Finding Internal Transitional Probabilities from Measurements of Exiting Probabilities

Kurt F. Krenz

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1 Preliminaries

Assume in our network we have a total of u nodes, m of which are boundary nodes and n of which are interior nodes where $u = m + n$. Then we have a $u \times u$ transition matrix P where $p_{i,j}$ is the probability of a particle moving directly from node i to node j. We will reorder P so that the rows and columns corresponding to boundary nodes come first, with the rows and columns corresponding to interior nodes coming last. Then P will have canonical form:

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}$$

where, I is the $m \times m$ identity matrix, 0 is the $m \times n$ matrix consisting of all 0's, R is the $n \times m$ matrix representing the probabilities of moving from the interior nodes to boundary nodes, and Q is the $n \times n$ matrix representing the probabilities of moving from one interior node to another.

From P, we can obtain the $n \times n$ matrix $N = (I - Q)^{-1}$ (In this formula I is $n \times n$). The probabilistic interpretation of N is $n_{i,j}$ is the expected number of times that a particle will be at node j before it is absorbed by a boundary node when it is started (input) at node i.

The final important matrix for our purposes is the $n \times m$ matrix B. The matrix B has the probabilistic interpretation that $b_{i,j}$ is the probability that a particle input at interior node i will exit the network at boundary node j.

2 Forward Problem

The forward problem consists of calculating each entry of the matrix B when we are given the matrix P. Since we have P, we implicitly have the matrix $(I-Q)$ and can calculate $N=(I-Q)^{-1}$. Then B can be found by using the formula:

$$B = NR = (I-Q)^{-1}R$$

I have written a FORTRAN program that will solve the forward problem for any legal network.

3 Inverse Problem

The inverse problem consists of using B to find the entries of P. The problem is to show that given B, there is a unique solution for P. This is done by showing that certain systems of equations (one system for each interior node of the network) are solvable in terms of the unknown probabilities $p_{i,j}$.

4 Case 1

The first network that I considered when trying to come up with a solution to the inverse problem was the following:

This network has three interior nodes (numbered 1, 2, and 3) and three boundary nodes (numbered $\hat{1}$, $\hat{2}$, and $\hat{3}$). Thus, P will be 6×6, R will be 3×3, $(I-Q)$ will be 3×3 (as will N), and B will be 3×3. I labelled the interior probabilities for this network in the following way:

This labelling will give the following matrices:

$$R = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix}$$

$$(I - Q) = \begin{bmatrix} 1 & -b_1 & -c_1 \\ -a_2 & 1 & -c_2 \\ -a_3 & -b_3 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix}$$

$B = NR$, so $R = (I - Q)B$ since $N^{-1} = (I - Q)$.
Using the formula $R = (I-Q)B$, we can write out the following equations:

$$\begin{aligned} r_1 &= b_{1,1} - b_1 b_{2,1} - c_1 b_{3,1} & (1) \\ 0 &= b_{1,2} - b_1 b_{2,2} - c_1 b_{3,2} & (2) \\ 0 &= b_{1,3} - b_1 b_{2,3} - c_1 b_{3,3} & (3) \end{aligned}$$

$$\begin{aligned} 0 &= -a_2 b_{1,1} + b_{2,1} - c_2 b_{3,1} & (4) \\ r_2 &= -a_2 b_{1,2} + b_{2,2} - c_2 b_{3,2} & (5) \\ 0 &= -a_2 b_{1,3} + b_{2,3} - c_2 b_{3,3} & (6) \end{aligned}$$

$$\begin{aligned} 0 &= -a_3 b_{1,1} - b_3 b_{2,1} + b_{3,1} & (7) \\ 0 &= -a_3 b_{1,2} - b_3 b_{2,2} + b_{3,2} & (8) \\ r_3 &= -a_3 b_{1,3} - b_3 b_{2,3} + b_{3,3} & (9) \end{aligned}$$

We can write these equations in the matrix form $Ap = b$ as follows:

$$\begin{bmatrix} 1 & b_{2,1} & b_{3,1} & & & & & & \\ 0 & b_{2,2} & b_{3,2} & & & & & & \\ 0 & b_{2,3} & b_{3,3} & & & & & & \\ & & & 1 & b_{1,2} & b_{3,2} & & & \\ & & & 0 & b_{1,1} & b_{3,1} & & & \\ & & & 0 & b_{1,3} & b_{3,3} & & & \\ & & & & & & 1 & b_{1,3} & b_{2,3} \\ & & & & & & 0 & b_{1,1} & b_{2,1} \\ & & & & & & 0 & b_{1,2} & b_{2,2} \end{bmatrix} \begin{bmatrix} r_1 \\ b_1 \\ c_1 \\ r_2 \\ a_2 \\ c_2 \\ r_3 \\ a_3 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_{1,1} \\ b_{1,2} \\ b_{1,3} \\ b_{2,2} \\ b_{2,1} \\ b_{2,3} \\ b_{3,3} \\ b_{3,1} \\ b_{3,2} \end{bmatrix}$$

Here, and throughout the paper, we use the convention that the blank spaces in A represent all 0's. We can consider $Ap = b$ as being three systems, each with 3 equations in 3 unknowns. They correspond to interior nodes 1, 2, and 3 respectively.

