The Idea of a Dual Network

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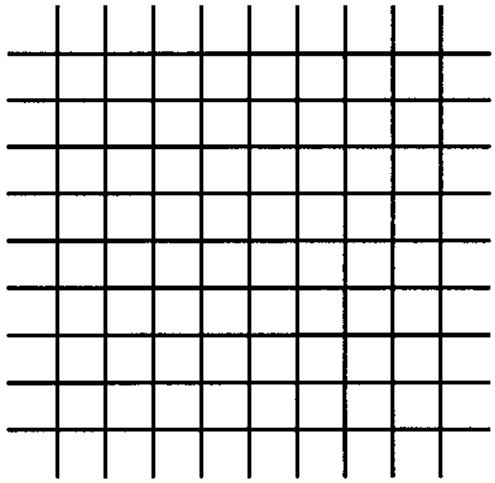
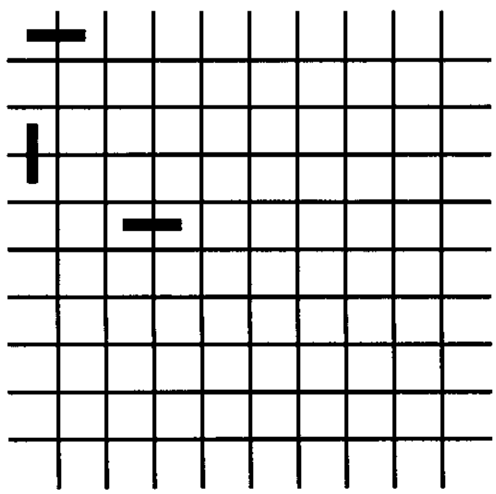
This paper describes the concept behind an unfinished problem developed during the summer of 1989 as a potential improvement for the inverse problem of determining the conductivity of a network of resistors given information from the boundary. The process involves constructing a version of a dual tessellation of a resistor network grid of the kind considered in the Curtis-Morrow technique, and then solving the inverse problem on this "dual graph." After describing the construction of the dual graph, I will describe certain of its characteristics which suggest its possible usefulness, followed by an account of the difficulties encountered in its actual implementation.

1 The Geometry of a Dual Graph

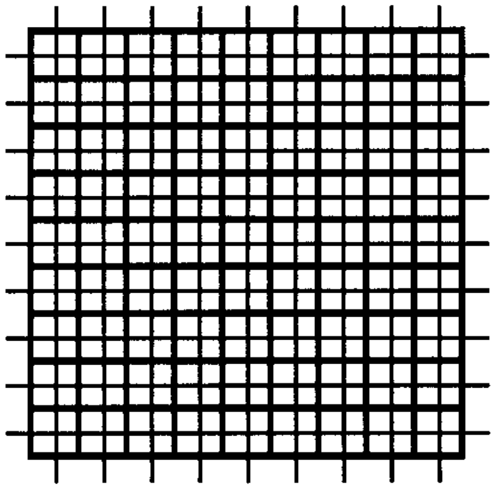
A dual graph, according to this plan, is understood only in relation to the original graph which generates it. An original graph consists of a network of resistors as defined by Curtis and Morrow, in which each boundary node is adjacent to exactly one other node in the network, resulting in a spiked appearance, as in the following representative 9 by 9 square network:



A node exists at both ends of each edge, at every intersection and at the end of each spike. Each individual edge has a specific conductivity. A dual graph in this context does not have the ordinary meaning given it in graph theory, where it would consist of the same nodes (vertices) as before and only those edges which had not connected two vertices in the original graph, but rather something more along the lines of a dual tessellation. For every edge in the original graph, a dual edge is defined across it, as demonstrated for three edges in the following figure.

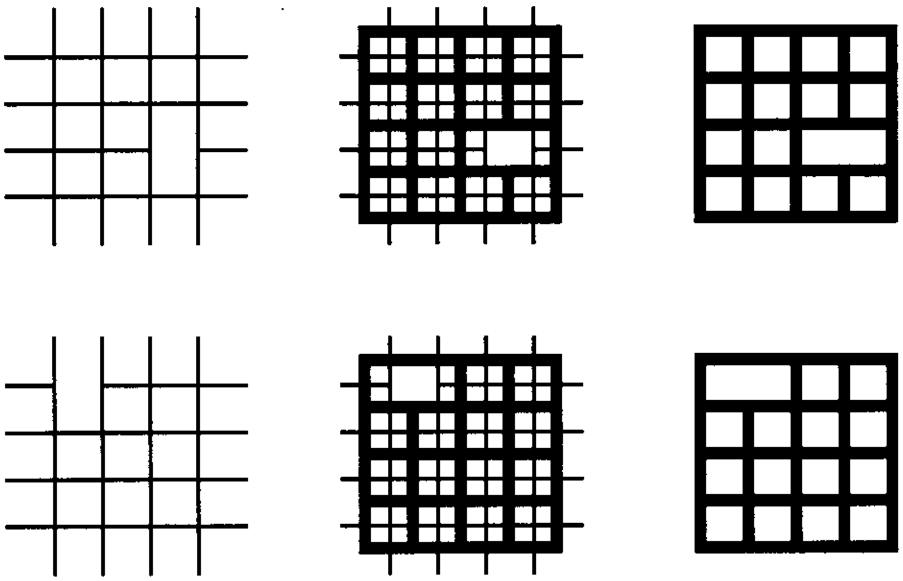


The new dual nodes are defined not only in the topological interior of each square but also in the divisions of the plane which would result if each border spike were extended indefinitely, producing a network whose boundary nodes all touch at least one other boundary node, a phenomenon which can be considered for simplicity as differentiating ordinary networks from boundary networks. Continuing this process for each edge results in an ordinary graph and its complete dual, as below:



where the dual is represented by the darker lines.

