Maximal Resistor Networks

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February 1991

1 Introduction

1.1 The Inverse Conductivity Problem

Given an object Ω of unknown electrical conductivity γ, can this conductivity be determined through measurements made only on $\partial\Omega$, the boundary of Ω?

Applying a voltage f to $\partial\Omega$ induces a uniquely determined potential u in the interior of Ω. The forward problem it to find the potential function u as the solution to the conductivity equation $\nabla(\gamma \cdot \nabla u) = 0$ where $u = f$ on $\partial\Omega$. The potential u can then be used to compute a map Λ of the boundary potential f to boundary current. The inverse problem asks whether the conductivity γ can be determined from Λ.

This problem has an analogous discrete version that can be used to model the continuous case. In this situation, Ω is a connected network of resistors, each assumed to be positive. The end points of these resistors are called nodes. The nodes are divided into two sets, boundary nodes and interior nodes. Voltages can be set and currents measured only at the boundary nodes.

In this paper I describe several resistor networks with n boundary nodes and $n(n-1)/2$ resistors, such that all of the resistors in the network can be determined from measurements of currents at the boundary generated by imposed voltages. I will call these *maximal* resistor networks. First I describe the solution to the forward problem for a general resistor network and explain how the question of existence of a maximal resistor network comes about. I then describe several maximal resistor networks, and show how the resistors in these networks are recovered.

1.2 An Explanation of the Graphs

The graphs that accompany the text represent resistor networks with specific configurations of boundary voltages used in recovering the values of the resistors in the network. All resistor networks considered in this article consist of positive valued resistors and contain no internal current sources or sinks. Boundary nodes are points in the network where current and voltage can be measured. Boundary voltages are imposed and uniquely determine all potential and current in the network. For each allowable set of resistor values, the variable voltages and boundary currents take on unique values in the graph configurations shown.

All boundary nodes are marked with both voltage and current (in parenthesis) and are located on the outer edges of the graphs. Interior nodes are marked with voltages, although these cannot be directly measured at an interior node. Voltages not known to be zero or one are indicated with the letter u and boundary currents not known to be zero are indicated with the letter I in parenthesis. All line segments in the graphs represent resistors. Corresponding to each configuration, resistors along which current flows have been draw with a thicker line than those with zero voltage drop. Resistors meet at a node only when such a node is indicated by a dot and a voltage.

2 General Resistor Networks

2.1 The Forward Problem

If the geometry of the resistor network and the values of the resistors are known, the forward problem can be solved. Kirchoff's laws are used to compute the potential induced in the interior of the network when a specific set of voltages is applied to the boundary. From this information a matrix is computed that is a linear map of boundary voltages to boundary currents. This matrix will be called a Λ matrix. The entries in the columns of the Λ matrix are the current at the boundary induced by applying a voltage of 1 to one boundary node and a voltage of 0 to all other boundary nodes. For all networks I am considering, the Λ map exists and is 1 to 1. The Λ matrix of a resistor network can be used to find the boundary currents induced in that network by any set of boundary voltages.

2.2 The Inverse Problem

The Inverse Problem asks whether or not the resistors in a network can be recovered from its Λ matrix, assuming that the geometry of the resistor network is know. For certain geometries of resistor networks the Λ matrix has been shown to contain enough information to reconstruct the values of all resistors in the network. The algorithms used are dependent on the existence of a configuration of boundary voltages that satisfies the following condition : m of the boundary nodes have zero current in the configuration and at most $m+1$ of the boundary nodes are not known to have zero voltage. If such a configuration exists, the exact value of these boundary voltages can be determined using linear equations taken from the Λ matrix and associated with each boundary node of zero current.

Depending on the geometry of the network, this configuration may give enough information about the potential in the interior of the network to allow resistor values to be computed using Kirchoff's and Ohm's laws. It does not, however, guarentee recoverability. If the number of non-zero boundary voltages is less that $m+1$, the system of equations used to compute non-zero boundary voltages is overdetermined, and there are values in the Λ matrix that can be found as linear combinations of other values. I will refer to such linearly dependent systems as *relations* in the Λ matrix.

2.3 Characteristics of the Λ Matrix

The question of existence of maximal resistor networks comes about by considering some special properties of the Λ matrix : it is symmetric about its diagonal axis, running from the top left to bottom right corners, and each entry along this axis is equal to the sum of all other entries in the column or row that it occupies. Therefore, an $n \times n$ Λ matrix, computed from a network with n boundary nodes, contains a set of at most $n(n-1)/2$ entries that cannot be immediately determined from other entries in the matrix. This set could be taken to be all the entries above the diagonal axis. Since the Λ matrix contains at most $n(n-1)/2$ independent values, this number also determines the maximum number of resistors recoverable from the Λ matrix of a resistor network with n boundary nodes. For most resistor networks the number of recoverable resistors per boundary node is much lower, because this set of entries is itself not free of linearly dependent relations. In that case, not all the $n(n-1)/2$ entries are needed to recover the network, since some of these entries can be deduced from other entries in the set.