Recall that in the inverse problem, the matrix B is known and we want to solve for the p's (r's, a's, b's, and c's in this case). To show that a unique solution to the inverse problem exists, we must show that each of the three sub-matrices has non-zero determinant. If so, then each sub-matrix will have rank 3 and we can solve $Ap = b$ uniquely for the unknown p's by separately solving each of the three systems of 3 equations in 3 unknowns.

The three sub-determinants are:

$$\det \begin{bmatrix} 1 & b_{2,1} & b_{3,1} \\ 0 & b_{2,2} & b_{3,2} \\ 0 & b_{2,3} & b_{3,3} \end{bmatrix} = \det \begin{bmatrix} b_{2,2} & b_{3,2} \\ b_{2,3} & b_{3,3} \end{bmatrix},$$

$$\det \begin{bmatrix} 1 & b_{1,2} & b_{3,2} \\ 0 & b_{1,1} & b_{3,1} \\ 0 & b_{1,3} & b_{3,3} \end{bmatrix} = \det \begin{bmatrix} b_{1,1} & b_{3,1} \\ b_{1,3} & b_{3,3} \end{bmatrix},$$

and

$$\det \begin{bmatrix} 1 & b_{1,3} & b_{2,3} \\ 0 & b_{1,1} & b_{2,1} \\ 0 & b_{1,2} & b_{2,2} \end{bmatrix} = \det \begin{bmatrix} b_{1,1} & b_{2,1} \\ b_{1,2} & b_{2,2} \end{bmatrix}$$

We need to show that each of these determinants is non-zero. Note that because of the symmetry of the network, it will suffice to show that any one of the three is non-zero. We will prove that the first determinant is non-zero.

Lemma

$$\det\begin{bmatrix} 1 & b_{2,1} & b_{3,1} \\ 0 & b_{2,2} & b_{3,2} \\ 0 & b_{2,3} & b_{3,3} \end{bmatrix} \neq 0$$

Proof

We know $\det B > 0$. so B is invertible. Then by Cramer's Rule

$$(B^{-1})_{1,1} = \frac{\det\begin{bmatrix} b_{2,2} & b_{2,3} \\ b_{3,2} & b_{3,3} \end{bmatrix}}{\det B} = \frac{\det\begin{bmatrix} b_{2,2} & b_{3,2} \\ b_{2,3} & b_{3,3} \end{bmatrix}}{\det B}$$

so

$$\det\begin{bmatrix} b_{2,2} & b_{3,2} \\ b_{2,3} & b_{3,3} \end{bmatrix} = (B^{-1})_{1,1} \det B$$

$\det B > 0$, so we need to show $(B^{-1})_{1,1} > 0$. Since $B = NR$, then $B^{-1} = R^{-1}N^{-1} = R^{-1}(I - Q)$. R is diagonal, so R^{-1} simply consists of the recipricals of the corresponding entries of R along its diagonal. Each of these entries of R^{-1} will be > 1 because the diagonal entries of R must be > 0, but < 1.

Then $(B^{-1})_{1,1} =$ (first row of R^{-1}) $\cdot$ (first column of$(I - Q)$)

$$= \begin{pmatrix} x & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -a_2 \\ -a_3 \end{pmatrix} = x$$

where $x = (r_1)^{-1} > 1$.

Incidently, we know that the first column of $(I - Q)$ is $\begin{pmatrix} 1 \\ -a_2 \\ -a_3 \end{pmatrix}$ because the matrix B is 3×3, but to take the first column of R^{-1} as $\begin{pmatrix} x & 0 & 0 \end{pmatrix}$ with $x > 1$, we need to know the shape of the network; in particular that node 1 is adjacent to exactly one boundary node.

Then $(B^{-1})_{1,1} > 0$ implies $(B^{-1})_{1,1} \cdot \det B > 0$ which in turn implies that

$$\det\begin{bmatrix} b_{2,2} & b_{3,2} \\ b_{2,3} & b_{3,3} \end{bmatrix} = \det\begin{bmatrix} 1 & b_{2,1} & b_{3,1} \\ 0 & b_{2,2} & b_{3,2} \\ 0 & b_{2,3} & b_{3,3} \end{bmatrix} > 0.$$

Thus each of the three sub-matrices of A has positive determinant and therefore rank 3, so each of the unknown probabilities can be found uniquely by solving separately each of the three systems of 3 equations in 3 unknowns.

