Annie's Survival Kit 5 - Math 324

1. (10 points) (a) (8 points) Let $\mathbf{F} = \langle 3x^2y, x^3+3y^2\rangle$ and let C be the path going along $x = y^2$ from $(4,2)$ to $(0,0)$. Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ either by doing so directly, by using path-independence to replace C by some other path or by using the fundamental theorem for line integrals, i.e. $\int_C \mathbf{F}\cdot d\mathbf{r} = f(P_1) - f(P_0)$ where $\nabla f = \mathbf{F}$ and P_0 and P_1 are the endpoints of C.

   (b) (2 points) Do it in another way.

a) $\text{curl}(\vec{F}) = N_x - M_y = 3x^2 - 3x^2 = 0 \Rightarrow$ can use any technique

+5)

• Fundamental thm

Want f s.t. $f_x = 3x^2y \Rightarrow f = x^3y + g(y)$

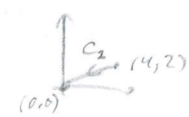
$f_y = x^3+3y^2 \quad \Downarrow \quad f_y = x^3 + g'(y) \Rightarrow g'(y) = 3y^2 \Rightarrow g = y^3 + C$

$\therefore f = x^3y + y^3 + C$

$\int_C \vec{F}\cdot d\vec{r} = f(0,0) - f(4,2) = C - (4^3\cdot 2 + 2^3 + C) = -136$

• Path-independence

- use straight line



$x = t$, $y = t/2$ "$4 \le t \le 0$" or even better $\boxed{x = 2t,\ y = t \text{ "}2 \le t \le 0\text{"}}$

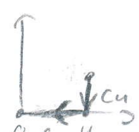
$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = \int_2^0 3x^2y\,dx + (x^3+3y^2)\,dy$$

$$= \int_2^0 3(2t)^2 t\, 2dt + ((2t)^3 + 3(t)^2)\,dt$$

$$= \int_2^0 \underbrace{24t^3 + 8t^3}_{32t^3} + 3t^2\,dt = \left[8t^4 + t^3\right]_2^0 = 0 - (8\cdot 2^4 + 8)$$

$$= -136$$

- use "xy"-path



$C_3: \begin{cases} x = t \\ y = 0 \end{cases}$ "$4 \le t \le 0$"

$C_4: \begin{cases} x = 4 \\ y = t \end{cases}$ "$2 \le t \le 0$"

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r} = \int_4^0 \underbrace{3\cdot t^2\cdot 0\cdot dt + (t^3 + 3\cdot 0^2)\cdot 0\,dt}_{0} + \int_2^0 \underbrace{3\cdot 4^2\cdot t\cdot 0\,dt}_{0} + (4^3 + 3\cdot t^2)\,dt$$

$$= \int_2^0 64 + 3t^2\,dt = \left[64t + t^3\right]_2^0 = 0 - (128 + 8) = -136$$

• Directly C: $x = t^2$, $y = t$ "$0 \le t \le 2$"

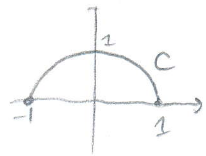
$$\int_C \vec{F}\cdot d\vec{r} = \int_2^0 3(t^2)^2 t\cdot 2t\,dt + ((t^2)^3 + 3\cdot t^2)\,dt = \int_2^0 \underbrace{6t^6 + t^6}_{7t^6} + 3t^2\,dt$$

$$= \left[t^7 + t^3\right]_2^0 = 0 - (2^7 + 2^3)$$

$$= -136$$

2. (10 points) Find the center of mass of a wire in the shape of the semi-circle $x^2 + y^2 = 1$ where $y \geq 0$, and whose density is proportional to the distance from $y = 1$.

$\delta = |1-y| = 1-y$ since all $y \leq 1$



$C: \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \leq t \leq \pi$

$$\text{Mass} = \int_C \delta \, ds$$

$$= \int_C 1-y \, ds$$

$$= \int_0^{\pi} (1-\sin t) \cdot \left| \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle \right| dt = \int_0^{\pi} (1-\sin t) \cdot |\langle -\sin t, \cos t \rangle| \, dt$$

$$= \int_0^{\pi} (1-\sin t) \underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2}}_{1} \, dt$$

$$= \int_0^{\pi} 1-\sin t \, dt$$

$$= \left[t + \cos t \right]_0^{\pi} = \pi + (-1-1) = \pi - 2$$

3. (10 points) Evaluate $\int_C x\sqrt{y}dy$ when C is the path going along $x = \frac{\cos^2(t)}{\sin(t)}$ and $y = \sin^2(t)$ for $t \in [\cancel{0}\,\frac{\pi}{4}, \frac{\pi}{2}]$.

$$\int_{\pi/4}^{\pi/2} \frac{\cos^2 t}{\sin t} \cdot \sqrt{\sin^2 t} \cdot 2\sin t \cdot \cos t \, dt$$

$$= \int_{\pi/4}^{\pi/2} 2\cos^3 t \sin t \, dt$$

$u = \cos^2 t$

$du = -2\cos t \sin t$

$$= \int -u \, du$$

$$= -\left[\frac{u^2}{2}\right] = -\left[\frac{\cos^4 t}{2}\right]_{\pi/4}^{\pi/2} = -\left(\underbrace{\frac{\cos^4(\frac{\pi}{2})}{2}}_{=0} - \frac{\cos^4(\frac{\pi}{4})}{2}\right) = \frac{(\frac{\sqrt{2}}{2})^4}{2} = \frac{1}{8}$$