For example, define a square network to a regular $m \times m$ square grid lattice with m boundary nodes connected to one resistor each on all four sides. These networks have m^2 interior nodes, giving a total of $2m(m+1)$ resistors in the network and a $4m \times 4m$ Λ matrix. Since the number $4m(4m-1)/2$ of potentially independent values in the Λ matrix is greater than the number of resistors inthe network, there exist relations in the Λ matrix in addition to the zero sum condition and symmetry. These relations consist of linearly dependent submatrices of the Λ matrix. It has been shown that a square resistor network can be reconstructed using these relations from just $2m(m+1)$ well chosen entries of the Λ matrix.

The number of linearly independent entries in a Λ matrix sets an upper limit for the number of recoverable resistors in the corresponding network. Since a maximal resistor network with n boundary nodes would contain $n(n-1)/2$ entries and be fully recoverable, the corresponding Λ matrix must contain no relations. Equivalently, any valid configuration of boundary voltages and currents of a maximal resistor network must have one more non-zero boundary voltage than zero boundary currents, unless the boundary voltages are all zero. In the following section I will show that Λ matrices without relations exist through examples of maximal resistor networks.

3 Maximal Resistor Networks

3.1 $N-$Fold Symmetric Maximal Networks

There are two families of planar maximal resistor networks with $n-$fold rotational symmetry, one with spikes at the boundary (one resistor coming from each boundary node) and the other with a smooth boundary (three resistors meeting at each boundary node). These networks are only maximal if n is odd, in which case $n(n-1)/2$ is a integer multiple of n. The parity of $(n-1)/2$ effects the internal geometry of the resistor network in each case, giving four basic types of n-fold symmetric maximal resistor networks. These are shown on the next four pages. Networks of these types could also be generalized to include networks with even numbers of boundary nodes, but these would not be maximal resistor networks as they would either contain too few resistors, or not be fully recoverable.

3.2 Maximal Resistor Networks with a Mixed Boundary

There exists a network geometry with boundary nodes that alternate between spiked and smooth that gives a maximal resistor network when the number of boundary nodes is $n = 4m+2, m \in \mathcal{Z}$. This geometry does not work when $n = 4m, m \in \mathcal{Z}$ because in this case there are relations in the network's Λ matrix and so it cannot contain enough information to recover all the resistors in the network. This is seen from the fact that a configuration of boundary voltages exists in which there are as many boundary nodes of zero current as there are variable boundary voltages. This is analogous to what happens when n is even for networks considered in the previous section.

3.3 Complete Resistor Networks

A complete resistor network is a network with n nodes such that each node is joined to every other node in the network by a resistor, so a complete resistor network contains $n(n-1)/2$ resistors. Since this number is the maximum number of resistors recoverable from the lambda matrix of a network with n nodes, all the n nodes of a complete resistor network must be considered boundary nodes. A complete resistor network is a maximal resistor network. These networks are non-planer when n is greater than 4.

Complete resistor networks are fully recoverable from their Λ matrices for any value of n. The Λ matrix is formed in the usual way, by setting the voltage at one boundary node equal to one and the voltage at all other nodes to zero, computing the currents at each boundary node and using these currents to form the columns of a the Λ matrix. In the case of a complete resistor network, setting the voltage at all nodes except for one equal to zero blocks current from flowing across any resistors except the ones connected to the node with non-zero voltage. Therefore, if the non-zero voltage is chosen to be 1, the current at each of the nodes of zero voltage is equal to minus the resistance across the resistor that connects it to the node of voltage 1. This means that in the complete case the inverse problem is immediately solved, since the resistors in a complete network can be read directly from its Λ matrix.

3.4 Reconstruction of an N−Fold Symmetric Maximal Network

Here I outline a reconstruction algorithm for a spiked maximal resistor network with eleven boundary nodes. Reconstruction algorithms for the other cases are analogous, as are the equations for calculating boundary values from the Λ matrix.

