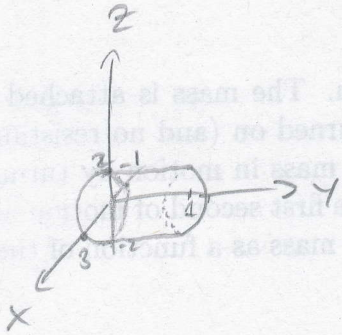
ASK 7 Solutions

1. $S: (x-1)^2 + z^2 = 4$ with $0 \le y \le 1$

a)



Cylinder parallel to y-axis of radius 2 centered around the line $\begin{cases} x = 1 + t \\ y = t \\ z = 0 \end{cases}$ between $y=0$ and $y=1$

b) $\begin{cases} x = 2\cos\theta + 1 \\ y = u \\ z = 2\sin\theta \end{cases}$ for $0 \le u \le 1$, $0 \le \theta \le 2\pi$

c) $\vec{r}_u = \langle 0, 1, 0 \rangle$

$\vec{r}_\theta = \langle -2\sin\theta, 0, 2\cos\theta \rangle$

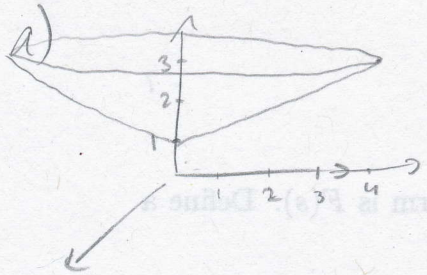
$\vec{r}_u \times \vec{r}_\theta = \langle 2\cos\theta, 0, 2\sin\theta \rangle$

$|\vec{r}_u \times \vec{r}_\theta| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = 2$

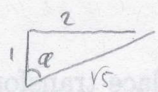
$$\text{Area}(S) = \iint_S 1\, dS$$
$$= \iint_D |\vec{r}_u \times \vec{r}_\theta|\, du\, d\theta$$
$$= \int_0^{2\pi} \int_0^1 2\, du\, d\theta$$
$$= 4\pi$$

2. $S: \quad z - 1 = \frac{1}{2}\sqrt{x^2+y^2} \quad \text{for} \quad z \leq 3$

a)



Cone with slope $\frac{1}{2}$ translated by 1 up centered around the z-axis between $1 \leq z \leq 3$



$\tan\varphi = 2$ $\quad \therefore \sin\varphi = \frac{2}{\sqrt{5}}$

$\varphi = \arctan(2)$ $\quad \cos\varphi = \frac{1}{\sqrt{5}}$

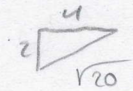
b) Either $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \frac{1}{2}r + 1 \end{cases}$ or $\begin{cases} x = \rho \cdot \frac{2}{\sqrt{5}}\cos\theta \\ y = \rho \cdot \frac{2}{\sqrt{5}}\sin\theta \\ z = \rho \cdot \frac{1}{\sqrt{5}} + 1 \end{cases}$

for $0 \leq \theta \leq 2\pi$

$0 \leq r \leq 4$

$0 \leq \rho \leq \sqrt{20}$

$0 \leq \theta \leq 2\pi$



c) $\vec{r}_r \times \vec{r}_\theta$ is the normal vector of the tangent plane

$\vec{r}_r = \langle \cos\theta, \sin\theta, \frac{1}{2} \rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$\vec{r}_r \times \vec{r}_\theta = \langle -\frac{1}{2}r\cos\theta, -\frac{1}{2}r\sin\theta, r \rangle$

Point $(1, 0, \frac{3}{2}) \Rightarrow \begin{array}{l} 1 = r\cos\theta \\ 0 = r\sin\theta \\ \frac{3}{2} = \frac{1}{2}r + 1 \end{array} \Rightarrow \begin{array}{l} r = 1 \\ \theta = 0 \end{array}$

$\therefore \vec{r}_r \times \vec{r}_\theta$ @ $(\overset{r}{1}, \overset{\theta}{0})$ is $\langle -\frac{1}{2}, 0, 1 \rangle$

$\therefore$ tangent plane is

$-\frac{1}{2}x + 0y + z = 1$

$\vec{r}_\rho \times \vec{r}_\theta$ is the normal vector of the tangent plane

$\vec{r}_\rho = \langle \frac{2}{\sqrt{5}}\cos\theta, \frac{2}{\sqrt{5}}\sin\theta, \frac{1}{\sqrt{5}} \rangle$

$\vec{r}_\theta = \langle -\frac{2}{\sqrt{5}}\rho\sin\theta, \frac{2}{\sqrt{5}}\rho\cos\theta, 0 \rangle$

