Operations On Networks Of Discrete And Generalized Conductors

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1 Introduction

The most basic unit of transaction will be the discrete *conductor*. A *conductor* is an element with two *nodes*, where electrical potentials may be applied, and a conductance γ that determines the current that flows across the nodes due to the potential difference. γ is a number in the set $\mathbf{C}^+ = \{z \in \mathbf{C} | Re(z) > 0\}$. *Ohm's Law* describes the conductance of a discrete element as the ratio

$$\gamma = I/V, \tag{1}$$

where I is the current flowing from a potential difference V across the conductor.

A *network* $\Omega = (\Omega_0, \Omega_1)$ is a set of points (the *nodes*) $\Omega_0 = \{p_i\}_1^k$, and a set of *edges* $\Omega_0 = \{\sigma_{ij}\}, \sigma_{ij} = p_i p_j$ connecting points of Ω_0. A *network of conductors* $\Gamma = (\Omega, \gamma)$ is a network together with a function $\gamma : \Omega_1 \to \mathbf{C}^+$ (the *conductivity*). The network operates as a set of conductors joined together. We allow potentials to be imposed on the subset $\partial\Omega_0 \subset \Omega_0$, called the *boundary* of Ω. The *interior* of Ω, $\Omega_0 - \partial\Omega_0$, is denoted $\mathbf{int}\ \Omega_0$. Let $\mathcal{N}(p_i) = \{p_j | \sigma_{ij} \in \Omega_1\}$ be the set of *neighbors* connected to p_i by edges in Ω_1. Potentials ϕ applied to the boundary of the network give rise to a potential u on the interior such that

$$\sum_{p_j \in \mathcal{N}(p_i)} \gamma(p_i p_j)(u(p_i) - u(p_j)) = 0 \qquad \forall p_i \in \mathbf{int}\ \Omega_0; \tag{2}$$

this is *Kirchoff's Law.* This causes currents I_ϕ to flow through each of the boundary nodes. The Dirichlet-Neumann map $\Lambda : \partial\Omega_0 \rightarrow \partial\Omega_0$ precisely describes this relationship by the equation $\Lambda\phi = I_\phi$. Λ is linear in ϕ and can be represented as a k by k matrix, where $k = |\partial\Omega_0|$. Λ is symmetric, its nondiagonal entries have nonpositive real part, and the sum of any row (or column) is zero.

We generalize further, to consider whole networks as the basic elements of larger configurations. A *finite k-block* Γ_k with *n faces* is a network of resistors where the boundary nodes are naturally partitioned into faces $\mathbf{F} = \cup_1^n F_j$, where each face consists of the nodes $F_j = \{p_i\}_{(j-1)k+1}^{jk}$. In this context, Λ is an n by n matrix of k by k matrices Λ_{ij}. Λ takes k-vector potentials ϕ_i on each of its faces to the corresponding k-vectors of currents. By requirements on networks, we see that $\Lambda_{ij} = \Lambda_{ji}^t$, and the sum of any row (or column) of matrices is also zero.

In the spirit of networks, we define a *configuration of blocks* to be a set of finite blocks where certain faces are joined together, and we are allowed to apply potentials at some subset of the faces. Λ is defined to be the potentials-to-currents map for the whole configuration, as was done with networks. Noticing that 1-blocks with two faces act exactly like conductors, we see that any theorem proved for configurations of blocks also must hold for networks of discrete conductors. For simplicity, we shall like to think of the faces as one-dimensional, though there is no reason why we could not have used some higher dimension.

Our main aim will be to find algebraic expressions that represent basic operations with configurations, and then to apply these expressions to find interesting results. For clarity we show proofs first in the case of networks, and then generalize to the case of finite blocks. In this paper the term *non-singular* applies to any matrix A (including proper rectangular matrices) such that $Au = 0 \Rightarrow u = 0$. Diagrams are included to clarify the operations, and we hold to the following representations: Empty circles represent boundary nodes, while filled circles represent interior nodes.

