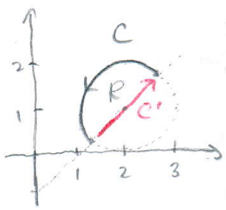
Annie's Survival Kit 6 - Math 324

1. (10 points) Use Green's theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \langle 2y + cos(x), 4x - e^y\rangle$ and C is given by $\mathbf{r}(t) = \langle \cos(t)+2, \sin(t)+1\rangle$ for $t \in [\frac{\pi}{4}, \frac{5\pi}{4}]$.



line

$y = x - 1$

so C': $\begin{cases} x = t \\ y = t-1 \end{cases}$ $dx = dt$, $dy = dt$

$2 - \frac{\sqrt{2}}{2} \le t \le \frac{\sqrt{2}}{2} + 2$

$x(\frac{5\pi}{4})$ $x(\frac{\pi}{4})$

To use Green's theorem,

I need to add C' to have a closed ccw curve

so $\int_C \vec{F}\cdot d\vec{r} = \oint_{C+C'} \vec{F}\cdot d\vec{r} - \int_{C'} \vec{F}\cdot d\vec{r}$

$$\underset{\text{Green}}{=} \iint_R \text{curl}(\vec{F})\, dA - \int_{2-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}+2} 2\cdot(t-1) + \cos(t)\, dt + 4t - e^{t-1}\, dt$$

$$= \iint_R \underset{N_x}{4} - \underset{M_y}{2}\, dA - \int_{2-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}+2} 6t - 2 + \cos t - e^{t-1}\, dt$$

$$= 2 \underbrace{\iint_R 1\, dA}_{\text{area of R}} - \left[3t^2 - 2t + \sin t - e^{t-1}\right]_{2-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}+2}$$

area of R which is half-disk $\Rightarrow \frac{1}{2}\pi$ of radius 1

$$= \pi - \left(3\left(\tfrac{\sqrt{2}}{2}+2\right)^2 - 2\left(\tfrac{\sqrt{2}}{2}+2\right) + \sin\left(\tfrac{\sqrt{2}}{2}+2\right) - e^{\frac{\sqrt{2}}{2}+1}\right)$$

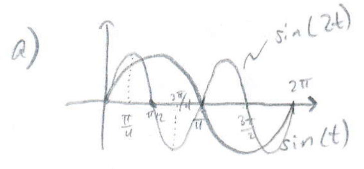
$$+ 3\left(2-\tfrac{\sqrt{2}}{2}\right)^2 - 2\left(2-\tfrac{\sqrt{2}}{2}\right) + \sin\left(2-\tfrac{\sqrt{2}}{2}\right) - e^{1-\frac{\sqrt{2}}{2}}$$

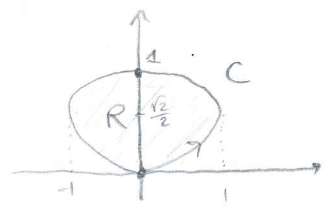
2. (10 points) (a) (7 points) Express the area of the region bounded by $\mathbf{r}(t) = \langle \overset{x}{\sin(2t)}, \overset{y}{\sin(t)} \rangle$ for $0 \leq t \leq \pi$ with an integral of the form $\int_{t_0}^{t_1} f(t)\,dt$. **Do not evaluate.** Hint: think about how $\sin(2t)$ and $\sin(t)$ behave (which increase? decrease?) when $t \in [0, \frac{\pi}{4}]$, when $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$, when $t \in [\frac{\pi}{2}, \frac{3\pi}{4}]$ and when $t \in [\frac{3\pi}{4}, \pi]$. Moreover, recall that the area of some region R is equal to $\int\int_R 1\,dA$. Finally, let D be some region and C its counterclockwise boundary, then Green's theorem states that, if $\mathbf{F}$ is defined and differentiable everywhere on D, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D \text{curl}(\mathbf{F})\,dA$ where $\text{curl}(\mathbf{F}) = N_x - M_y$ for $\mathbf{F} = \langle M, N \rangle$.

(b) (3 points) Using the following trigonometric identities, evaluate the integral you found in (a):

$$\sin^3(t) = \frac{3\sin(t) - \sin(3t)}{4}, \sin(2t) = 2\sin(t)\cos(t), \cos(2t) = \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t).$$

a)





t	$\overset{x}{\sin(2t)}$	$\overset{y}{\sin(t)}$	
0	0	0	both x, y augment, x faster
$\frac{\pi}{4}$	1	$\sqrt{2}/2$	x decreases y increases
$\frac{\pi}{2}$	0	1	x, y decrease
$\frac{3\pi}{4}$	-1	$\sqrt{2}/2$	x increases y decreases
π	0	0	

$\text{Area}(R) = \iint_R 1\,dA$

How can I set up the bounds for R? I only know C as parametric equations!