If a node or edge is missing in the original graph, the loss will produce a similar change in the dual, as in the following examples:



A dual network is constructed in exactly the same way for a rectangular network, and similarly for a hexagonal, triangular, or even an Archimedean network. However, the dual of a hexagonal network (six neighboring nodes to an interior node) will be triangular (three neighboring nodes to an interior node), and vice versa.

2 The Conductivity of a Dual Graph

Various possibilities exist for the values attached to the dual graph. The simplest suggestion for the conductivities of the edges would be to let the conductivity of any given dual edge be the same as the conductivity of the edge from which it was generated in the original graph. Other plausible ideas include some form of proportionality; either the conductivity of the dual edge will be directly or inversely proportional to the conductivity of the original edge. The value of the imposed currents or voltages for the dual nodes presents a greater problem, as there is no nice one-to-one correspondence between nodes as between edges. The dual nodes might be given the value of the mean of the ordinary nodes surrounding them, or perhaps some weighted average of the surrounding ordinary nodes multiplied by the conductivities of the adjacent dual edges. Care may be necessary at the corners of the dual graph, since whatever experimental algorithms are used to assign values to the dual edges and nodes might result in a violation of Kirkhoff's Law, rendering the network inconsistent. In the formation of the lambda matrix, however, where all but one node is zeroed, such complications should not arise.

3 Relation between the Ordinary and Dual Networks

Clearly, both ordinary and dual networks will have the same number of edges, although in significantly different configurations. Also, the geometry and conductances of the dual network will be unique to a particular original graph. Depending on the complexity of the algorithm chosen for determining the values associated with the dual nodes, these values may not be unique to a single original network, but any two original networks which produced identical dual networks would almost certainly have to contain very similar information.

While the number of edges and thus conductances remains the same, the dual graph contains considerably fewer nodes. To be specific, an m by n original network, containing $m(n+2) + n(m+2)$ nodes, yields a dual network with only $m(n+1) + n(m+1)$ nodes. Further, the number of exterior nodes remains the same in both original and dual graphs; the loss is only in the number of interior nodes. (These last statements exclude the unusual cases of a missing boundary node, or the removal of an interior node which is adjacent to two corner boundary nodes. Even in these cases, however, the dual network should be simpler to solve than the original.)

Here, then, is the basic idea. In the inverse problem, we are given information about the boundary and asked to determine the conductances inside the network. If the boundary information could be recombined into dual boundary information, and the dual conductances, once determined, translated back into original conductances, then the problem should be solved using considerably fewer calculations, as there are fewer interior nodes to compute.

Admittedly, there is no evidence that this trick should work. However, the equivalent number of inputs (pieces of boundary information and boundary nodes) and outputs (conductances and edges) is a coincidence which is too convenient to be passed over without examination. The trick, given appropriate algorithms for the dual values, has the potential to be a technique.

4 Testing the Hypothesis

Unfortunately, so far I have no data with which to either support or cast doubt upon this idea. After taking a great deal of time to solve the forward and inverse problem for the ordinary spiked m by n rectangular case, I was unable to program the forward problem for a dual network (one in which the boundary points are directly connected by edges). Programming the computer to create a dual network given a spiked network was quickly done. However, I first laboured under the misconception that the boundary nodes needed to be included in addition to the interior nodes in the A matrix, the matrix which computes the values (voltages, in the Dirichlet to Neumann case) of the interior nodes, which resulted only in great confusion on the computer's part (computations of NAN's). Once that main mistake was corrected, the program seemed to work, except that somehow, during computation, the computer changed the value of either m or n, the variable corresponding to the number of rows or of columns, to zero. A little debugging revealed that the problem occurred when the linear algebra library program which solved the A matrix was called. This problem did not occur with the ordinary spiked case and was not restricted to any particular terminal. I was unable to proceed without using the library programs, so the hypothesis remains, unfortunately, completely untested.

The greatest challenge to the likelihood of the dual solution technique bearing fruit is the apparent irrelevance of the values of the dual corner nodes when solving the values of the interior nodes. Nevertheless, one would not expect the dual nodes which correspond to the planar region separated by corner pair spikes to behave in the same way as other dual nodes because of the unusual behavior of these corner spike pairs, and in any case the values of the dual corner nodes will influence the adjacent connected nodes. Despite the apparent difficulties involved, and despite my inability to test the dual networks, the idea merits further study.