The Direct Method for a Square Network

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In this report I discuss the implementation and testing of the Curtis-Morrow direct method (described in [5]) for reconstructing the conductivity of a resistor network, given the measurements of currents and voltages at the boundary.

1 Implementation

Troy Holly, Laura Smithies and I have written programs which use the direct method to recover the conductivity of the resistors in a square grid. I assume the reader to be familiar with the principles of the Curtis-Morrow method, and I proceed by discussing the implementation of the algorithm.

First, given a network of resistors, I calculate a matrix Λ^{-1} which contains the Neumann-Dirichlet data. Second, I describe the algorithm which we used to reconstruct the conductances from the boundary data. Finally, I report on some tests that were run.

1.1 Generation of a Matrix Λ^{-1}

We consider a square grid with n nodes in each interior row. The number of interior nodes is n^2, and the number of exterior nodes is $4n$. A square grid with $n = 5$ is shown in figure 1.

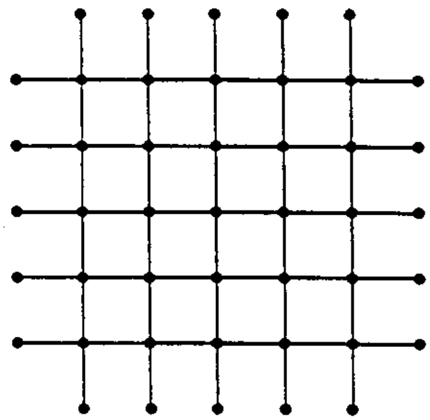


figure 1

A matrix Λ^{-1} is generated as follows. The nodes are numbered from 1 to $4n$. For each $1 \leq j \leq 4n-1$, let ϕ_j be the boundary current: the current at a node i_j is set equal to 1, the current at a node i_{j+1} is set equal to -1, the current at all other exterior nodes is set equal to zero. The voltages corresponding to ϕ_j are then calculated, with node $4n$ is taken as ground; that is, the voltage at node $4n$ is set equal to 0. The voltages at the boundary nodes form the j^{th} column of the matrix Λ^{-1} .

1.2 Reconstruction of the Conductances

The region is divided into 4 wedges as in figure 2a.

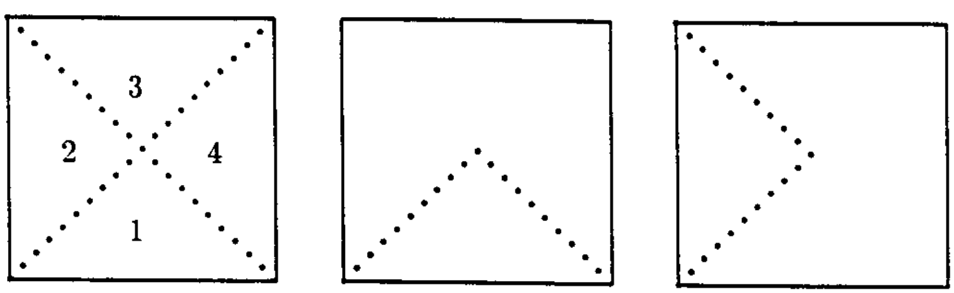


figure 2a figure 2b figure 2c

The reconstruction algorithm is similar for each wedge: it is sufficient to rotate the wedge and the ground node by a 90 degree angle, as shown in figures 2b and 2c. Therefore, I describe only the process of reconstruction for the wedge No. 1. For such a wedge, the Curtis-Morrow method consists of n steps. Since the method is inductive, it is only necessary to describe the algorithm for one step. Suppose we are at step k. The current at a node l is set equal to 1, and the current at a node m_j, where j varies from 1 to k, is set equal to -1, as in figure 3.

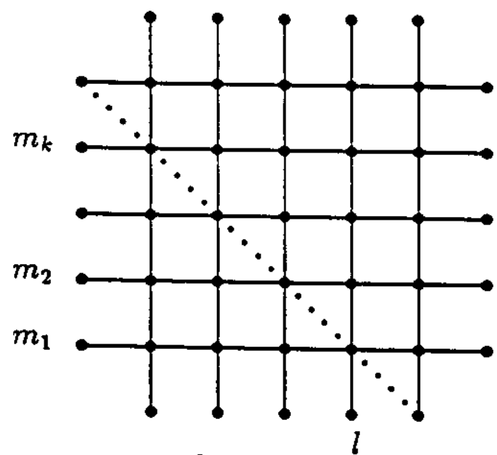


figure 3

The current at all the remaining exterior nodes is set equal to zero. For each m_j, we solve a $4n-1$ by $4n-1$ linear system

$$Ax = b$$

where A, constitutes a base for the space of the current flow, and where the vector b contains the current flow pattern corresponding to m_j.