Now, to complete the proof we need to show $det\ B > 0$. $B = NR$, and $det\ R > 0$ because R is diagonal with positive entries, so if $det\ N > 0$ then $det\ B > 0$ will follow. Now $N = (I - Q)^{-1}$ so if $det\ (I - Q) > 0$, then $det\ N > 0$ and hence $det\ B > 0$.

$$(I - Q) = \begin{bmatrix} 1 & -b_1 & -c_1 \\ -a_2 & 1 & -c_2 \\ -a_3 & -b_3 & 1 \end{bmatrix}$$

Let

$$F(t) = det \begin{bmatrix} 1 & -b_1 t & -c_1 t \\ -a_2 t & 1 & -c_2 t \\ -a_3 t & -b_3 t & 1 \end{bmatrix}$$

for $0 \leq t \leq 1$.

When t=0, $F(t)$=1. By assumption, the a's, b's, and c's are positive and < 1. We can go from one $(I - Q)$ to any other by varying t. $F(t) > 0$ when $t = 0$, therefore $F(t)$ is always > 0.

Thus $det\ (I - Q) > 0$, $det\ N > 0$, and

$$detB = detN \cdot detR > 0$$

This completes the proof.

5 General Complete Graph Case

We will now prove the uniqueness of the inverse solution in the general case of a complete graph network where we have n interior nodes, all mutually adjacent, and each adjacent to exactly one boundary node and vice versa.

$B = NR$, and $R = (I - Q)B$, so in this case

$$\begin{bmatrix} r_1 & & & & & \\ & r_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & r_n \end{bmatrix} = \begin{bmatrix} 1 & -b_1 & \cdot & \cdot & \cdot & -n_1 \\ -a_2 & 1 & \cdot & \cdot & \cdot & -n_2 \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \\ -a_n & -b_n & \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \cdot & \cdot & \cdot & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdot & \cdot & \cdot & b_{2,n} \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ b_{n,1} & b_{n,2} & \cdot & \cdot & \cdot & b_{n,n} \end{bmatrix}$$

This will produce n^2 equations. Writing them in the matrix form $Ap = b$ will give:

$$\begin{bmatrix} 1 & b_{2,1} & \cdot & \cdot & b_{n,1} & & & & & & & & \\ 0 & b_{2,2} & \cdot & \cdot & b_{n,2} & & & & & & & & \\ \cdot & \cdot & & & & & & & & & & & \\ \cdot & \cdot & & & & & & & & & & & \\ 0 & b_{2,n} & \cdot & \cdot & b_{n,n} & & & & & & & & \\ & & & & & \cdot & & & & & & & \\ & & & & & & \cdot & & & & & & \\ & & & & & & & \cdot & & & & & \\ & & & & & & & & 1 & b_{1,n} & \cdot & \cdot & b_{n-1,n} \\ & & & & & & & & 0 & b_{1,1} & \cdot & \cdot & b_{n-1,1} \\ & & & & & & & & \cdot & \cdot & & & \\ & & & & & & & & \cdot & \cdot & & & \\ & & & & & & & & 0 & b_{1,n-1} & \cdot & \cdot & b_{n-1,n-1} \end{bmatrix} \begin{bmatrix} r_1 \\ b_1 \\ \cdot \\ \cdot \\ n_1 \\ \cdot \\ \cdot \\ \cdot \\ r_n \\ a_n \\ \cdot \\ \cdot \\ (n-1)_n \end{bmatrix} = \begin{bmatrix} b_{1,1} \\ b_{1,2} \\ \cdot \\ \cdot \\ b_{1,n} \\ \cdot \\ \cdot \\ \cdot \\ b_{n,n} \\ b_{n,1} \\ \cdot \\ \cdot \\ b_{n,n-1} \end{bmatrix}$$

We can think of $Ap = b$ as being n systems, each with n equations in n unknowns. As before, to show that there is a unique solution, we need to show that each of the n sub-matrices of A has non-zero determinant. If so, then each sub-matrix will have rank n and we can solve $Ap = b$ uniquely for the p's (r's, a's, b's, $\cdots$, and n's in this case) by separately solving each of the n systems of n equations in n unknowns.