$\vec{r}_\rho \times \vec{r}_\theta = \langle \frac{-2}{5}\rho\cos\theta, \frac{-2}{5}\rho\sin\theta, \frac{4}{5}\rho \rangle$

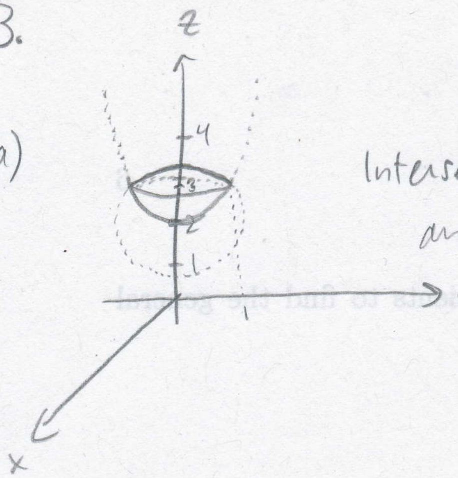
Point $(1, 0, \frac{3}{2}) \Rightarrow \begin{array}{l} 1 = \rho\frac{2}{\sqrt{5}}\cos\theta \\ 0 = \rho \cdot \frac{2}{\sqrt{5}}\sin\theta \\ \frac{3}{2} = \rho \cdot \frac{1}{\sqrt{5}} + 1 \end{array} \Rightarrow \begin{array}{l} \rho = \frac{\sqrt{5}}{2} \\ \theta = 0 \end{array}$

$\therefore \vec{r}_\rho \times \vec{r}_\theta$ at $(\frac{\sqrt{5}}{2}, 0)$ is $\langle \frac{-1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \rangle$

$\therefore$ tangent plane is

$-\frac{1}{\sqrt{5}}x + 0y + \frac{2}{\sqrt{5}}z = \frac{2}{\sqrt{5}}$

3.



a) Intersection of a ball of radius $\sqrt{2}$ centered at $(0,0,2)$ and a paraboloid with vertex at $(0,0,2)$

They intersect when $z-2 = 2-(z-2)^2$

$$z-2 = 2 - z^2+4z-4$$
$$0 = -z^2+3z$$
$$0 = z(-z+3)$$
$$\therefore z = 3$$

so we have a disk of radius 1

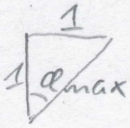
We take S to be the shell of this solid.

b) S is parametrized in two parts

S_1: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2+2 \end{cases}$

D_1: $0 \le r \le 1$, $0 \le \theta \le 2\pi$

S_2: $\begin{cases} x = \sqrt{2}\sin\varphi\cos\gamma \\ y = \sqrt{2}\sin\varphi\sin\gamma \\ z = \sqrt{2}\cos\varphi + 2 \end{cases}$



D_2: $0 \le \varphi \le \pi/4$, $0 \le \gamma \le 2\pi$

c) $\vec{r}_r = \langle \cos\theta, \sin\theta, 2r \rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$\vec{r}_r \times \vec{r}_\theta = \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle$

$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{4r^2 + r^2} = r\sqrt{5}$

$\vec{r}_\varphi = \langle \sqrt{2}\cos\varphi\cos\gamma, \sqrt{2}\cos\varphi\sin\gamma, -\sqrt{2}\sin\varphi \rangle$

$\vec{r}_\gamma = \langle -\sqrt{2}\sin\varphi\sin\gamma, \sqrt{2}\sin\varphi\cos\gamma, 0 \rangle$

$\vec{r}_\varphi \times \vec{r}_\gamma = \langle 2\sin^2\varphi\cos\gamma, 2\sin^2\varphi\sin\gamma, 2\cos\varphi\sin\varphi \rangle$

$|\vec{r}_\varphi \times \vec{r}_\gamma| = \sqrt{4\sin^4\varphi + 4\cos^2\varphi\sin^2\varphi}$
$= \sqrt{4\sin^2\varphi} = 2\sin\varphi$

$$\text{Area} = \int_0^{2\pi}\int_0^1 r\sqrt{5}\,dr\,d\theta + \int_0^{2\pi}\int_0^{\pi/4} 2\sin\varphi\,d\varphi\,d\theta$$
$$= \sqrt{5}\cdot\frac{1}{2}\cdot 2\pi + 2\left(\frac{-\sqrt{2}}{2}+1\right)\cdot 2\pi = \sqrt{5}\,\pi + 4\left(1-\frac{\sqrt{2}}{2}\right)\pi$$