Prove or disprove: If $\{\mathbf{a}_1, ..., \mathbf{a}_n\}$ is a set of vectors in $\mathbb{R}^n$ that spans $\mathbb{R}^n$, then $\{\mathbf{a}_1, ..., \mathbf{a}_n\}$ is linearly independent.

Proof. If $\{\mathbf{a}_1, ..., \mathbf{a}_n\}$ is linearly dependent, then one of them is in the span of the others, say it's $\mathbf{a}_1$.
Then $\text{span}\{\mathbf{a}_1, ..., \mathbf{a}_n\} = \text{span}\{\mathbf{a}_2, ..., \mathbf{a}_n\}$. But $n - 1 < n$, and so $\{\mathbf{a}_2, ..., \mathbf{a}_n\}$ cannot span $\mathbb{R}^n$.
In particular, $\text{span}\{\mathbf{a}_1, ..., \mathbf{a}_n\} \neq \mathbb{R}^n$, and so the set doesn't span.
Thus we have shown that if the set of vectors spans $\mathbb{R}^n$, then it must be linearly independent.
□

Prove or disprove: If $\{\mathbf{a}_1, ..., \mathbf{a}_n\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then $\{\mathbf{a}_1, ..., \mathbf{a}_n\}$ spans $\mathbb{R}^n$.

Proof. Suppose the set does not span $\mathbb{R}^n$. Then there exists $\mathbf{a} \in \mathbb{R}^n$ that is not in $\text{span}\{\mathbf{a}_1, ..., \mathbf{a}_n\}$.
Since $n + 1 > n$, we know that the set $\{\mathbf{a}, \mathbf{a}_1, ..., \mathbf{a}_n\}$ is linearly dependent. I claim that this implies that $\{\mathbf{a}_1, ..., \mathbf{a}_n\}$ is also linearly dependent.
Since $\{\mathbf{a}, \mathbf{a}_1, ..., \mathbf{a}_n\}$ is linearly dependent, there exist $c, c_1, ..., c_n$ not all zero with
$c\mathbf{a} + c_1\mathbf{a}_1 + ... + c_n\mathbf{a}_n = \mathbf{0}$.
But notice that we have to have $c = 0$, because otherwise we could solve for $\mathbf{a}$ in terms of $\mathbf{a}_1, ..., \mathbf{a}_n$, and this isn't possible since it's not in the span. Therefore we actually have $c_1\mathbf{a}_1 + ...c_n\mathbf{a}_n = \mathbf{0}$, where $c_1, ..., c_n$ are not all zero, and so $\{\mathbf{a}_1, ..., \mathbf{a}_n\}$ is linearly dependent.
Thus we have shown that if the set is linearly independent, it must also span $\mathbb{R}^n$. □