Chapter 4: Subspaces

This chapter is all about subspaces. Again, this review is intended to be useful, but not comprehensive. The midterm will cover sections 3.1-3.3 and 4.1-4.3 from the textbook.
A subspace is a subset of $\mathbb{R}^n$ that satisfies certain conditions. You must know the conditions, and know that one way to check if something is a subspace is to check each of the conditions.

Fact 1: The span of a set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
Why is it true?: You can just check that each of the conditions of being a subspace is satisfied.
How is it helpful?: If you want to know if a set is a subspace, sometimes it is clear that it can be written as the span of a set of vectors, and then you know it is a subspace.
Sometimes, though, using the definition is an easier way. For example, we showed that the null space of a matrix is always a subspace by just using the definition. You should know what the null space is, know that it is a subspace, and be able to show why (see proof of theorem 4.3).

You should also know the definition of basis and dimension. Two important facts about bases: (remark: bases is the plural of basis).
Fact 2: If a set of vectors forms a basis for a subspace S, then every element of S can be written uniquely as a linear combination of those vectors.
Why is it true?: Since a basis spans, there is at least one way to represent any vector. Since a basis is linearly independent, there can't be more than one way. (Check this!)
How is it helpful?: We will occasionally find it useful to express vectors in terms of bases other than the standard one. It is helpful that there is a unique way to do this.

Fact 3: If S is a subspace of $\mathbb{R}^n$, then any basis of S has the same number of vectors. (There are lots of different choices of basis, but they all have the same number of elements. This is called the dimension.)
Why is it true?: We will probably skip this proof in class, but see the proof of theorem 4.12 in the book if you are interested.
How is it helpful?: The dimension of a subspace is a very interesting property. It kind of tells us what it looks like. For example, a 2-dimensional subspace of $\mathbb{R}^3$ is a plane in $\mathbb{R}^3$ that goes through the origin. (Try to think of an example, and find a basis for it. Remember the definition of dimension is the size of a basis.) The subspace looks kind of like $\mathbb{R}^2$.

Fact 4: If A and B are equivalent matrices, then the span of the rows of A is the same as the span of the rows of B.
Fact 5: If B is a matrix in echelon form, then the nonzero rows of B form a linearly independent set.
Why is it true?: Use the definition of linearly independent to show this. If you had a linear combination of them that summed to zero, you can see that the coefficient of the first row has to be zero so that the first position of the sum is zero. Then you just keep moving down and show that the next coefficient has to be zero.
How is it helpful?: Using facts 4 and 5, we have a way to find a basis for a subspace if we have a spanning set. We just use the spanning set to form the rows of a matrix, row reduce, and then the nonzero rows form a basis. (This is method 1 to find a basis.)

Fact 6: If U and V are equivalent matrices, then any linear dependence that exists among the columns of U also exists among the columns of V. (See theorem 4.11)
Why is it true?: You could show this by showing that linear dependences are preserved when we apply any elementary row operation. You don't need to know this for the exam though.

Fact 7: Suppose $S = \text{span}\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ is a subspace. Then we can find a basis for S by (this is the second method to find a basis from a spanning set):

1. Using $\mathbf{u}_1, ..., \mathbf{u}_m$ to form the columns of a matrix A
2. Row reducing that matrix to echelon form B
3. The basis is the columns of A corresponding to pivot columns of B.

Why is it true?: The idea is that the pivot columns of B are linearly independent, and the other columns of B are spanned by them. Since the dependences of A and B are the same, the same holds for A, and so those columns are linearly independent and span S. (We will talk about this in class on Monday.)
Why is it helpful?: This method may be a little harder to understand, but it is helpful because it gives you a basis that is actually a subset of the original set of vectors.
To see examples of both methods, look at examples 1 and 2 in section 4.2.

Fact 8: If a set of vectors in a subspace S is linearly independent, either it is a basis or vectors can be added to it to form a basis. If the set spans S, either it is a basis or vectors can be removed from it to form a basis.
Why is it true?: If they are linearly independent but don't span, there is a vector that is not in the span, and adding that vector keeps the set linearly independent. We can keep doing this until we get a linearly independent spanning set. If we have a spanning set, we can find a basis that is a subset of that set by using the second method to find a basis.
How is it helpful?: This is useful to prove some of the other theorems. Also just good to understand that a linearly independent set is at most the size of a basis, and a spanning set is at least the size of the basis, stuff like that.

Fact 9: Suppose U is a set of m vectors in a subspace of dimension k. If $m < k$, then U does not span S. If $m > k$ then U is not linearly independent. If $m = k$ then U is linearly independent if and only if it spans.
Why is it true?: Follows from Facts 3 and 8. See proof of Theorem 4.15 in the book for more details on the first part. Similar proof to the case for $\mathbb{R}^n$.
How is it helpful?: If you know the dimension of a subspace, you can say things about a set of vectors without knowing what the specific vectors are. If you want to check if m vectors form a basis, you only need to check if they span or are linearly independent (and then the other one is the same),

Fact 10: Suppose S_1, S_2 subspaces of $\mathbb{R}^n$ and $S_1 \subset S_2$. Then $\dim(S_1) \leq \dim(S_2)$ and if the dimensions are equal then $S_1 = S_2$.
Why is it true?: We will skip this proof. But basically, if $b_1, ..., b_k$ is a basis for S_1, then it is also a linearly independent set in S_2. If the dimension of S_2 is also k, then it is a basis for S_2. If not, then it can be extended to a basis for S_2.
How is it helpful?: For example, this tells us that the only subspace of $\mathbb{R}^n$ of dimension n is $\mathbb{R}^n$ itself. You should definitely know this.

Finally, the stuff from section 4.3. You should know the definitions of row space, column space, and rank, and know that the rank is the dimension of both the row space and column space (they always match up! Why is this?) You should also know the definition of nullity, and know the rank-nullity theorem. You should also have some idea why the rank nullity theorem holds (See example 3 in section 4.3 and the proof of theorem 4.23). There are also a lot of connections with the big theorem at the end of this section, but I'll go over that in the "big theorem" writeup.