By multiplying the solution vector x by the matrix Λ^{-1} we obtain the boundary voltages corresponding to the current flow pattern stored in vector b. We produce k boundary voltage vectors by repeating this process k times, as j varies from 1 to k (See figure 3). At the next step, we form a k by k matrix H, by extracting from each boundary voltage vector values corresponding to the nodes $q_1 \dots q_{k-1}$, as in figure 4.

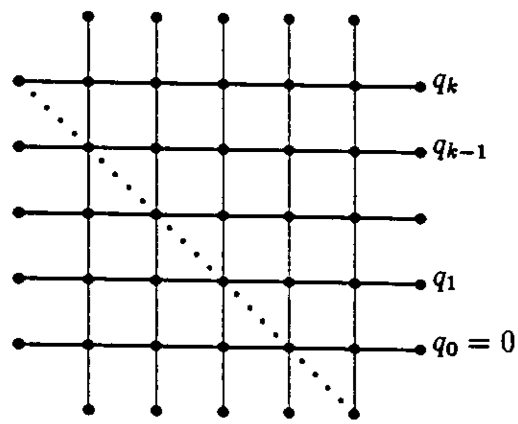


figure 4

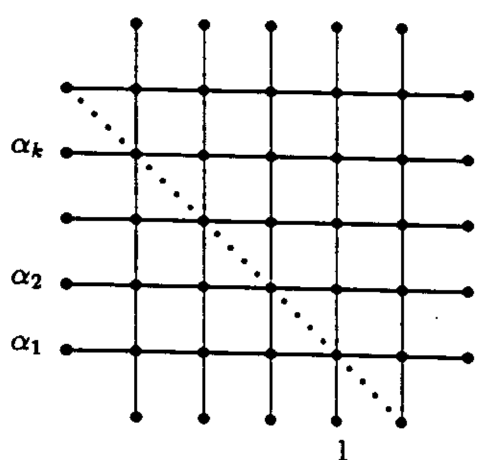
The last row of the matrix H is set equal to 1. We have a k by k linear system

$$H\alpha = w$$

where w is a zero vector except for its last entry, which is set equal to -1. The last equation of the system corresponds to the equation

$$\alpha_1 + \alpha_2 + \ldots + \alpha_k = -1$$

By solving this system, we obtain boundary currents as indicated in figure 5.



k^{th} step

figure 5

This causes all the voltages on and above the diagonal to be zero. We use Λ^{-1} to calculate the the boundary voltages corresponding to this boundary current. Next, using this boundary data, conductances as calculated from the previous $k-1$ steps and Kirkoff's law, we compute the interior voltages within a wedge. See figure 6a, where the conductivities along the edges marked with a ○ have already been calculated.

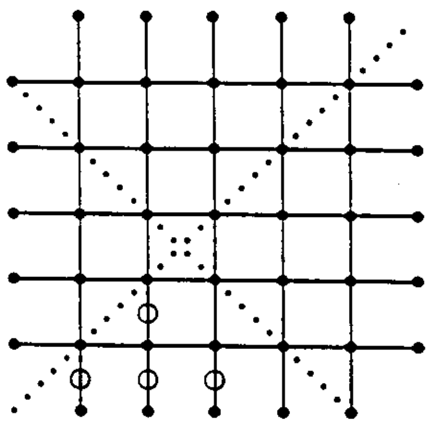


figure 6a

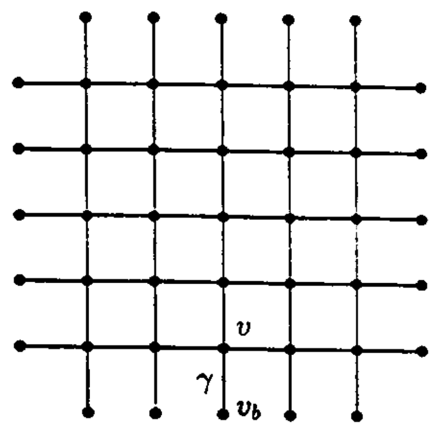


figure 6b

Voltages at the nodes neigbouring the exterior ones are computed from the formula:

$$(v_b - v)\gamma = 0$$

where v is the unknown voltage, v_b is the voltage at its exterior neighbour, and γ is the resistor connecting the two nodes, as in figure 6b.