The n sub-determinants are:

$$\det \begin{bmatrix} 1 & b_{2,1} & \cdot & \cdot & \cdot & b_{n,1} \\ 0 & b_{2,2} & \cdot & \cdot & \cdot & b_{n,2} \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ 0 & b_{2,n} & \cdot & \cdot & \cdot & b_{n,n} \end{bmatrix}, \cdots, \det \begin{bmatrix} 1 & b_{1,n} & \cdot & \cdot & \cdot & b_{n-1,n} \\ 0 & b_{1,1} & \cdot & \cdot & \cdot & b_{n-1,1} \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ 0 & b_{1,n-1} & \cdot & \cdot & \cdot & b_{n-1,n-1} \end{bmatrix}$$

We need to show that each of these n determinants is non-zero. It will again suffice to show that any one of these determinants is non-zero because of the symmetry of the network. We will show that the first determinant is non-zero.

Lemma

$$\det \begin{bmatrix} 1 & b_{2,1} & \cdot & \cdot & \cdot & b_{n,1} \\ 0 & b_{2,2} & \cdot & \cdot & \cdot & b_{n,2} \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ 0 & b_{2,n} & \cdot & \cdot & \cdot & b_{n,n} \end{bmatrix} \neq 0$$

Proof

This proof will be very similar to the proof of the first case. $\det B > 0$ (because $B = NR$ and R has positive determinant because its only non-zero entries are positive entries on the diagonal, and N has positive determinant because $N = (I-Q)^{-1}$ and $(I-Q)$ has positive determinant. Thus, $\det B$ is the product of two positive determinants), and thus B is invertible.

Then

$$(B^{-1})_{1,1} = \frac{\det \begin{bmatrix} b_{2,2} & b_{2,3} & \cdot & \cdot & \cdot & b_{2,n} \\ b_{3,2} & b_{3,3} & \cdot & \cdot & \cdot & b_{3,n} \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ b_{n,2} & b_{n,3} & \cdot & \cdot & \cdot & b_{n,n} \end{bmatrix}}{\det B} = \frac{\det \begin{bmatrix} b_{2,2} & b_{3,2} & \cdot & \cdot & \cdot & b_{n,2} \\ b_{2,3} & b_{3,3} & \cdot & \cdot & \cdot & b_{n,3} \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ b_{2,n} & b_{3,n} & \cdot & \cdot & \cdot & b_{n,n} \end{bmatrix}}{\det B}$$

so

$$\det \begin{bmatrix} b_{2,2} & b_{3,2} & \cdot & \cdot & \cdot & b_{n,2} \\ b_{2,3} & b_{3,3} & \cdot & \cdot & \cdot & b_{n,3} \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ b_{2,n} & b_{3,n} & \cdot & \cdot & \cdot & b_{n,n} \end{bmatrix} = (B^{-1})_{1,1} \det B$$

We know $\det B > 0$, so we need to show $(B^{-1})_{1,1} > 0$. Now since $B = NR$, then $B^{-1} = R^{-1}(I - Q)$. R is diagonal, so R^{-1} consists of the recipricals - each of which is > 1 - of the corresponding entries of R along its diagonal. We know the general form of $(I - Q)$, (namely that it has dimension $n \times n$), from the dimension of B.

Then $(B^{-1})_{1,1} =$ (first row of R^{-1}) $\cdot$ (first column of $(I - Q)$)

$$= \begin{pmatrix} x & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -a_2 \\ -a_3 \\ \cdot \\ \cdot \\ \cdot \\ -a_n \end{pmatrix} = x$$

where $x = (r_1)^{-1} > 1$.

Then $(B^{-1})_{1,1} > 0$ implies $(B^{-1})_{1,1} \cdot \det B > 0$ which implies

$$\det \begin{bmatrix} b_{2,2} & b_{3,2} & \cdot & \cdot & \cdot & b_{n,2} \\ b_{2,3} & b_{3,3} & \cdot & \cdot & \cdot & b_{n,3} \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ b_{2,n} & b_{3,n} & \cdot & \cdot & \cdot & b_{n,n} \end{bmatrix} = \det \begin{bmatrix} 1 & b_{2,1} & b_{3,1} & \cdot & \cdot & \cdot & b_{n,1} \\ 0 & b_{2,2} & b_{3,2} & \cdot & \cdot & \cdot & b_{n,2} \\ 0 & b_{2,3} & b_{3,3} & \cdot & \cdot & \cdot & b_{n,3} \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & & \\ 0 & b_{2,n} & b_{3,n} & \cdot & \cdot & \cdot & b_{n,n} \end{bmatrix} > 0$$

Thus each of the n sub-matrices of matrix A has positive determinant and therefore rank n, and each of the unknown probabilities can be found uniquely by solving separately each of the n systems of n equations in n unknowns. This completes the proof.

