Math 308L **Name (Print):** ____________________
Autumn 2016
Midterm 1
October 21, 2016
Time Limit: 50 Minutes Instructor ____________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a sheet of notes, 8.5×11 and handwritten by you, but no other devices, books, or notes are permitted.

Unless otherwise stated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	7	
2	10	
3	6	
4	9	
5	8	
Total:	40	

1. (a) (3 points) Suppose $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Give an example of a matrix A with $A\mathbf{x} = \mathbf{b}$.

Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) (4 points) Let $A = \begin{bmatrix} 3 & 1 \\ 8 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{x}_1 = \begin{bmatrix} 15 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 2 \\ 18 \end{bmatrix}$ and $\mathbf{x}_4 = \begin{bmatrix} -17 \\ -12 \end{bmatrix}$.
Compute $A\mathbf{x}_1 + A\mathbf{x}_2 + A\mathbf{x}_3 + A\mathbf{x}_4$.

Notice that $A\mathbf{x}_1 + A\mathbf{x}_2 + A\mathbf{x}_3 + A\mathbf{x}_4 = A(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4)$, and $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Therefore we multiply the matrix by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to get

$$A\mathbf{x}_1 + A\mathbf{x}_2 + A\mathbf{x}_3 + A\mathbf{x}_4 = \begin{bmatrix} 4 \\ 10 \\ 2 \end{bmatrix}$$

2. Suppose a space ship is moving along some fixed line with constant acceleration. If it has acceleration a, initial velocity b, and initial position c, then its position at time t is given by $p(t) = \frac{1}{2}at^2 + bt + c$. The goal of this problem is to find a, b and c from the following data. Assume that we know that $p(1) = 0, p(2) = 1$, and $p(3) = 4$.

   (a) (4 points) Set up a system of linear equations and find the corresponding augmented matrix.

   The system of equations is

$$\begin{aligned} \frac{1}{2}a + b + c &= 0 \\ 2a + 2b + c &= 1 \\ \frac{9}{2}a + 3b + c &= 4 \end{aligned}$$

   and the corresponding augmented matrix is

$$\begin{bmatrix} \frac{1}{2} & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 \\ \frac{9}{2} & 3 & 1 & 4 \end{bmatrix}.$$

   (b) (3 points) Put the augmented matrix into echelon form.

   There are many possibilities for this. One is the following.

$$\begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

   (c) (3 points) Solve for a, b and c. Please check your work.

   Using back substitution, we end up with $a = 2$, $b = -2$, and $c = 1$.

3. Decide whether the following statements are true or false, and circle the explanation that best explains why.

(a) (3 points) If $m < n$ then any two consistent systems of linear equations with m equations and n variables are equivalent.

1. True. Since both systems are consistent with $m < n$, they both have infinitely-many solutions and so the solution sets are equal.
2. True. We will be able to find a sequence of elementary operations transforming one system into the other, and so they are equivalent.
3. **False**. Consider for example the linear systems (with 1 equation, 2 variables)

$$\begin{array}{ccc} \text{System A:} & \text{and} & \text{System B:} \\ x_1 + x_2 = 5 & & x_1 + x_2 = 7. \end{array}$$

4. False. The corresponding augmented matrices have to be equivalent, but the systems do not.

(b) (3 points) For any vectors $u, v, w \in \mathbb{R}^n$, we have $\text{span}\{u, v\} \subset \text{span}\{u, v, w\}$.

1. **True**. We can always set the coefficient in front of w to be zero, and so any linear combination of u and v is also a linear combination of u, v and w.
2. True. Since the dimensions are all the same, we have $\text{span}\{u, v\} = \text{span}\{u, v, w\}$, and in particular $\text{span}\{u, v\} \subset \text{span}\{u, v, w\}$.
3. False. Consider for example

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

4. False. Consider the case when u, v and w are all the zero vector.

4. Read each of the following statements carefully, and decide whether it is true or false. **Justify your answers.**

   (a) (3 points) The set $\left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 15 \\ 0 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

   False. Notice that $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is not in the span of these four vectors, and so they do not span $\mathbb{R}^4$. By the big theorem, a set of 4 vectors in $\mathbb{R}^4$ is linearly independent if and only if it spans $\mathbb{R}^4$.

   (b) (3 points) The set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ is linearly independent.

   False. You can see this by row reducing the matrix and seeing that the system will have infinitely many solutions. You can also see that if the vectors are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, then $2\mathbf{u}_2 - \mathbf{u}_1 - \mathbf{u}_3 = \mathbf{0}$.

   (c) (3 points) Suppose a linear system has 5 variables and 3 equations, and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ is a solution to the system. Then the system has infinitely-many solutions.

   True. Since the system is consistent we can put it into echelon form, and then there will be free variables since there are more variables than equations.

5. Let h be a real number, and consider the set of vectors in $\mathbb{R}^3$

$$\left\{ \begin{bmatrix} 0 \\ h \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1-h \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} \right\}.$$

In case it is helpful for your computations, the following two matrices are equivalent.

$$\begin{bmatrix} 0 & 2 & 0 & -2 \\ h & 1 & 1-h & 0 \\ 1 & 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1-3h & -3h \\ 0 & 0 & -2+6h & -2+6h \end{bmatrix}$$

(a) (4 points) For what values of h is the set of vectors linearly independent? Explain your reasoning.

For no values of h. A set of 4 vectors in $\mathbb{R}^3$ cannot be linearly independent.

(b) (4 points) For what values of h does the set of vectors span $\mathbb{R}^3$? Explain your reasoning.

For $h \neq \frac{1}{3}$ the vectors span $\mathbb{R}^3$. Imagine we augmented the matrix on the left with the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then if we perform row operations to put the full augmented matrix into echelon form, we see that if $h \neq \frac{1}{3}$ the system will be consistent, regardless of what a, b and c were. Therefore the vectors span.

On the other hand, there is some choice of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ so that the right hand side of the reduced matrix will become $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. If $h = \frac{1}{3}$, then this system is inconsistent. Therefore the vectors do not span.