The remaining interior voltages are computed row by row, progressing from the bottom of the wedge upward. The voltages in each row are calculated using the following formula:

$$(v_{i,j}-v_{i-1,j})\gamma_1+(v_{i-1,j+1}-v_{i-1,j})\gamma_2+(v_{i-2,j}-v_{i-1,j})\gamma_3+(v_{i-1,j-1}-v_{i-1,j})\gamma_4=0$$

where $v_{i,j}$ is the unknown voltage, $v_{i-1,j}, v_{i-1,j+1}, v_{i-2,j}$ and $v_{i-1,j-1}$ are known from the previous calculations of step k, as in figure 7, and conductances $\gamma_1 \cdots \gamma_4$ are known from the calculations of the previous $k-1$ steps.

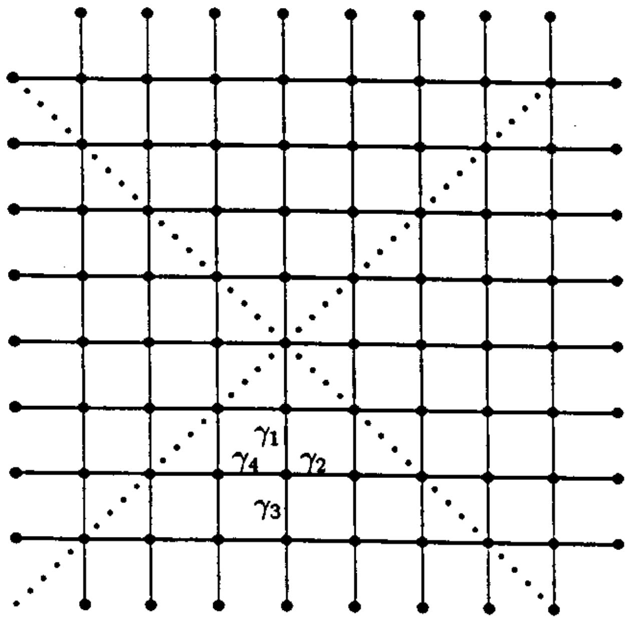


figure 7

Finally, using the boundary data, Kirkoff's law, known conductances and interior voltages, we reconstruct the conductances within the band defined by the two dotted line as in figure 8. We start at a node l and progress upward within the band until we encounter the end of the wedge.

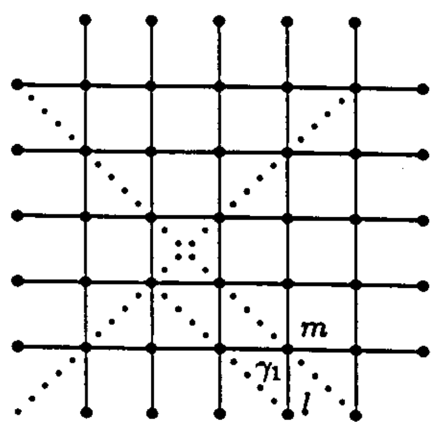


figure 8

The conductivity of the resistor connecting the nodes l and m in figure 8 is calculated from the formula

$$v_q \gamma_1 = 1$$

where v_q is the voltage at the node l. The remaining conductances are computed as follows. When the resistor is horizontal, we use the equation

$$v_y \gamma_2 + v_x \gamma_1 = 0$$

where γ_1 is the unknown, and γ_2 is known from the previous computations of the kth step, as in figure 9.