6 A Different Type of Network

Now we will examine the following case:

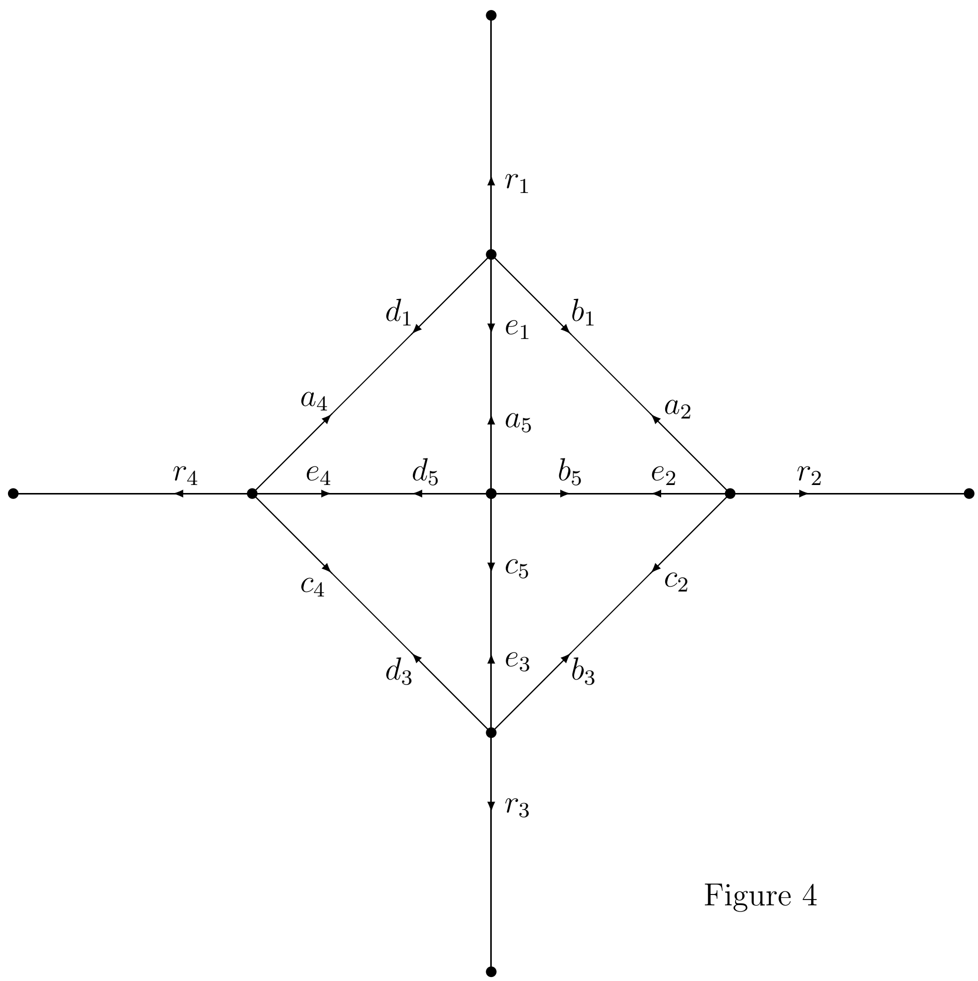

Figure 4

Note that here we have an interior node which isn't adjacent to a boundary node. In this case, P will be 9×9, $(I - Q)$ will be 5×5, N will be 5×5, R will be 5×4, and B will be 5×4.

$B = NR$, so $R = (I - Q)B$. Writing out $R = (I - Q)B$ gives:

$$\begin{bmatrix} r_1 & & & \\ & r_2 & & \\ & & r_3 & \\ & & & r_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -b_1 & 0 & -d_1 & -e_1 \\ -a_2 & 1 & -c_2 & 0 & -e_2 \\ 0 & -b_3 & 1 & -d_3 & -e_3 \\ -a_4 & 0 & -c_4 & 1 & -e_4 \\ -a_5 & -b_5 & -c_5 & -d_5 & 1 \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \\ b_{5,1} & b_{5,2} & b_{5,3} & b_{5,4} \end{bmatrix}$$

Multiplying this out will give us 20 equations. The equations can be written in the matrix form $Ap = b$ as follows:

$$\begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & A_3 & & \\ & & & A_4 & \\ & & & & A_5 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}$$

Where each A_i is a 4×4 matrix of entries from the matrix B, each P_i is a 4-component column vector of unknown probabilities corresponding to interior node i, and each B_i is a 4-component column vector of entries from the matrix B.