Let u be a function that gives the voltage at each node in the network and is a valid solution to the boundary conditions shown in the graph. If the boundary nodes are numbered as shown, then the boundary conditions are the following :

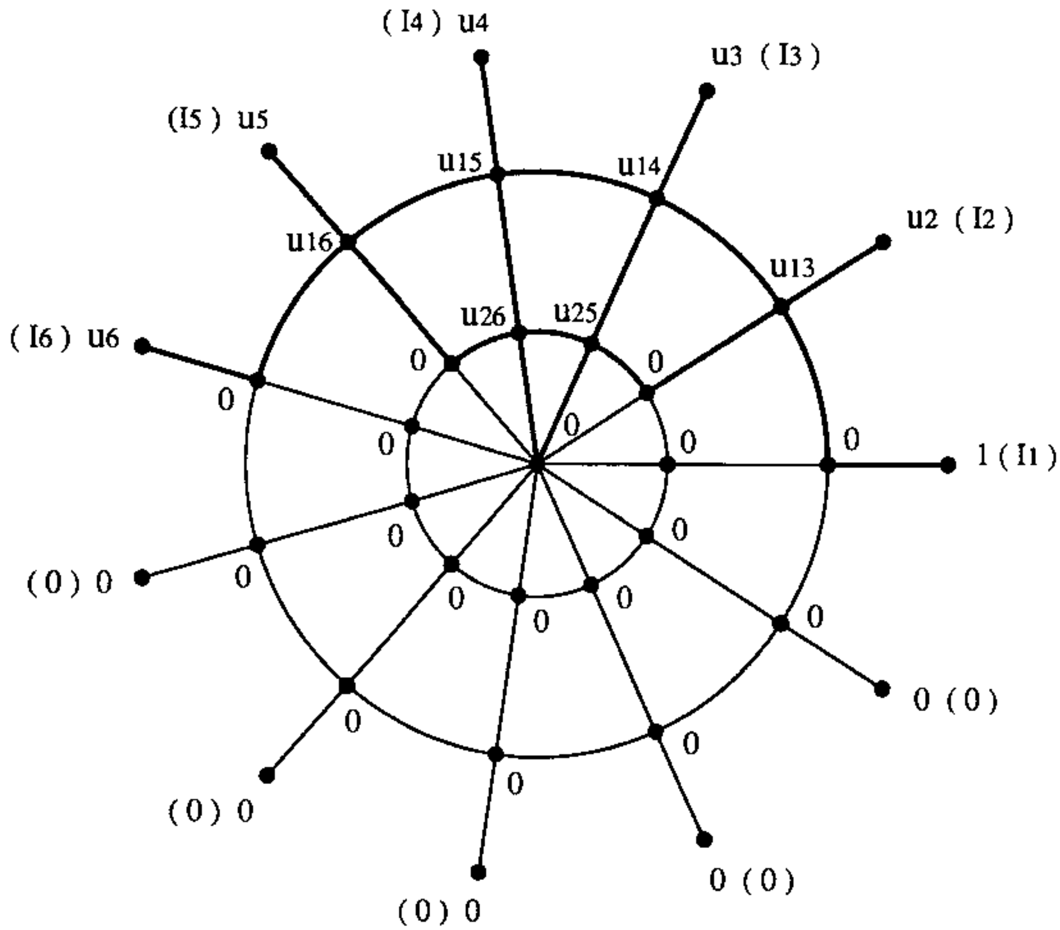
$$\Lambda_{7,2}u_2 + \Lambda_{7,3}u_3 + \Lambda_{7,4}u_4 + \Lambda_{7,5}u_5 + \Lambda_{7,5}u_6 = -\Lambda_{7,1}$$
$$\Lambda_{8,2}u_2 + \Lambda_{8,3}u_3 + \Lambda_{8,4}u_4 + \Lambda_{8,5}u_5 + \Lambda_{8,6}u_6 = -\Lambda_{8,1}$$
$$\Lambda_{9,2}u_2 + \Lambda_{9,3}u_3 + \Lambda_{9,4}u_4 + \Lambda_{9,5}u_5 + \Lambda_{9,6}u_6 = -\Lambda_{9,1}$$
$$\Lambda_{10,2}u_2 + \Lambda_{10,3}u_3 + \Lambda_{10,4}u_4 + \Lambda_{10,5}u_5 + \Lambda_{10,6}u_6 = -\Lambda_{10.1}$$
$$\Lambda_{11,2}u_2 + \Lambda_{11,3}u_3 + \Lambda_{11,4}u_4 + \Lambda_{11,5}u_5 + \Lambda_{11,6}u_6 = -\Lambda_{11,1}$$
$$u_1 = 1 \qquad u_7 = u_8 = u_9 = u_{10} = u_{11} = 0$$

The row of five boundary nodes with zero voltage and current in this configuration force all the zero voltages shown at interior nodes. Since there is a voltage drop from 1 to 0 over the resistor joining nodes $u_1 = 1$ and $u_{12} = 0$, there is a current across this resistor, equal to the current across the resistor joining nodes u_{12} and u_{13}. Given the voltage drop across this resistor, $u_{12} = 0$ implies $u_{13} \neq 0$. Likewise there is a voltage drop across and thus current through the resistor connecting u_{13} and $u_{24} = 0$, and the resistor joining u_{24} to u_{25}, so $u_{25} \neq 0$, and across the resistor connecting u_{25} and u_{34}. The knowledge that $u_{13} \neq 0$ and $u_{25} \neq 0$, and therefore that current flows across a 'ladder' connecting node u_1 to node u_{34}, is central to the reconstruction algorithm.