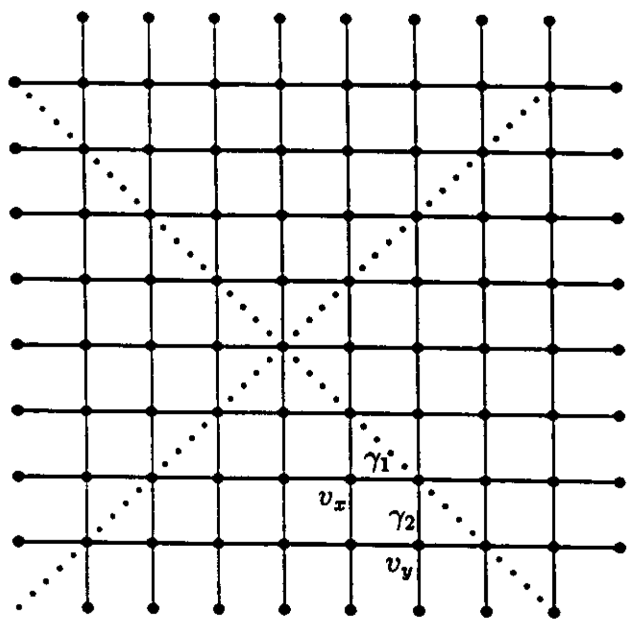


figure 9

When the resistor is vertical, we use the formula

$$(v_w - v_c)\gamma_2 + (v_s - v_c)\gamma_3 - v_c\gamma_4 - v_c\gamma_1 = 0$$

where γ_1 is unknown, and $\gamma_1 \ldots \gamma_3$ are known from the previous computations of the kth step, as in figure 10.

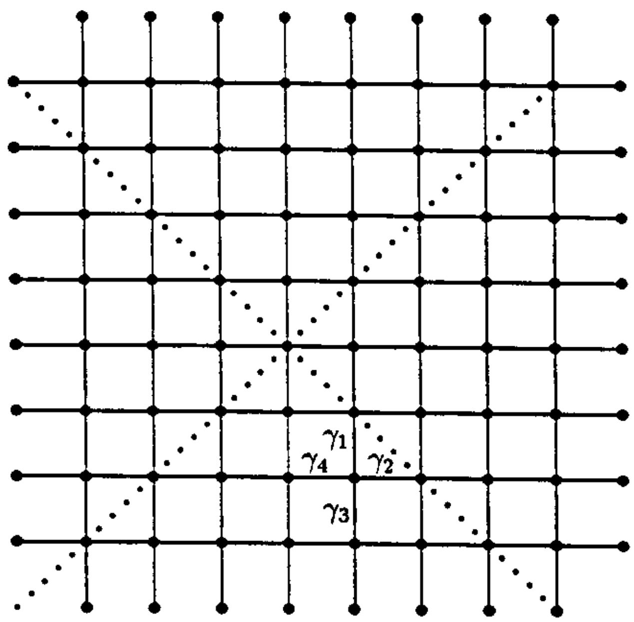


figure 10

2 Testing the Reconstruction Method

I performed two sets of tests. The first set tested the program with various types of conductivity distributions, and the second set was obtained by using various accuracies of Λ^{-1}.

2.1 Conductivity distributions

2.1.1 Constant Conductivity

The $k+1$ by k matrix H contained an extra row obtained by extracting from each of the k boundary voltage vectors values corresponding to the node q_k (See figure 6). To solve this overdetermined system, our program originally called a LINPACK least squares solver that uses QR decomposition (which finds an approximate solutions for overdetermined systems). We were able to recover a network of 264 resistors (an 11 by 11 grid with 121 interior nodes) with 90% accuracy, and less accuracy for larger grids.

After replacing the least squares solver by a LINPACK general solver, which uses LU decomposition for solving systems of equations, all resistors in a 420-resistor network (a 14 by 14 grid with 196 interior nodes), were reconstructed with an accuracy of at least 99.7%. For a network of 480 resistors (a 15 by 15 grid with 225 interior nodes), 478 resistors were reconstructed with an accuracy of at least 90%, and the remaining 2 resistors were reconstructed with an accuracy of 60%.

2.1.2 Randomly Generated Conductivity

The program was ren using 3 sets of randomly generated conductivity distributions, and the results are presented in Table 1.

Table 1

range of γ values	largest recovered grid	accuracy range
1 - 10	13 x 13	89%-100%
1 - 100	11 x 11	80%-100%
1 - 1000	9 x 9	99%-100%

Note: By "recovered grid" I mean a grid, for which the reconstructed conductivities are greater than zero. The reconstructed conductivities of negative values tend to occur in the region where the original conductance values were the greatest.