As in the previous cases, if we can prove that each of the five sub-matrices of A has non-zero determinant, then each sub-matrix will have rank 4 and we can solve $Ap = b$ uniquely for the unknown p's (r's, a's, b's, c's, d's, and e's in this case) by separately solving each of the five systems of 4 equations in 4 unknowns.

The five systems of equations are: $A_1 P_1 = B_1$, or

$$\begin{bmatrix} 1 & b_{2,1} & b_{4,1} & b_{5,1} \\ 0 & b_{2,2} & b_{4,2} & b_{5,2} \\ 0 & b_{2,3} & b_{4,3} & b_{5,3} \\ 0 & b_{2,4} & b_{4,4} & b_{5,4} \end{bmatrix} \begin{bmatrix} r_1 \\ b_1 \\ d_1 \\ e_1 \end{bmatrix} = \begin{bmatrix} b_{1,1} \\ b_{1,2} \\ b_{1,3} \\ b_{1,4} \end{bmatrix}$$

$A_2 P_2 = B_2$, or

$$\begin{bmatrix} 1 & b_{1,2} & b_{3,2} & b_{5,2} \\ 0 & b_{1,1} & b_{3,1} & b_{5,1} \\ 0 & b_{1,3} & b_{3,3} & b_{5,3} \\ 0 & b_{1,4} & b_{3,4} & b_{5,4} \end{bmatrix} \begin{bmatrix} r_2 \\ a_2 \\ c_2 \\ e_2 \end{bmatrix} = \begin{bmatrix} b_{2,2} \\ b_{2,1} \\ b_{2,3} \\ b_{2,4} \end{bmatrix}$$

$A_3P_3 = B_3$, or

$$\begin{bmatrix} 1 & b_{2,3} & b_{4,3} & b_{5,3} \\ 0 & b_{2,1} & b_{4,1} & b_{5,1} \\ 0 & b_{2,2} & b_{4,2} & b_{5,2} \\ 0 & b_{2,4} & b_{4,4} & b_{5,4} \end{bmatrix} \begin{bmatrix} r_3 \\ b_3 \\ d_3 \\ e_3 \end{bmatrix} = \begin{bmatrix} b_{3,3} \\ b_{3,1} \\ b_{3,2} \\ b_{3,4} \end{bmatrix}$$

$A_4P_4 = B_4$, or

$$\begin{bmatrix} 1 & b_{1,4} & b_{3,4} & b_{5,4} \\ 0 & b_{1,1} & b_{3,1} & b_{5,1} \\ 0 & b_{1,2} & b_{3,2} & b_{5,2} \\ 0 & b_{1,3} & b_{3,3} & b_{5,3} \end{bmatrix} \begin{bmatrix} r_4 \\ a_4 \\ c_4 \\ e_4 \end{bmatrix} = \begin{bmatrix} b_{4,4} \\ b_{4,1} \\ b_{4,2} \\ b_{4,3} \end{bmatrix}$$

and $A_5P_5 = B_5$, or

$$\begin{bmatrix} b_{1,1} & b_{2,1} & b_{3,1} & b_{4,1} \\ b_{1,2} & b_{2,2} & b_{3,2} & b_{4,2} \\ b_{1,3} & b_{2,3} & b_{3,3} & b_{4,3} \\ b_{1,4} & b_{2,4} & b_{3,4} & b_{4,4} \end{bmatrix} \begin{bmatrix} a_5 \\ b_5 \\ c_5 \\ d_5 \end{bmatrix} = \begin{bmatrix} b_{5,1} \\ b_{5,2} \\ b_{5,3} \\ b_{5,4} \end{bmatrix}$$

Notice that by proving that any one of A_1, A_2, A_3, or A_4 (which correspond to the interior nodes numbered 1-4, respectively) has positive determinant, we will have proved that all four have positive determinants because of the symmetry of the network. Thus we are left to prove that the fifth determinant, along with one of the first four determinants are positive.

Proof

Consider the fifth determinant:

$$\det \begin{bmatrix} b_{1,1} & b_{2,1} & b_{3,1} & b_{4,1} \\ b_{1,2} & b_{2,2} & b_{3,2} & b_{4,2} \\ b_{1,3} & b_{2.3} & b_{3,3} & b_{4,3} \\ b_{1,4} & b_{2,4} & b_{3,4} & b_{4,4} \end{bmatrix} = \det \begin{bmatrix} n_{1,1}r_1 & n_{2,1}r_1 & n_{3,1}r_1 & n_{4,1}r_1 \\ n_{1,2}r_2 & n_{2,2}r_2 & n_{3,2}r_2 & n_{4,2}r_2 \\ n_{1,3}r_3 & n_{2,3}r_3 & n_{3,3}r_3 & n_{4,3}r_3 \\ n_{1,4}r_4 & n_{2,4}r_4 & n_{3,4}r_4 & n_{4,4}r_4 \end{bmatrix}$$

$$= (r_1 \cdot r_2 \cdot r_3 \cdot r_4) \det \begin{bmatrix} n_{1,1} & n_{2,1} & n_{3,1} & n_{4,1} \\ n_{1,2} & n_{2,2} & n_{3,2} & n_{4,2} \\ n_{1,3} & n_{2,3} & n_{3,3} & n_{4,3} \\ n_{1,4} & n_{2,4} & n_{3,4} & n_{4,4} \end{bmatrix}$$

since $B = NR$.