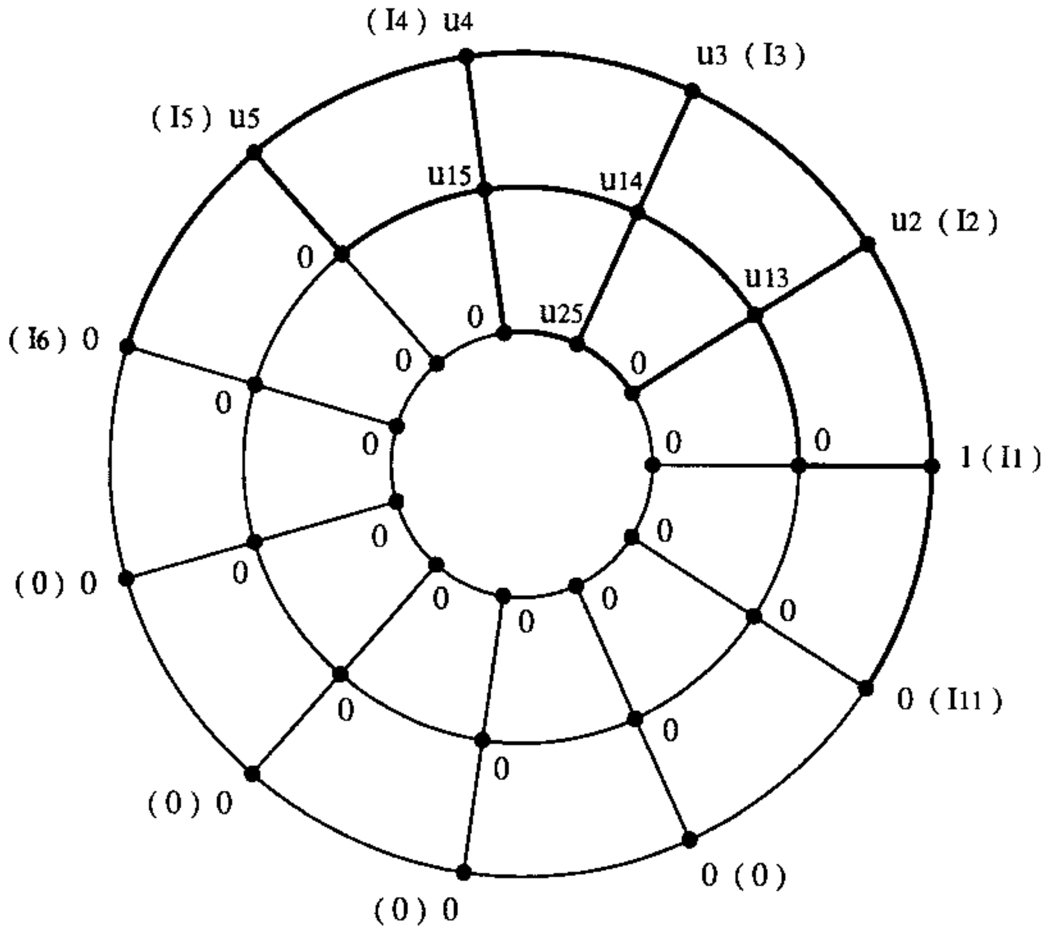
Let γ represent the set of resistors in the network, γ_m representing the boundary resistor connecting to node u_m and γ_{m+11} representing the resistor on the exterior ring of the network meeting γ_m on the right. With all the boundary values of u for the given configuration in hand, The set I of boundary currents can be computed from the Λ matrix.

$$I_1 = \Lambda_{1,1} + \Lambda_{1,2}u_2 + \Lambda_{1,3}u_3 + \Lambda_{1,4}u_4 + \Lambda_{1,5}u_5 + \Lambda_{1,6}u_6$$

