Participation Quiz: Friday Oct 28 in class

Problem 1. *Suppose $T : \mathbb{R}^m \to \mathbb{R}^k$ and $S : \mathbb{R}^k \to \mathbb{R}^n$ are linear transformations, and define the function $R : \mathbb{R}^m \to \mathbb{R}^n$ by $R(x) = S(T(x))$. Show that R is a linear transformation.*

Proof. Suppose $\mathbf{u}$ and $\mathbf{v}$ are elements of $\mathbb{R}^m$. Then we have

$$\begin{aligned} R(\mathbf{u}+\mathbf{v}) &= S(T(\mathbf{u}+\mathbf{v})) && \text{(by definition of } R\text{)} \\ &= S(T(\mathbf{u})+T(\mathbf{v})) && \text{(since } T \text{ is a linear transformation)} \\ &= S(T(\mathbf{u}))+S(T(\mathbf{v})) && \text{(since } S \text{ is a linear transformation)} \\ &= R(\mathbf{u})+R(\mathbf{v}) && \text{(by definition of } R\text{).} \end{aligned}$$

Now suppose that $\mathbf{u}$ is an element of $\mathbb{R}^m$ and r is a real number (i.e. $r \in \mathbb{R}$). Then we have

$$\begin{aligned} R(r\mathbf{u}) &= S(T(r\mathbf{u})) && \text{(by definition of } R\text{)} \\ &= S(rT(\mathbf{u})) && \text{(since } T \text{ is a linear transformation)} \\ &= rS(T(\mathbf{u})) && \text{(since } S \text{ is a linear transformation)} \\ &= rR(\mathbf{u}) && \text{(by definition of } R\text{).} \end{aligned}$$

Therefore R satisfies the conditions of being a linear transformation, and so we are done. □

Problem 2. *Suppose $S(x) = Ax$ and $T(x) = Bx$, where $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$. By Problem 1, we know that $R(x) = S(T(x))$ is a linear transformation, and by theorem 3.8 there is some matrix C so that $R(x) = Cx$ for all x. Find such a matrix C.*

Proof. First note that C should be a 2×2 matrix since if we compare with problem 1 we see that $m = n = 2$ and $k = 3$. Now, by the proof of theorem 3.8, we know that C will be the matrix with columns $R(\mathbf{e}_1)$ and $R(\mathbf{e}_2)$. So we can find C by computing these vectors. We have $R(\mathbf{e}_1) = S(T(\mathbf{e}_1))$ and $R(\mathbf{e}_2) = S(T(\mathbf{e}_2)$. Now, $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and so

$$S(T(\mathbf{e}_1)) = AT(\mathbf{e}_1) = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Similarly, we have $T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and so

$$S(T(\mathbf{e}_2)) = AT(\mathbf{e}_2) = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore we have $C = [T(\mathbf{e}_1)\, T(\mathbf{e}_2)] = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$ □