We know $(r_1 \cdot r_2 \cdot r_3 \cdot r_4)$ is positive. We also know N is always invertible. Thus

$$(N^{-1})_{5,5} = (I - Q)_{5,5} = 1 = \frac{\det \begin{bmatrix} n_{1,1} & n_{2,1} & n_{3,1} & n_{4,1} \\ n_{1,2} & n_{2,2} & n_{3,2} & n_{4,2} \\ n_{1,3} & n_{2,3} & n_{3,3} & n_{4,3} \\ n_{1,4} & n_{2,4} & n_{3,4} & n_{4,4} \end{bmatrix}}{\det N}$$

and

$$\det \begin{bmatrix} n_{1,1} & n_{2,1} & n_{3,1} & n_{4,1} \\ n_{1,2} & n_{2,2} & n_{3,2} & n_{4,2} \\ n_{1,3} & n_{2,3} & n_{3,3} & n_{4,3} \\ n_{1,4} & n_{2,4} & n_{3,4} & n_{4,4} \end{bmatrix} = 1 \cdot \det N = \det N$$

which is positive since $N = (I - Q)^{-1}$, and $\det(I - Q) > 0$.

So the fifth sub-determinant of A is the product of two positive numbers and hence positive. Thus, the fifth sub-matrix of A has rank 4.

Now we need to show that one of the first four sub-determinants is positive. We will prove that the third sub-determinant ($\det A_3$) is positive. We want to show

$$\det \begin{bmatrix} 1 & b_{2,3} & b_{4,3} & b_{5,3} \\ 0 & b_{2,1} & b_{4,1} & b_{5,1} \\ 0 & b_{2,2} & b_{4,2} & b_{5,2} \\ 0 & b_{2,4} & b_{4,4} & b_{5,4} \end{bmatrix} = \det \begin{bmatrix} b_{2,1} & b_{4,1} & b_{5,1} \\ b_{2,2} & b_{4,2} & b_{5,2} \\ b_{2,4} & b_{4,4} & b_{5,4} \end{bmatrix} \neq 0.$$

I created the new matrices

$$\tilde{B} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & 1 \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & 0 \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & 0 \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & 0 \\ b_{5,1} & b_{5,2} & b_{5,3} & b_{5,4} & 0 \end{bmatrix} \text{ and } \tilde{R} = \begin{bmatrix} r_1 & & & & v_1 \\ & r_2 & & & v_2 \\ & & r_3 & & v_3 \\ & & & r_4 & v_4 \\ & & & & v_5 \end{bmatrix}$$

which are just the matrices B and R with the extra columns added. The v_i's in $\tilde{R}$ are unknown.

Now we want to make $\tilde{R}$ such that $\tilde{B} = N\tilde{R}$. We do this by setting

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = N \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = (I - Q) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and solving for the v_i's.

This gives $v_1 = 1$, $v_2 = -a_2$, $v_3 = -a_3$, $v_4 = -a_4$, and $v_5 = -a_5$.

Now we have $\tilde{B} = N\tilde{R}$. We know N is invertible, and $\tilde{R}$ is upper triangular so

$$\det \tilde{R} = (r_1 \cdot r_2 \cdot r_3 \cdot r_4 \cdot -a_5) \neq 0$$

so $\tilde{R}$ is invertible. We calculate $\tilde{R}^{-1}$ to be:

$$\tilde{R}^{-1} = \begin{bmatrix} r_1^{-1} & & & & \frac{r_1^{-1}}{a_5} \\ & r_2^{-1} & & & \frac{-r_2^{-1}a_2}{a_5} \\ & & r_3^{-1} & & \frac{-r_3^{-1}a_3}{a_5} \\ & & & r_4^{-1} & \frac{-r_4^{-1}a_4}{a_5} \\ & & & & \frac{-1}{a_5} \end{bmatrix}$$

