit: 50 Minutes

Name (Print): ______________________

Instructor ______________________

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a sheet of notes, 8.5×11 and handwritten by you, but no other devices, books, or notes are permitted.

Unless otherwise stated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	9	
2	10	
3	11	
4	10	
Total:	40	

1. Suppose V is a subset of $\mathbb{R}^n$. For V to be a subspace, we require the three conditions.

   1. $\mathbf{0} \in V$.
   2. If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$. (closed under addition.)
   3. If $\mathbf{u} \in V$, $r \in \mathbb{R}$, then $r\mathbf{u} \in V$. (closed under scalar multiplication.)

   For each of the following subsets V, decide whether each condition is satisfied. For example, you might say that conditions 1 and 3 hold, but condition 2 does not.

   (a) (3 points) Let $V \subseteq \mathbb{R}^3$ be the set of all vectors of the form $\begin{bmatrix} x \\ x^2 \\ x^3 \end{bmatrix}$ for $x \in \mathbb{R}$.

   Solution: Condition 1 is satisfied, but not 2 or 3.

   (b) (3 points) Let $V \subseteq \mathbb{R}^2$ be the set of all vectors of the form $\begin{bmatrix} x \\ 2x \end{bmatrix}$ for $x \geq 0$.

   Solution: Conditions 1 and 2 are satisfied, but not 3.

   (c) (3 points) Let $V \subseteq \mathbb{R}^3$ be the set of all vectors of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where y and z are any real numbers and x is an integer.

   Solution: Conditions 1 and 2 are satisfied, but not 3.

2. Read each of the following statements carefully, and decide whether it is true or false. You are not required to justify your answers, but I recommend justifying them to yourself.

(a) (2 points) If A and B are equivalent matrices, then $\text{col}(A) = \text{col}(B)$.

False. For example, consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The column spaces (which are just the spans of the columns) are not equal.

(b) (2 points) If A and B are $n \times n$ diagonal matrices, then $AB = BA$.

True. Write out two diagonal matrices - you will see that the product is a diagonal matrix whose diagonal is the product of the original two. Since multiplication of numbers is commutative, so is this multiplication.

(c) (2 points) Suppose $m > n$ and $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation with $T(\mathbf{x}) = A\mathbf{x}$. Then for any $\mathbf{b} \in \mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ is consistent.

False. The condition $A\mathbf{x} = \mathbf{b}$ is equivalent to T being onto, which isn't necessarily true just because $m > n$. For example, $T : \mathbb{R}^3 \to \mathbb{R}^2$ could be the map sending everything to 0.

(d) (2 points) If A and B are $n \times n$ matrices with AB invertible, then A and B are also invertible.

True. If B is not invertible, then there is a nontrivial x with $Bx = 0$. But then $ABx = 0$ as well, so AB couldn't have been invertible. Therefore B has to be invertible. Now, if A is not invertible, there is a nontrivial x so that $Ax = 0 \implies AB(B^{-1}x) = 0$. Since B is invertible, we must have that $B^{-1}x$ is nonzero, and so AB can't be invertible. Therefore if AB is invertible, then both A and B are as well.

(e) (2 points) Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, and $V \subseteq \mathbb{R}^n$ is a subspace. Then the set of all $\mathbf{x} \in \mathbb{R}^m$ with $T(\mathbf{x}) \in V$ is a subspace of $\mathbb{R}^m$.

True. We can check all conditions of being a subspace, and they will hold since T is linear and V is a subspace.

3. (a) (3 points) Suppose A and B are $n \times n$ matrices and that A, B, and $A+B$ are invertible. Simplify the following expression as much as possible.

$$(AB)^{-1}BA + (A+B)^T(A^T+B^T)^{-1}$$

Solution: $B^{-1}A^{-1}BA + I$.

(b) (3 points) Solve the following equation for the matrix X, where A,B and C are all $n \times n$ matrices and A and B are invertible.

$$AXB = C.$$

$X = A^{-1}CB^{-1}$.

(c) (3 points) Give an example of an onto linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Consider the linear transformation $T(x) = Ax$ where A is the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$.

(d) (2 points) Give an example of two 2×2 invertible matrices A and B whose sum $A+B$ is not invertible.

Let $A = I$ and $B = -I$, so that $A + B = 0$.

4. Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & -2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 3 \end{bmatrix}.$$

(a) (5 points) Find a basis for null(A).

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(b) (5 points) Find a basis for col(A).

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$