2 Operations With Networks: Networks Of Individual Conductors

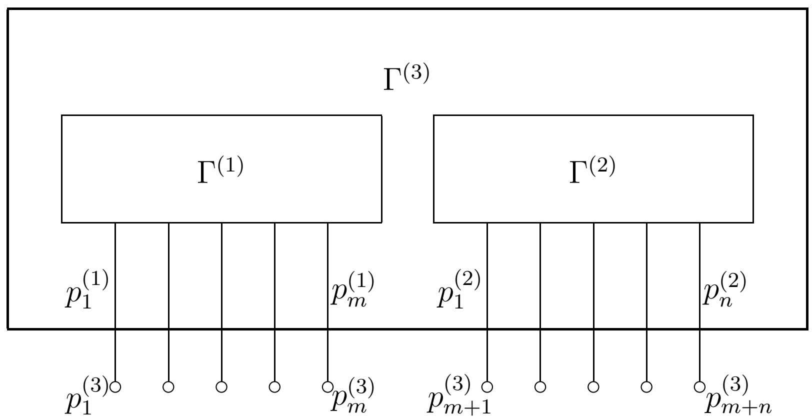


FIGURE 1

Lemma 1 *Let $\Gamma^{(1)}, \Gamma^{(2)}$ be a pair of networks with associated maps $\Lambda^{(1)}, \Lambda^{(2)}$ and boundary nodes $\{p_i^{(1)}\}_1^m, \{p_i^{(2)}\}_1^n$. Take $\Gamma^{(3)}$ to be the network consisting of both $\Gamma^{(1)}$ and $\Gamma^{(2)}$, as in Figure 1, with boundary nodes $\{p_i^{(3)}\}_1^{m+n}$ corresponding to*

$$p_i^{(3)} \longleftrightarrow \begin{cases} p_i^{(1)}, & i \leq m \\ p_{i-m}^{(2)} & i > m \end{cases}$$

Then

$$\Lambda^{(3)} = \begin{pmatrix} \Lambda^{(1)} & 0 \\ 0 & \Lambda^{(2)} \end{pmatrix} \tag{3}$$

Proof. Let e_i $(i = 1 \ldots m+n)$, $e_i^{'}$ $(i = 1 \ldots m)$ and $e_i^{''}$ $(i = 1 \ldots n)$ be the euclidean bases for $\mathbf{C}^{m+n}$, $\mathbf{C}^m$ and $\mathbf{C}^n$.

$$e_i^t \Lambda^{(3)} e_j = I_{e_j}^{(3)}(p_i) = \left\{ \begin{array}{ll} I_{e_j^{'}}^{(1)}(p_i^{(1)}), & i,j \leq m \\ I_{e_{j-m}^{''}}^{(2)}(p_{i-m}^{(2)}), & i,j > m \\ 0 & \text{otherwise} \end{array} \right\}$$

$$
= \left\{ \begin{array}{ll} (e_i^{'})^t \Lambda^{(1)} e_j^{'}, & i,j \leq m \\ (e_{i-m}^{''})^t \Lambda^{(2)} e_{j-m}^{''} & i,j > m \\ 0 & \text{otherwise} \end{array} \right\}.
$$

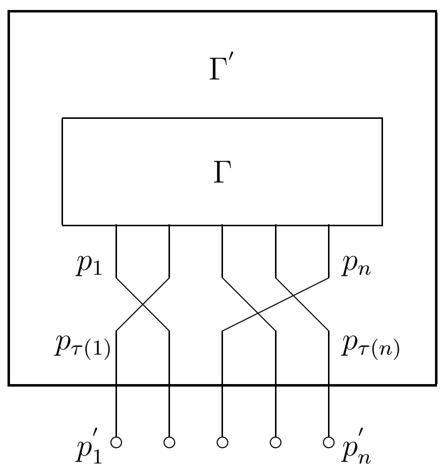


FIGURE 2: $\tau = (12)(345)$

Lemma 2 *Let Γ be a network with map Λ and boundary nodes $\{p_i\}_1^n$. Let $\tau \in S_n$ be a permutation. Let $\Gamma^{'}$ be the network with boundary nodes $\{p_i^{'}\}_1^n \longleftrightarrow \{p_{\tau(i)}\}_1^n$, as in Figure 2. Then*

$$
\Lambda^{'} = A^t \Lambda A \tag{4}
$$

where

$$
A = (a_{ij}) = \left\{ \begin{array}{ll} 1, & i = \tau(j) \\ 0 & \textit{otherwise} \end{array} \right.
$$