2.1.3 Regions of Low Conductivities

Resistors of low conductivity were located in the center of a 14 by 14 grid; the results are presented in Table 2.

Table 2

γ values in the center	γ values in the backgr.	accuracy range for the center	accuracy range for the background
10^{-2}	1	99.9%-100%	99.9%-100%
10^{-3}	1	60%-100%	99.8%-100%
10^{-4}	1	1 value negative, rest: 60%-100%	99.9%-100%

2.1.4 Regions of High Conductivity

Resistors of high conductivity were located in the center of a 14 by 14 grid; the results are presented in Table 3.

Table 3

γ values in the center	γ values in the backgr.	accuracy range for the center	accuracy range for the background
9	1	99%-100%	99.6%-100%
99	1	70%-97%	97%-100%
999	1	1 value negative, rest: 3%-84%	1 value negative, rest: 20%-100%

2.1.5 Regions of High and Low Conductivity

Resistors of high conductivity in the center of the grid are surrounded by resistors of low conductivity; the conductivity of the background is 1. For a 12 by 12 grid (312 resistors), the results are presented in Table 4. (For networks of size 13 by 13 and larger, the conductivities of central resistors were only partly recovered.)

Table 4

γ values in center	γ values around center	accuracy range in center	accuracy range around center	accuracy range in background
10^1	10^{-2}	93.4%-100%	100%	100%
10^1	10^{-3}	1 value negative, rest:0.2%-100%	92%-100%	99%-100%
10^2	10^{-2}	10%-99.9%	99.6%-100%	100%
10^2	10^{-3}	1 value negative, rest:0.02%-99.8%	1 value negative, rest:25-100%	85%-100%
10^3	10^{-2}	1 value negative, rest:0.01%-98%	50%-100%	97%-100%

2.2 Inaccurate Λ^{-1}

Our program computes Λ^{-1} with a double precision accuracy. To simulate inaccurate measurements of the boundary values, Λ^{-1} was rounded off to various numbers of decimal places. Using the rounded matrix, I tried to recover a network of constant conductivity. I found out that if Λ^{-1} is rounded off to d digit accuracy, then approximately, a d by d grid can be reconstructed. I also tried to recover a network with randomly located resistors of conductances. In one case the conductances had values 0.5, 1, and 2; in another case the conductances ranged from 1 to 10. The reconstructed grids were somewhat smaller then for the constant case. These results are presented in Table 5. $||\delta\Lambda||$ denotes an absolute value of the largest entry in the matrix $\Lambda^{-1} - \Lambda_2^{-1}$, where Λ_2^{-1} is computed using the reconstructed conductances.

Table 5

No. of decimal places in Λ^{-1}	γ values	largest recovered grid	accuracy range	order of $\|\|\delta\Lambda^{-1}\|\|$
1	1	2 x 2	89%-100%	10^{-2}
	0.5,1,2	2 x 2	89%-100%	10^{-2}
	1-10	-	-	-
2	1	3 x 3	71%-100%	10^{-2}
	0.5.1,2	3 x 3	70%-100%	10^{-2}
	1-10	2 x 2	92%-100%	10^{-3}
3	1	4 x 4	89%-100%	10^{-3}
	0.5,1,2	4 x 4	89%-100%	10^{-2}
	1-10	3 x 3	86%-100%	10^{-2}
4	1	5 x 5	88%-100%	10^{-4}
	0.5,1,2	5 x 5	87%-100%	10^{-3}
	1-10	4 x 4	91%-100%	10^{-3}
5	1	6 x 6	89%-100%	10^{-3}
	0.5,1,2	6 x 6	30%-100%	10^{-2}
	1-10	5 x 5	90%-100%	10^{-3}
6	1	7 x 7	74%-100%	10^{-3}
	0.5,1,2	7 x 7	65%-100%	10^{-3}
	1-10	6 x 6	51%-100%	10^{-3}
7	1	8 x 8	40%-100%	10^{-4}
	0.5,1,2	7 x 7	88%-100%	10^{-4}
	1-10	6 x 6	90%-100%	10^{-4}
8	1	9 x 9	28%-100%	10^{-3}
	0.5,1,2	8 x 8	71%-100%	10^{-4}
	1-10	7 x 7	88%-100%	10^{-4}
9	1	9 x 9	90%-100%	10^{-4}
	0.5,1,2	8 x 8	86%-100%	10^{-4}
	1-10	7 x 7	97.3%-100%	10^{-4}
10	1	10 x 10	99%-100%	10^{-4}
	0.5,1,2	9 x 9	98%-100%	10^{-5}
	1-10	8 x 8	70.2%-100%	10^{-4}
11	1	11 x 11	90%-100%	10^{-5}
	0.5,1,2	9 x 9	99.9%-100%	10^{-5}
	1-10	9 x 9	89%-100%	10^{-5}
12	1	12 x 12	75%-100%	10^{-4}
	0.5,1,2	11 x 11	88%-100%	10^{-4}
	1-10	10 x 10	90.5%-100%	10^{-5}

