1. (10 points) Set up a triple integral to find the volume of the region bounded by \( z \leq x^2 + y^2, x^2 + y^2 \leq 3 \) and \( z \geq 0 \) using spherical coordinates. (Recall that volume is \( \iiint_R 1 \, dV \).) **Do not evaluate.**

**Answer:** The region is within the cylinder \( x^2 + y^2 = 3 \): below the paraboloid \( z = x^2 + y^2 \) and above the plane \( z = 0 \). Therefore, fixing \( \phi \) and \( \theta \), a half-line starting at the origin hits the paraboloid first (where \( \rho = \frac{\cos(\phi)}{\sin(\phi)} \) since \( z = x^2 + y^2 \) is \( \rho \cos(\phi) = \rho^2 \sin^2(\phi) \) in spherical coordinates) and then the cylinder (where \( \rho = \frac{x}{\sin(\phi)} \) since \( x^2 + y^2 = 3 \) is \( \rho^2 \sin^2(\phi) = 3 \)). The paraboloid and the cylinder intersect at \( z = 3 \) in a circle of radius \( \sqrt{3} \). Thus, \( \phi_{\text{min}} = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \), and since we are considering the region over \( z = 0 \), \( \phi_{\text{max}} = \frac{\pi}{2} \). Finally, \( \theta \) is from 0 to \( 2\pi \) since we have a full revolution around the \( z \)-axis. Therefore, the volume is

\[
\iiint_0^{2\pi} \int_0^{2\pi} \int_0^{\sqrt{3}} 1 \rho^2 \sin(\phi) \, d\rho d\phi d\theta.
\]

2. (10 points) Switch \( \iiint_0^{2\pi} \int_0^{\sqrt{3}} \int_2^3 zr^4 \, dz \, dr \, d\theta + \iint_0^{2\pi} \int_0^{\sqrt{3}} \int_2^{\sqrt{4-r^2+2}} zr^4 \, dz \, dr \, d\theta \) to spherical coordinates.

**Answer:** The region in the first triple integral is part of a solid cylinder of radius \( \sqrt{3} \) centered around the \( z \)-axis of height 1 (with \( 2 \leq z \leq 3 \)). The region in the second triple integral is the part of the ball of radius two centered at \( (0, 0, 2) \) (since \( z = \sqrt{4 - r^2 + 2} \)) between \( 2 \leq z \leq 3 \) and outside the aforementioned cylinder. Therefore, together, the region is the ball of radius two centered at \( (0, 0, 2) \) cut with the planes \( z = 2 \) and \( z = 3 \).

With spherical coordinates, fixing \( \phi \) and \( \theta \), the half-line always enters through the plane \( z = 2 \) (where \( \rho = \frac{2}{\cos(\phi)} \)), but either comes out on the sphere (where \( \rho = 4 \cos(\phi) \) since \( x^2 + y^2 + (z-2)^2 = 4 \) which is equivalent to \( x^2 + y^2 + z^2 - 4z + 4 = 4 \) and thus \( \rho^2 - 4 \rho \cos(\phi) = 0 \)) or on the plane \( z = 3 \) (where \( \rho = \frac{3}{\cos(\phi)} \)). We will thus need two triple integrals here too.

When \( \phi = 0 \), we come out on \( z = 3 \) and continue to do so until angle \( \frac{\pi}{6} \). Then from \( \frac{\pi}{6} \) to \( \frac{\pi}{4} \), we come out on the sphere. Moreover, \( 0 \leq \theta \leq 2\pi \) since we have a full revolution around the \( z \)-axis.

Finally, note that \( zr^4 \, dz \, dr \, d\theta = zr^3 \, dV = \rho \cos(\phi) \rho^3 \sin^3(\phi) \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta \). Thus, we obtain

\[
\iiint_0^{2\pi} \int_0^{\sqrt{3}} \int_0^{\rho(\cos(\phi))} \rho^6 \sin^4(\phi) \, d\rho d\phi d\theta + \iiint_0^{2\pi} \int_0^{\sqrt{3}} \int_{\rho(\cos(\phi))}^{\rho(\cos(\phi))} \rho^6 \sin^4(\phi) \, d\rho d\phi d\theta.
\]

3. (10 points) Find the area of the ellipse \( (2x + 5y - 7)^2 + (3x - 7y + 1)^2 \leq 1 \).

**Answer:** Let \( u = 2x + 5y - 7 \) and \( v = 3x - 7y + 1 \).

The Jacobian is

\[
\begin{pmatrix}
u_x & u_y \\
v_z & v_y
\end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 3 & -7 \end{pmatrix} = -14 - 15 = -29. \quad \text{Thus } |dudv| = |-29|dxdy, \text{ so } \int_R 1 \, dA \text{ becomes}
\int \int_{u^2 + v^2 \leq 1} \frac{1}{29} \, dudv = \frac{1}{29} \pi 1^2 = \frac{\pi}{29}
\]