Proof.

$$
e_i^t \Lambda^{'} e_j = e_{\tau(i)}^t \Lambda e_{\tau(j)} = e_i^t A^t \Lambda A e_j.
$$

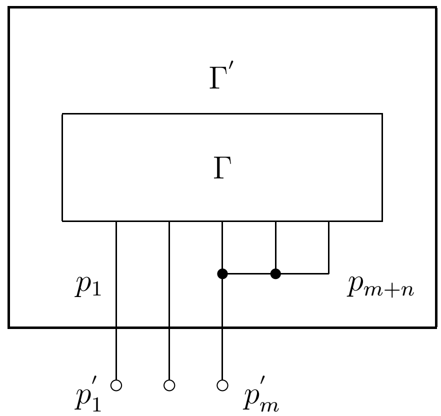


FIGURE 3

Lemma 3 *Let Γ be a network with map Λ and boundary nodes $\{p_i\}_1^{n+k}$. Let Γ' be the network that results from shorting the last k nodes together and treating them as a single node, as in Figure 3. Let $\{p_i'\}_1^{n+1}$ be the set of boundary nodes of Γ', and let Λ' be the map for Γ'. Then*

$$\Lambda' = B^t \Lambda B \tag{5}$$

where B is the $n+k$ by $n+1$ matrix

$$B = (b_{ij}) = \begin{cases} \delta_{ij}, & i \leq n+1 \\ 1, & i > n+1 \text{ and } j = n+1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let e_i $(i = 1 \ldots n+k)$ and e_i' $(i = 1 \ldots n)$ be the euclidean bases for $\mathbf{C}^{n+k}$ and $\mathbf{C}^n$. Applying the potentials e_j' to Γ' impresses the potentials

$$\phi_j = \begin{cases} e_j, & j \leq n \\ \sum_{j>n} e_j, & j = n+1 \end{cases}$$

on the network Γ. The currents $\Lambda\phi_j$ from Γ give the currents

$$I'_{e_j'}(p_i') = \begin{cases} e_i^t \Lambda \phi_j, & i \leq n \\ \sum_{j>n} e_i^t \Lambda \phi_j, & i = n+1 \end{cases}$$

on Γ'. $\Lambda' e_j' = I\Lambda' e_i' = B^t \Lambda \phi_i$, and $\Lambda' = \Lambda' I = B^t \Lambda B$.

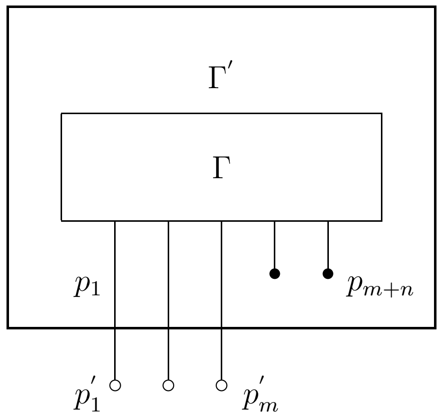


FIGURE 4

Lemma 4 *Let Γ be a connected network with map Λ and boundary nodes $\{p_i\}_1^{n+k}$. Let Γ' be the network that results from restricting $\partial\Omega_0$ to the first n nodes, so the the last k are effectively interior, as in Figure 4. Write*

$$\Lambda = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix},$$

where A is n by n and C is k by k. Then

$$\Lambda' = A - BC^{-1}B^t. \tag{6}$$

Proof. Let e_i $(i = 1 \ldots n + k)$ and e_i' $(i = 1 \ldots n)$ be the euclidean bases for $\mathbf{C}^{n+k}$ and $\mathbf{C}^n$. For each potential e_i' on Γ', the nodes $\{p_j\}_{n+1}^{n+k}$ achieve the potentials $u_i^t = \begin{pmatrix} 0 & \cdots & 0 & u_i(p_{n+1}) & \cdots & u_i(p_{n+k}) \end{pmatrix}$ such that the current at those nodes vanishes. u_i is the solution of the equation

$$\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} (e_i + u_i) = 0, \qquad i = n+1 \ldots n+k$$

To solve for all the u_i's at once, let V be the k by n matrix

$$V = (v_{ij}) = u_j(p_{i+k})$$

By the previous equation, V is the solution of

$$B^t + CV = 0$$

As stated in the introduction, the entries on the diagonal of Λ are $\Lambda_{ii} = -\sum_{j\neq i}\Lambda_{ij}$, so for $n > 0$, the diagonal entries of C satisfy the inequality $Re(c_{ii}) > -Re(\sum_{j\neq i} c_{ij})$ when the network is connected. Therefore C is diagonally dominant, C^{-1} exists, and $V = -C^{-1}B^t$.