For the last test, we considered a network of 60 resistors; ($n = 5$; the pattern is as shown in figure 1) The original values of the conductivities gamma are as follows.

3.000000 7.000000 1.000000 8.000000 1.000000
1.000000 4.000000 3.000000 5.000000 6.000000 4.000000
1.000000 3.000000 8.000000 7.000000 5.000000
1.000000 2.000000 1.000000 5.000000 3.000000 7.000000
1.000000 9.000000 4.000000 3.000000 4.000000
1.000000 7.000000 1.000000 6.000000 1.000000 2.000000
1.000000 1.000000 1.000000 4.000000 2.000000
1.000000 3.000000 5.000000 1.000000 5.000000 1.000000
1.000000 1.000000 6.000000 7.000000 3.000000
1.000000 4.000000 1.000000 8.000000 4.000000 1.000000
1.000000 1.000000 1.000000 1.000000 7.000000

For this network, Λ^{-1} is calculated by the Neumann solver. Then Λ^{-1} is made inaccurate by rounding its values to 5, 6, 7, 8, 9, 10, 11 and 12 places. The results are listed below.

Λ^{-1} is rounded to 5 decimal places, the recovered values of conductivities:

3.000030 6.997676 1.000091 8.018972 1.000000
1.000000 4.002458 2.990327 4.949424 6.033999 4.000000
0.964402 3.053650 8.343167 6.550438 5.009314
1.008368 1.991478 0.945085 5.121097 3.002047 6.996144
0.931063 9.211617 4.527863 3.011582 3.995535
1.003313 6.901683 1.027161 5.749884 0.998677 2.000854
0.966330 1.034061 1.031957 3.928177 1.995806
1.001106 3.105529 5.063558 0.998784 4.994728 1.000325
0.994415 1.000835 5.941277 6.995690 2.989534
1.000000 4.002761 1.000264 7.998434 3.995147 1.000000
1.000000 0.999898 1.000016 1.000032 7.000350

Λ^{-1} is rounded to 6 decimal places, the recovered values of conductivities:

3.000003 6.999813 0.999999 7.998099 1.000000
1.000000 4.000254 2.999911 4.998851 6.004736 4.000000
1.000190 3.001256 8.019076 6.984170 4.999649
0.999952 2.002154 0.996184 4.976418 2.998608 7.000083
0.998873 9.015000 4.108281 3.000210 3.999827
1.000067 6.994498 0.991504 6.023826 0.999681 2.000046
0.999458 1.000888 1.001032 4.013853 1.999770
1.000056 3.000343 5.000334 0.999147 4.998272 1.000027
0.999724 0.999721 5.997201 6.998054 2.999116
1.000000 4.000264 1.000300 8.000929 4.000220 1.000000
1.000000 0.999990 0.999995 1.000002 7.000007

Λ^{-1} is rounded to 7 decimal places, the recovered values of conductivities:

3.000000 6.999955 1.000000 8.000239 1.000000
1.000000 4.000049 2.999951 4.999693 5.998605 4.000000
0.999383 3.000380 8.002257 6.993762 4.999978
1.000141 1.999334 0.999530 5.001925 2.999890 7.000039
0.999131 9.000405 3.996184 3.000047 3.999874
1.000044 6.998662 0.999808 5.995793 0.999975 2.000019
0.999516 1.000531 1.000279 3.998563 1.999926
1.000017 3.001465 4.999920 0.999927 5.000049 1.000004
0.999916 0.999944 5.999564 7.000178 2.999865
1.000000 4.000078 1.000071 8.000134 4.000169 1.000000
1.000000 0.999997 0.999999 0.999999 7.000002