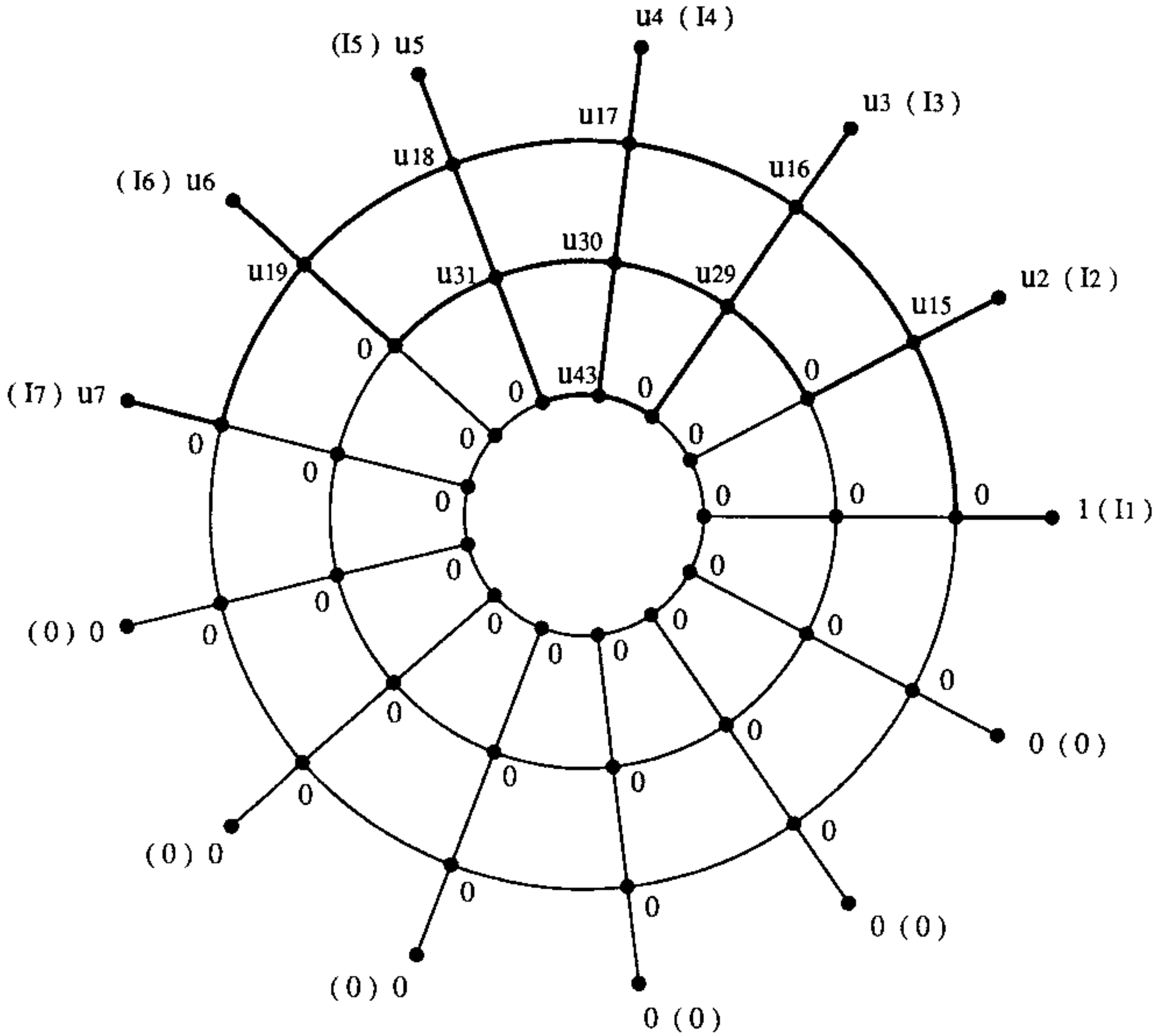
Since $u_{12} = 0$, $\gamma_1 = I_1/u_1$ can then be computed. This configuration of zero and non-zero boundary voltages can be rotated around the graph, next setting $u_2 = 1$, and finally $u_{11} = 1$. In each rotation, new non-zero boundary voltages are found from the Λ matrix and the ladder of current used in reconstruction passes across a new set of resistors, so the entire set of boundary spokes, γ_1 through γ_{11}, can be found. Once these are known, the value of u at nodes on the first ring of the network can computed, since $u_{m+11} = u_m - I_m/\gamma_m$. When the boundary spike resistors and resistors on the outer ring are known, the problem is analogous to the original situation; the first ring of inner spokes can be considered a new boundary on which all voltages and currents are deduced from known values, and values of γ in the interior of the network can be calculated just as were the exterior values.