For each $e_i^{'}$, the currents at $\Gamma^{'}$ are

$$\Lambda^{'} e_i^{'} = \begin{pmatrix} I & 0 \end{pmatrix} \Lambda(e_i + u_i), \qquad i = 1 \ldots n.$$

Solving for the currents by substituting for u_i we get

$$\Lambda^{'} I = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \begin{pmatrix} I \\ -C^{-1}B^t \end{pmatrix} = A - BC^{-1}B^t.$$

We will refer to this notion as *interiorization* in the remainder of this article. This proof may be extended to networks of other disjoint networks, as follows: If the nodes interiorized are part of a network connected to some other network that will retain at least two boundary nodes, this proof still works. Interiorizing an entire disjoint network simply removes those rows columns from the matrix (see Lemma 1). If the interiorized network is part of a larger network that will end up with a single boundary node, the rows and columns of the interiorized network are removed, and the row and column related to the remaining node are all zeroed out, since no current will flow into that node.

3 Generalization: Operations On Configurations Of Blocks

Lemma 5 *Let $\Gamma_k^{(1)}, \Gamma_k^{(2)}$ be a pair of configurations of k-blocks with associated maps $\Lambda^{(1)}, \Lambda^{(2)}$ and faces $\{F_i^{(1)}\}_1^m, \{F_i^{(2)}\}_1^n$. Take $\Gamma_k^{(3)}$ to be the configuration consisting of both $\Gamma_k^{(1)}$ and $\Gamma_k^{(2)}$, with faces $\{F_i^{(3)}\}_1^{m+n}$ corresponding to*

$$F_i^{(3)} \longleftrightarrow \begin{cases} F_i^{(1)}, & i \leq m \\ F_{i-m}^{(2)} & i > m \end{cases}$$

Then

$$\Lambda^{(3)} \quad = \quad \begin{pmatrix} \Lambda^{(1)} & 0 \\ 0 & \Lambda^{(2)} \end{pmatrix} \tag{7}$$

Proof. Follows immediately from Lemma 1.

Lemma 6 *Let Γ_k be a configuration with map Λ and faces $\{F_i\}_1^n$. Let $\tau \in S_n$ be a permutation. Let $\Gamma_k^{'}$ be the configuration with faces $\{F_i^{'}\}_1^n \longleftrightarrow \{F_{\tau(i)}\}_1^n$, as in Fig. 2.6. Then*

$$\Lambda^{'} = A^t \Lambda A \tag{8}$$

where A is the n by n matrix of k by k matrices A_{ij}

$$A = (A_{ij}) = \begin{cases} I, & i = \tau(j) \\ 0 & \mathit{otherwise} \end{cases}$$

Proof. Follows from Lemma 2, permuting the nodes n at a time.

Definition 1 *Let $\{p_i^j\}$ be the set of boundary nodes of some Γ_k, so that p_i^j corresponds to the ith node on the jth face. A* twist *of a face F_j is a renumbering of the nodes on that face so that $\{p_i^j\}$ becomes $\{p_{k+1-i}^j\}$.*

Lemma 7 *Let Γ_k be a configuration with map Λ and faces $\{F_i\}_1^m$. Let $\Gamma_k^{'}$ be the network that we arrive at by twisting the nth face of Γ_k. Then*

$$\Lambda^{'} = B^t \Lambda B \tag{9}$$

where B is the m by m matrix of k by k matrices B_{ij}

$$B = (B_{ij}) = \begin{cases} I, & i = j \neq n \\ \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 1 & 0 \\ 0 & 1 & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} & i = j = n \\ 0 & \mathit{otherwise} \end{cases}$$

Proof. Obvious from Lemma 2.