Λ^{-1} is rounded to 8 decimal places, the recovered values of conductivities:

3.000000 7.000000 1.000000 8.000032 1.000000
1.000000 4.000000 2.999987 4.999958 5.999945 4.000000
0.999973 3.000061 8.000187 6.999562 5.000021
1.000006 1.999953 0.999981 5.000284 2.999994 6.999990
0.999975 8.999883 3.999691 3.000071 4.000004
1.000002 6.999943 0.999985 6.000329 0.999999 2.000000
0.999982 1.000023 0.999990 4.000077 2.000002
1.000001 3.000045 4.999955 1.000004 4.999979 1.000000
0.999994 1.000003 6.000054 7.000032 2.999996
1.000000 4.000008 1.000001 8.000004 4.000006 1.000000
1.000000 1.000000 1.000000 1.000000 7.000000

Λ^{-1} is rounded to 9 decimal places, the recovered values of conductivities:

3.000000 6.999999 1.000000 8.000000 1.000000
1.000000 4.000001 3.000002 5.000005 5.999989 4.000000
0.999991 2.999993 8.000014 6.999961 5.000000
1.000002 1.999992 0.999994 4.999958 2.999998 7.000000
0.999987 9.000019 4.000075 2.999999 3.999999
1.000001 6.999977 0.999993 5.999953 1.000000 2.000000
0.999993 1.000008 1.000001 3.999983 1.999999
1.000000 3.000022 5.000002 1.000000 5.000001 1.000000
0.999999 1.000000 5.999997 6.999997 2.999999
1.000000 4.000001 1.000000 8.000000 4.000000 1.000000
1.000000 1.000000 1.000000 1.000000 7.000000

Λ^{-1} is rounded to 10 decimal places, the recovered values of conductivities:

```
        3.000000   7.000000   1.000000   8.000000   1.000000
1.000000   4.000000   3.000000   5.000001   6.000000   4.000000
        0.999999   2.999999   8.000000   6.999995   5.000000
1.000000   1.999999   0.999999   4.999999   3.000000   7.000000
        0.999999   9.000001   4.000006   3.000000   4.000000
1.000000   6.999999   0.999999   5.999999   1.000000   2.000000
        1.000000   1.000001   1.000000   3.999999   2.000000
1.000000   3.000002   5.000000   1.000000   5.000000   1.000000
        1.000000   1.000000   5.999999   7.000000   3.000000
1.000000   4.000000   1.000000   8.000000   4.000000   1.000000
        1.000000   1.000000   1.000000   1.000000   7.000000
```

Λ^{-1} is rounded to 11 decimal places, the recovered values of conductivities:

```
        3.000000   7.000000   1.000000   8.000000   1.000000
1.000000   4.000000   3.000000   5.000000   6.000000   4.000000
        1.000000   3.000000   8.000000   6.999999   5.000000
1.000000   2.000000   1.000000   5.000000   3.000000   7.000000
        1.000000   9.000000   4.000000   3.000000   4.000000
1.000000   7.000000   1.000000   6.000000   1.000000   2.000000
        1.000000   1.000000   1.000000   4.000000   2.000000
1.000000   3.000000   5.000000   1.000000   5.000000   1.000000
        1.000000   1.000000   6.000000   7.000000   3.000000
1.000000   4.000000   1.000000   8.000000   4.000000   1.000000
        1.000000   1.000000   1.000000   1.000000   7.000000
```

Λ^{-1} is rounded to 12 decimal places, the recovered values of conductivities:

```
     3.000000  7.000000  1.000000  8.000000  1.000000
1.000000  4.000000  3.000000  5.000000  6.000000  4.000000
     1.000000  3.000000  8.000000  7.000000  5.000000
1.000000  2.000000  1.000000  5.000000  3.000000  7.000000
     1.000000  9.000000  4.000000  3.000000  4.000000
1.000000  7.000000  1.000000  6.000000  1.000000  2.000000
     1.000000  1.000000  1.000000  4.000000  2.000000
1.000000  3.000000  5.000000  1.000000  5.000000  1.000000
     1.000000  1.000000  6.000000  7.000000  3.000000
1.000000  4.000000  1.000000  8.000000  4.000000  1.000000
     1.000000  1.000000  1.000000  1.000000  7.000000
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