Thus $\det \tilde{B} \neq 0$ (because $\det N \cdot \det \tilde{R} \neq 0$) and $\tilde{B}$ is also invertible. Then

$$(\tilde{B}^{-1})_{3,3} = \frac{\det \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,4} & 1 \\ b_{2,1} & b_{2,2} & b_{2,4} & 0 \\ b_{4,1} & b_{4,2} & b_{4,4} & 0 \\ b_{5,1} & b_{5,2} & b_{5,4} & 0 \end{bmatrix}}{\det \tilde{B}} = \frac{\det \begin{bmatrix} b_{2,1} & b_{2,2} & b_{2,4} \\ b_{4,1} & b_{4,2} & b_{4,4} \\ b_{5,1} & b_{5,2} & b_{5,4} \end{bmatrix}}{\det \tilde{B}}$$

$$= \frac{\det \begin{bmatrix} b_{2,1} & b_{4,1} & b_{5,1} \\ b_{2,2} & b_{4,2} & b_{5,2} \\ b_{2,4} & b_{4,4} & b_{5,4} \end{bmatrix}}{\det \tilde{B}}$$

so

$$\det\begin{bmatrix} b_{2,1} & b_{2,2} & b_{2,4} \\ b_{4,1} & b_{4,2} & b_{4,4} \\ b_{5,1} & b_{5,2} & b_{5,4} \end{bmatrix} = (\tilde{B}^{-1})_{3,3} \cdot \det\tilde{B}$$

We know $\det\tilde{B} \neq 0$, so we need to show $(\tilde{B}^{-1})_{3,3} \neq 0$.
$(\tilde{B}^{-1})_{3,3} =$ (third row of $\tilde{R}^{-1}$) $\cdot$ (third column of $(I-Q)$)

$$= \begin{pmatrix} 0 & 0 & r_3^{-1} & 0 & \frac{-a_3}{r_3 a_5} \end{pmatrix} \cdot \begin{pmatrix} -c_1 \\ -c_2 \\ 1 \\ -c_4 \\ -c_5 \end{pmatrix} = r_3^{-1} + \frac{a_3 c_5}{r_3 a_5} = r_3^{-1} > 1$$

since $a_3 = 0$. So $(\tilde{B}^{-1})_{3,3} \cdot \det\tilde{B} > 0$ which implies:

$$\det\begin{bmatrix} b_{2,1} & b_{4,1} & b_{5,1} \\ b_{2,2} & b_{4,2} & b_{5,2} \\ b_{2,4} & b_{4,4} & b_{5,4} \end{bmatrix} = \det\begin{bmatrix} 1 & b_{2,3} & b_{4,3} & b_{5,3} \\ 0 & b_{2,1} & b_{4,1} & b_{5,1} \\ 0 & b_{2,2} & b_{4,2} & b_{5,2} \\ 0 & b_{2,4} & b_{4,4} & b_{5,4} \end{bmatrix} > 0$$

Thus the third sub-matrix (and also sub-matrices 1, 2, and 4 by symmetry) of A has positive determinant and therefore rank 4. This, along with the fact that the fifth sub-matrix of A also has rank 4 ensures that each of the unknown probabilities can be found uniquely by solving separately each of the five systems of 4 equations in 4 unknowns. This completes the proof.

7 An Idea Concerning the Recoverability of P from B for any Gerneral Network

Given any network N, create a new network $\acute{N} = N \cup \{$"sink node" z with an incoming edge from each boundary node in $N\}$

Let each edge in $\acute{N}$ have a flow capacity of 1 (each edge is capable of handling a maximum of 1 unit of flow), where edges that can be traversed in either direction have a flow capacity of 1 in either direction. A *flow* can be thought of as a particle moving from some source node to some sink node. For each interior node a of N, we will maximize the $a - z$ flow in $\acute{N}$, where a is the source node and z is the sink node, such that the flow in each edge does not exceed that edge's capacity (1). Because of the requirement that each edge has unit capacity, the paths that each "particle" takes in an $a - z$ flow will be edge-disjoint.

We can find the value of the maximum $a - z$flow for each interior node a by using the Augmenting Flow Algorithm as seen in Tucker, $AppliedCombinatorics$.

By the Max Flow-Min Cut Theorem, the value of a maximal $a - z$ flow is equal to the capacity of a minimal $a - z$ cut (minimal number of edges whose removal disconnects a and z).

Conjecture - Given a network N with interior nodes a_i, if a_i has k_i transitional probabilities (is adjacent to k_i other nodes), then those k_i probabilities of leaving node a_i can be recovered from B if and only if

$$\text{maximum } a_i - z \text{ flow in } \acute{N} = k_i$$

Corollary - The entire matrix P for a network N is recoverable from B if and only if

$$\text{maximum } a_i - z \text{ flow in } \acute{N} = k_i$$

for all interior nodes a_i.