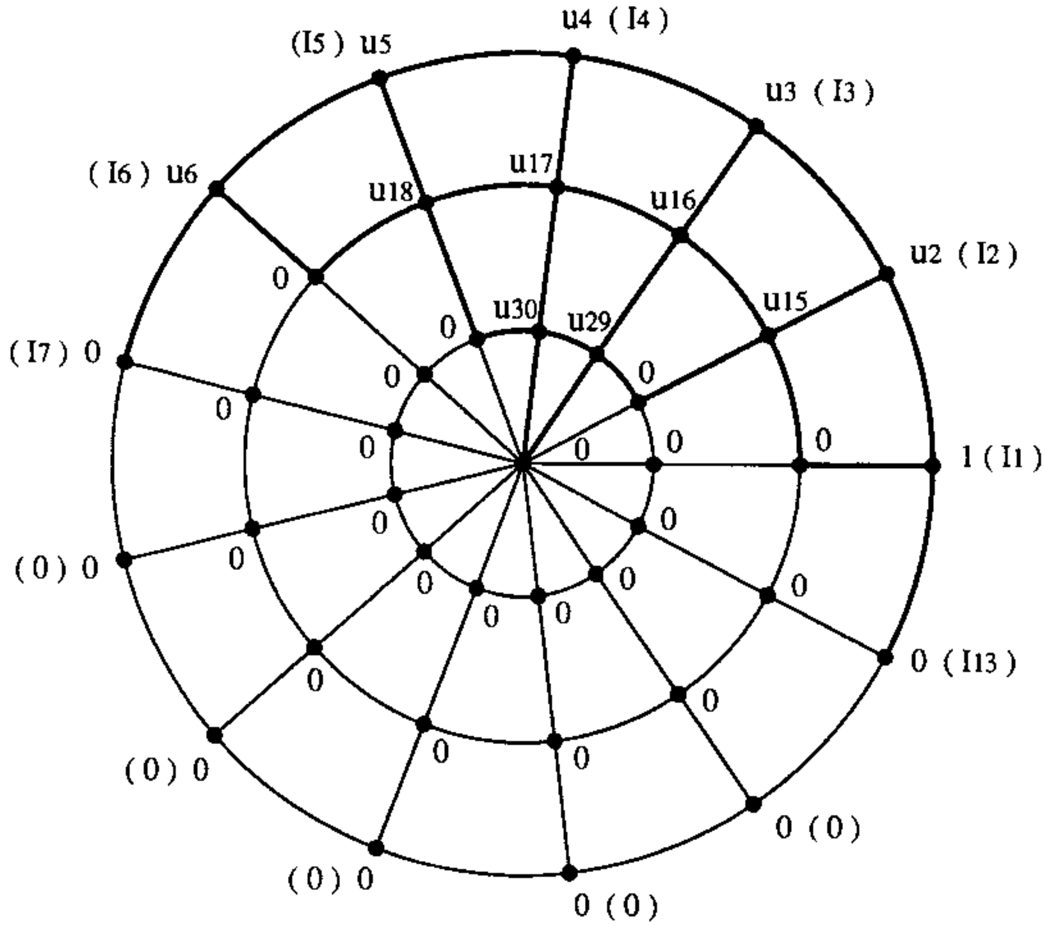
3.3.1 **Spiked network with 11 boundary nodes, a prototype for a general spiked maximal network with $n = 4m+3$ boundary nodes, $m \in \mathcal{Z}$. These networks have $(n-3)/4$ rings and a central node.**



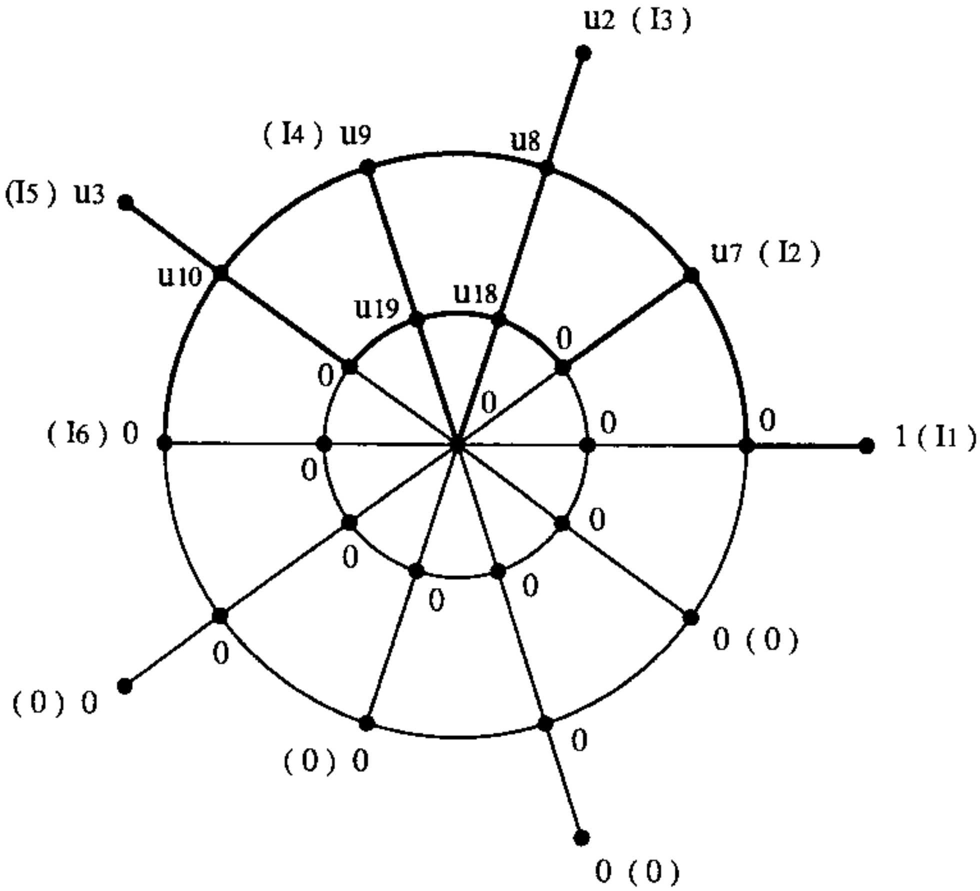
3.3.2 Smooth network with 11 boundary nodes, a prototype for a general smooth maximal network with $n = 4m+3$ boundary nodes, $m \in \mathcal{Z}$. These networks have $(n+1)/4$ rings



