# Chapter 1: Systems of Linear Equations

## Advice: Learn how to solve systems, but also understand how things work in general and what we can prove about general systems

In this chapter you will learn how to set up systems of linear equations from word problems and how to solve them. In addition, I expect you to be able to

- Describe and defend the steps of solving a linear system
- Draw conclusions about linear systems (such as number of solutions it will have) based on traits of the system
- Describe solution sets, even when they are infinite size

To do these things, you need to learn definitions. Definitions in math are very precise, and **I expect you to use the precise definitions**. This doesn't mean you need to memorize every single small definition (and in fact on exams you will be allowed notesheets). But if you don't have something memorized and it shows up, look it up. If you think you won't be able to remember something and it seems important, write it down (and work on committing it to memory). Often students write completely nonsensical things, and I believe it's because they don't really know what things mean.
For example, in order to really understand how we solve linear systems, you need to know the definitions of solutions, elementary operations and equivalent systems as well as echelon form and reduced echelon form.To talk about solution sets, you need to know what a set is, what it means for sets to be equal, etc.

# Chapter 2: Euclidean Space

## Advice: Work with examples and build your intuition, but also get comfortable with the more abstract notation/terminology

In chapter 2 we learn about vectors, linear combinations of vectors, span, and linear independence. You should be reasonably comfortable with all of those terms. What do I mean by this? When you hear one of them, you should be able to think about examples, think about what they mean geometrically for vectors in two and three dimensions, **and be able to give formal definitions** (for any dimension). How do you get comfortable?

- Practice adding and scalar multiplying vectors, and make sure it makes sense to you why all the algebraic properties in theorem 2.3 hold. Also take a look at theorem 2.16 and make sure that makes sense to you. Neither of the proofs is difficult but even if you don't go through the proofs, going through examples can help you feel more comfortable with these facts.

- Go through lots of examples, and make sure you understand everything you are doing. Try picking two vectors in $\mathbb{R}^3$ or $\mathbb{R}^4$ and think about what the span is. Pick another vector and decide if it is in the span of the others or not.
- Try defining span and linear independence in different ways, without changing the meaning and without losing precision.
- Go through some examples of $n$ vectors in $\mathbb{R}^n$ and decide if they span *and* if they are linearly independent. Can you see why they coincide? Make sure you know how span and linear independence are related, but know that they are not the same thing.
- Go through the theorems in sections 2.2 and 2.3 and make sure you understand what they say, why they're useful, and at least a sense of why they're true. The most interesting ones are probably theorems 2.7, 2.8, 2.13, 2.14, and 2.17, along with what I wrote out in the next section regarding the "big theorem".
- Make sure you are comfortable with the matrix notation. Again, go through some examples to see why $Ax = b$ is essentially the same as the system given by the augmented matrix $[A|b]$.

## Big theorem

In the book, the "big theorem" (theorem 2.19 is version 1) only applies when $m = n$. But even in the case when $m$ and $n$ are different, some things will be equivalent so I wrote them out here. When we go into further sections and add to the big theorem, we will also add to each of the parts.
Note: the word "equivalent" here means that the statements are all true or all false - it doesn't mean they mean the same thing. It is important that you know what each part means.

**Theorem 1.** *Suppose $A$ is an $n \times m$ matrix with $A = [\mathbf{a}_1 ... \mathbf{a}_m]$. If $m > n$, none of the following hold. If $m \leq n$, the following are equivalent. (i.e. they are all true or all false.)*

1. *The columns of $A$ are linearly independent (i.e. $\{\mathbf{a}_1, ..., \mathbf{a}_m\}$ is a linearly independent set.)*
2. *$A\mathbf{x} = \mathbf{b}$ has at most one solution $\forall \mathbf{b} \in \mathbb{R}^n$.*

**Theorem 2.** *Suppose $A$ is an $n \times m$ matrix with $A = [\mathbf{a}_1 ... \mathbf{a}_m]$. If $m < n$, none of the following hold. If $m \geq n$, the following are equivalent. (i.e. they are all true or all false.)*

1. *$\{\mathbf{a}_1, ..., \mathbf{a}_m\}$ spans $\mathbb{R}^n$ (i.e. $\text{span}\{\mathbf{a}_1, ..., \mathbf{a}_m\} = \mathbb{R}^n$.)*
2. *$A\mathbf{x} = \mathbf{b}$ has at least one solution $\forall \mathbf{b} \in \mathbb{R}^n$.*

**Theorem 3.** *Now suppose we are in the situation of theorems 1 and 2, with $m = n$. Then all four statements of the conditions of theorems 1 and 2 are equivalent.*
*We can summarize this as saying that the following are equivalent (i.e. they are either all true or all false.)*

*1.* $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for all* $\mathbf{b} \in \mathbb{R}^n$*.*

*2. The columns of A are linearly independent (i.e.* $\{\mathbf{a}_1, ..., \mathbf{a}_m\}$ *is a linearly independent set.)*

*3.* $\{\mathbf{a}_1, ..., \mathbf{a}_m\}$ *spans* $\mathbb{R}^n$ *(i.e.* $\text{span}\{\mathbf{a}_1, ..., \mathbf{a}_m\} = \mathbb{R}^n$*.)*

# 1 General Advice

My biggest advice is to spend more time on the exam thinking about problems and how best to approach them. Try to find ways to do things efficiently. Some might call this taking shortcuts, but really it's just about finding the best/easiest way to approach a problem. If you're finding yourselves doing a lot of messy computation there might be an easier/nicer way - see if you can find one. This is true generally in life, but especially true on exams.
In terms of this specific exam, definitely take a look at true/false questions, and make sure you know how to justify your answers. There are true/false questions in the book and on Webassign. Constantly ask yourself why and what if. For example, what if you got rid of the homogeneous assumption in some true false question? Would it still be true? Why?