Use Green's theorem: need to figure out what is $\vec{F}$ so that it holds,

$$\iint_R 1\,dA = \oint_C \vec{F} \cdot d\vec{r} = \oint_C x\,dy = \int_0^{\pi} \sin(2t) \cdot \cos(t)\,dt$$

$\boxed{\text{Green if } \text{curl}\,\vec{F} = 1}$

Many possible $\vec{F}$:

$\boxed{\langle 0, x \rangle}$

$\langle -y, 0 \rangle$

$\langle \frac{-y}{2}, \frac{x}{2} \rangle$

$\langle y, 2x \rangle$

⋮

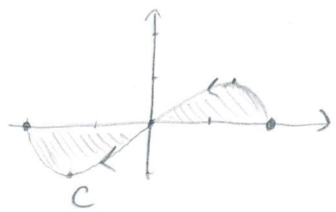
Can pick any!

b) $= \int_0^{\pi} 2\sin(t)\cos^2(t)\,dt$

$= -\frac{2}{3}\left[\cos^3 t\right]_0^{\pi} = -\frac{2}{3}\big(\underbrace{\cos^3\pi - \cos^3 0}_{-2}\big)$

$= \frac{4}{3}$

3. (10 points) Let $\mathbf{r}(t) = \langle 2\cos(t), \sin(2t) \rangle$ for $0 \le t \le \pi$. Express the mass of the region between $\mathbf{r}(t)$ and the x-axis with density equal to twice the distance from the y-axis with integrals of the form $\int_{t_0}^{t_1} f(t)\, dt$. Make sure $f(t)$ is free of absolute values. **Do not evaluate.**

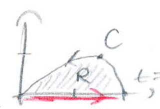


t	x: $2\cos(t)$	y: $\sin(2t)$	
0	2	0	x decreases, y increase
$\frac{\pi}{4}$	$\sqrt{2}$	1	
$\pi/2$	0	0	Both decrease
$\frac{3\pi}{4}$	$-\sqrt{2}$	-1	Both decrease
π	-2	0	x decreases, y increase

$\delta = 2 \cdot |x|$

Note that the mass of the right upper region is the same as the mass of the lower left region since the contribution of point $\vec{r}(t^*)$ is $2 \cdot |2\cos(t^*)|$ and so is the contribution of point $\vec{r}(\pi - t^*)$ since $2 \cdot |2 \cdot \cos(\pi - t^*)| = 2 \cdot |2 \cdot (-\cos(t^*))| = 2 \cdot |2 \cdot \cos(t^*)|$ and for any $t^* \in [0, \pi]$, $\pi - t^*$ is also in $[0, \pi]$.

Therefore, the mass $= 2 \cdot \underbrace{\iint_R \overbrace{2 \cdot x}^{\delta \text{ always } \ge 0 \text{ in } R}\, dA}_{\text{mass of } R}$ where R:



$C' \to \begin{cases} x = t \\ y = 0 \end{cases}$ $\quad dx = dt,\ dy = 0 \quad 0 \le t \le 2$

Here again, I don't know how to set the bounds, so I need to use Green's theorem. To do so, I need to close C with C'

$$4\iint_R x\, dA \underset{\boxed{\text{Green if } \operatorname{curl}(\vec{F}) = x}}{=} 4\oint_{C+C'} \vec{F}\cdot d\vec{r} = \underbrace{4\int_0^{\pi/2} \overbrace{-2\cos(t)}^{-x} \cdot \overbrace{\sin(2t)}^{y} \cdot \overbrace{2\cdot(-\sin t)\, dt}^{dx}}_{4\int_C \vec{F}\cdot d\vec{r}} + \underbrace{4 \cdot \int_0^2 \overbrace{-t}^{-x} \cdot \overbrace{0}^{y} \cdot \overbrace{dt}^{dx}}_{4\cdot\int_{C'} \vec{F}\cdot d\vec{r} = 0}$$

ex: $\vec{F} = \langle 0, \frac{x^2}{2} \rangle$

or $\boxed{\langle -xy, 0 \rangle}$ $\therefore \vec{F}\cdot d\vec{r} = -xy\, dx$

or $\langle xy, x^2 \rangle$

⋮

can pick any!

$$\boxed{\therefore \text{ mass} = 16 \int_0^{\pi/2} \cos(t)\sin(t)\sin(2t)\, dt}$$