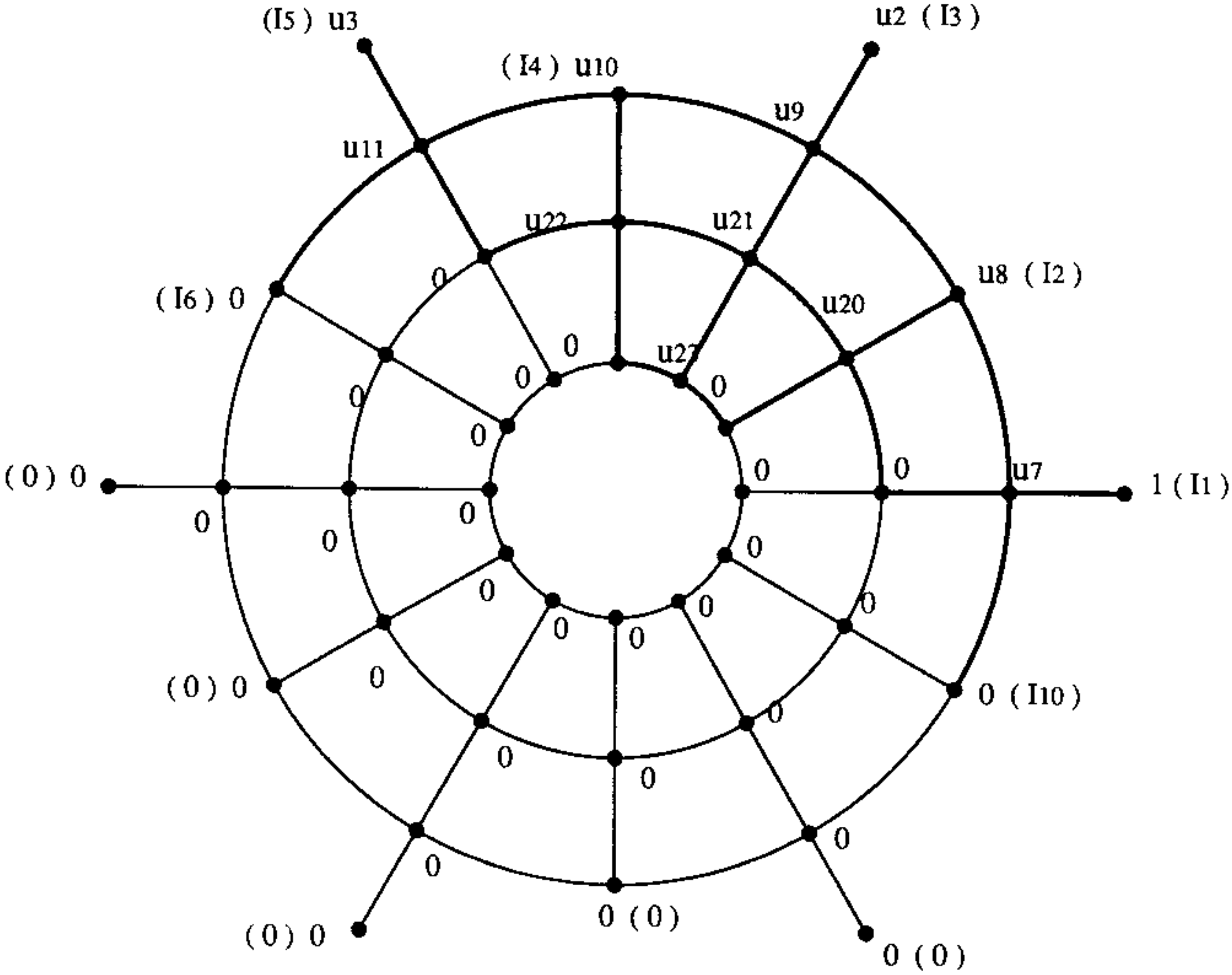
3.3.3 **Spiked network with 13 boundary nodes, a prototype for a general spiked maximal network with $n = 4m+1$ boundary nodes, $m \in \mathcal{Z}$. These networks have $(n-1)/4$ rings**



