Annie's Survival Kit 4 - Math 324

1. (10 points) Let $f(x, y) = x^2 - y^2 + 4xy$. Recall that $D_{\hat{\mathbf{u}}} f = \frac{df}{ds|_{\hat{\mathbf{u}}}} = \nabla f \cdot \hat{\mathbf{u}}$ and $\nabla f = \langle f_x, f_y \rangle$.

   (a) (3 points) In which direction does f **decrease** the fastest at $(2, 1)$?

   (b) (1 point) For which unit vector does f **increase** the fastest at $(2, 1)$?

   (c) (3 points) What is the rate of change of f at $(2, 1)$ in the direction of the fastest **decrease**?

   (d) (3 points) Find all points at which the direction of fastest change of f is the same as in (a).

a) $\nabla f \cdot \hat{u} = |\nabla f| \cdot |\hat{u}| \cdot \cos(\theta)$

θ = angle between ∇f and $\hat{u}$

This is minimum when $\theta = \pi$

So direction is $-\nabla f = -\langle 2x+4y, -2y+4x \rangle \overset{(2,1)}{=} -(8,6) = \boxed{(-8,-6)}$

b) Increase fastest when $\theta = 0$,

so in direction $\nabla f = (8,6)$

$\therefore \hat{u} = \frac{1}{\sqrt{8^2+6^2}} = \frac{1}{10}(8,6) = \boxed{\left(\frac{4}{5}, \frac{3}{5}\right)}$

c) The rate of change is $|\nabla f| \cdot |\hat{u}| \cdot \cos(\pi) = 10 \cdot 1 \cdot (-1) = -10$

d) Want $\nabla f = (8,6) \cdot k$ for $k \neq 0$

$\therefore (2x+4y, 4x-2y) = (8k, 6k)$

$\therefore$
$2x + 4y = 8k$
$4x - 2y = 6k$

Solving:
$10y = 10k$
$y = k$
$x = 2k$

All points (x,y) such that

$\therefore$ $\boxed{y = \frac{1}{2}x \text{ except for } (0,0)}$

2. (10 points) Let $u = x^2 + y^2$, $v = \frac{y}{x}$ and $f = f(u, v)$.

(a) (7 points) Express $xf_x + yf_y$ in terms of f_u and f_v.

(b) (3 points) Find $xf_x + yf_y$ when $f(u, v) = u^3$.

By the chain rule:

a) $f_x = f_u \cdot u_x + f_v \cdot v_x = f_u \cdot (2x) + f_v \cdot \left(\frac{-y}{x^2}\right)$

$f_y = f_u \cdot u_y + f_v \cdot v_y = f_u \cdot (2y) + f_v \cdot \left(\frac{1}{x}\right)$

$\therefore \quad x \cdot f_x + y \cdot f_y = 2x^2 f_u + \left(-\frac{y}{x}\right) \cdot f_v + 2y^2 f_u + \frac{y}{x} f_v$

$= 2(x^2 + y^2) f_u = 2u f_u$

b) If $f(u,v) = u^3$, then $f_u = 3u^2$

and $xf_x + yf_y = 2u \cdot 3u^2 = 6u^3$

3. (10 points) (a) (6 points) Find the tangent plane on $z = 2\sqrt{x^2+y^2}$ at the point $(1, -1, \sqrt{8})$.

(b) (2 points) Is $(6, -8, -5)$ a normal vector for the tangent plane at some point of this surface? (Hint: the length and direction of the normal are irrelevant; only its orientation matters.) If so, find all such points on the surface. Otherwise, explain why.

(c) (2 points) Is $(1, 1, 1)$ a normal vector for the tangent plane at some point of this surface? (Hint: the length and direction of the normal are irrelevant; only its orientation matters.) If so, find all such points on the surface. Otherwise, explain why.

a) $z = 2\sqrt{x^2+y^2} \Rightarrow z^2 = 4x^2+4y^2$

Let $w = 4x^2+4y^2-z^2$. Consider the level curve $w=0$.

$\nabla w = \langle 8x, 8y, -2z \rangle \underset{(1,-1,\sqrt{8})}{=} \langle 8, -8, -2\sqrt{8} \rangle$

Plane: $8x - 8y - 2\sqrt{8}z = 0$

b) Want $\langle 8x, 8y, -2z \rangle = k \cdot \langle 6, -8, -5 \rangle$ for $k \neq 0$ and $z = 2\sqrt{x^2+y^2}$

$\therefore$ $8x = 6k$, $8y = -8k$, $-2z = -5k$ $\Rightarrow$ $x = \frac{3k}{4}$, $y = -k$, $z = \frac{5k}{2}$

Does $\left(\frac{5k}{2}\right) = 2\sqrt{\left(\frac{3k}{4}\right)^2 + (-k)^2}$?

$\frac{5k}{2} = 2\sqrt{\frac{9k^2}{16} + \frac{16k^2}{16}}$

$\frac{5k}{2} = 2 \cdot \sqrt{\frac{25k^2}{16}} = 2 \cdot \frac{5k}{4} = \frac{5k}{2}$ ✓ for all k

$\therefore$ Points $\left(\frac{3k}{4}, -k, \frac{5k}{2}\right)$ $\forall k \neq 0$

c) Want $\langle 8x, 8y, -2z \rangle = k \langle 1, 1, 1 \rangle$ for $k \neq 0$ and $z = 2\sqrt{x^2+y^2}$

$8x = k$, $8y = k$, $-2z = k$ $\Rightarrow$ $x = k/8$, $y = k/8$, $z = -k/2$

Does $-\frac{k}{2} = 2\sqrt{\left(\frac{k}{8}\right)^2 + \left(\frac{k}{8}\right)^2}$?

$-\frac{k}{2} = 2\sqrt{\frac{2k^2}{64}} = \frac{2k}{8} \cdot \sqrt{2} = \frac{k\sqrt{2}}{4}$?

No, except if $k=0$, but here $k \neq 0$.

No point on the cone has a tangent plane with normal vector $(1,1,1)$