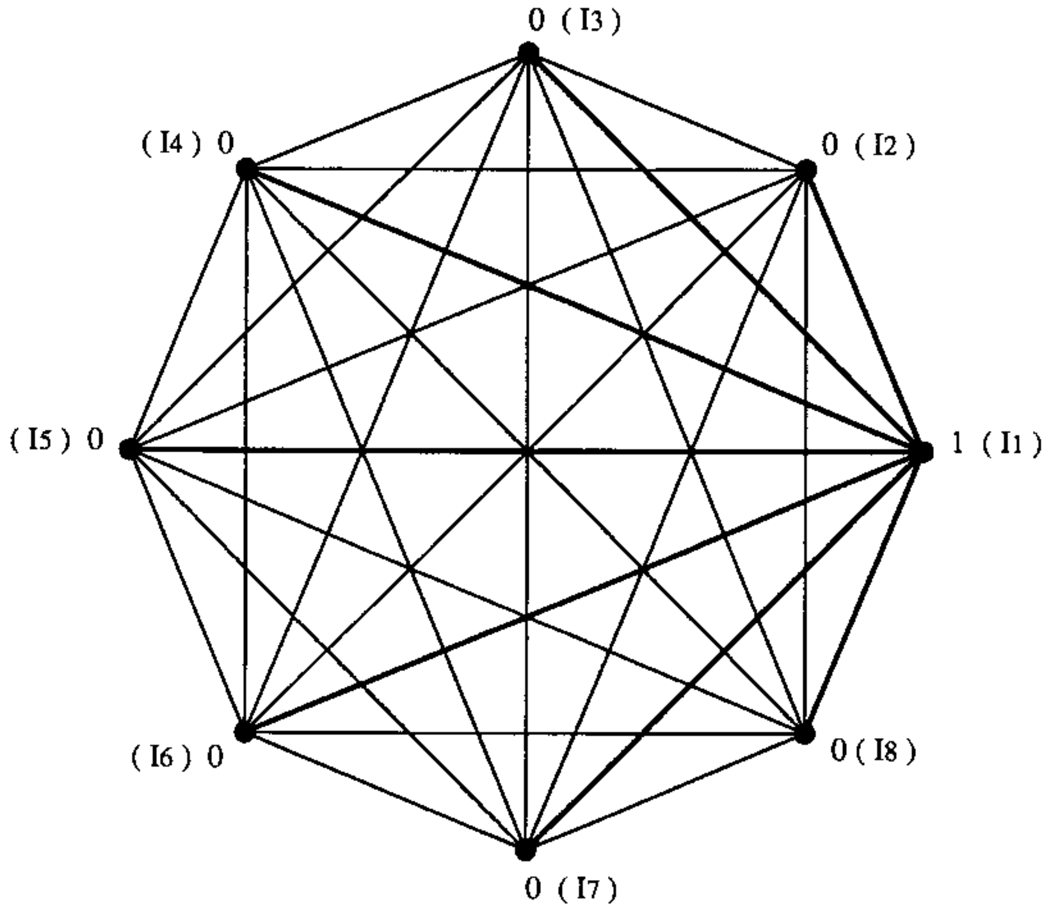
3.3.4 Smooth network with 13 boundary nodes, a prototype for a general smooth maximal network with $n = 4m + 1$ boundary node, $m \in \mathcal{Z}$. These networks have $(n-1)/4$ rings and a central node.



3.3.5 A maximal resistor network with 10 boundary nodes, a prototype for a general mixed boundary maximal resistor network with $n = 4m + 2$ boundary nodes, $m \in \mathcal{Z}$. These networks have $(n-2)/4$ rings and a centeral node.



3.3.6 A resistor network with 12 boundary nodes. This network is *not* fully recoverable, as may be seen from the marked boundary values. No resistor networks of this type, with $n = 4m + 2$ boundary nodes, $m \in \mathcal{Z}$, are fully recoverable.



3.3.7 A complete resistor network with 8 boundary nodes, a prototype for a general complete resistor network with n boundary nodes.

REFERENCES

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