Similar transformations exist in higher dimensions to handle the permutations corresponding to face rotation, and the transforming matrix B could be calculated from Lemma 2 also.

Lemma 8 *Let Γ_k be a configuration with map Λ and faces $\{F_i\}_1^{m+n}$, where an individual node p_a^b is the ath node on the bth side. Let $\Gamma_k^{'}$ be the configuration that results from shorting the last n faces together and treating them as a single face. Let $\{F_i^{'}\}_1^{m+1}$ be the set of faces of $\Gamma_k^{'}$, and let $\Lambda^{'}$ be the map for $\Gamma_k^{'}$. Then*

$$\Lambda^{'} = C^t \Lambda C \tag{10}$$

where C is the $m+n$ by $m+1$ matrix of k by k matrices

$$C = (C_{ij}) = \begin{cases} I, & i = j, \textit{ or } i > m+1 \textit{ and } j = m+1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. A modification of Lemma 3 will work here. Let e_i $(i = 1 \ldots k(m+n))$ and $e_i^{'}$ $(i = 1 \ldots k(m+1))$ be the euclidean bases for $\mathbf{C}^{k(m+n)}$ and $\mathbf{C}^{k(m+1)}$. Applying the potentials $e_j^{'}$ to $\Gamma^{'}$ impresses the potentials

$$\phi_j = \begin{cases} e_j, & j \leq km \\ \sum_{a=0}^{n-1} e_{j+ak}, & j > km \end{cases}$$

on the network Γ. The currents $\Lambda\phi_j$ from Γ give the currents

$$I_{e_j^{'}}^{'}(p_a^b) = \begin{cases} e_{a+bk}^t \Lambda \phi_j, & a+bk \leq km \\ \sum_{h=0}^{n-1} e_{a+(b+h)k}^t \Lambda \phi_j, & a+bk > km \end{cases}$$

on $\Gamma^{'}$. $\Lambda^{'} e_j^{'} = I \Lambda^{'} e_i^{'} = C^t \Lambda \phi_i$, and $\Lambda^{'} = \Lambda^{'} I = C^t \Lambda C$.

Lemma 9 *Let Γ_k be a configuration with map Λ and faces $\{F_i\}_1^{m+n}$. Let $\Gamma_k^{'}$ be the network that results from restricting $\mathbf{F}$ to the first m faces, so the the last n are effectively interior. Write*

$$\Lambda = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix},$$

where A is the km by km matrix

$$A = \begin{pmatrix} \Lambda_{1,1} & \cdots & \Lambda_{1,m} \\ \vdots & \ddots & \vdots \\ \Lambda_{m,1} & \cdots & \Lambda_{m,m} \end{pmatrix},$$

B is the km by kn matrix

$$B = \begin{pmatrix} \Lambda_{1,m+1} & \cdots & \Lambda_{1,m+n} \\ \vdots & \ddots & \vdots \\ \Lambda_{m,m+1} & \cdots & \Lambda_{m,m+n} \end{pmatrix},$$

and C is the kn by kn matrix

$$C = \begin{pmatrix} \Lambda_{m+1,m+1} & \cdots & \Lambda_{m+1,m+n} \\ \vdots & \ddots & \vdots \\ \Lambda_{m+1,m+1} & \cdots & \Lambda_{m+n,m+n} \end{pmatrix}.$$

Then

$$\Lambda' = A - BC^{-1}B^t. \tag{11}$$

Proof. Follows from Lemma 4, interiorizing the last kn nodes.

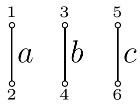
4 The Constructibility Theorem

Theorem 1 *For any finite configuration of finite blocks, there is a three-step algorithm for computing its Dirichlet-Neumann map Λ, assuming that the map Λ_i for each constituent block is known.*

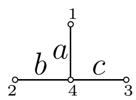
Proof. Take a configuration with n blocks. Apply Lemma 5 $n-1$ times, first to Λ_1 and Λ_2, then to the resultant Λ and Λ_3 , and so on. Permuting and twisting the faces as necessary with Lemmas 6 and 7, attach the required faces together with Lemma 8. Finally, interiorize the faces as required via Lemma 9.

By the statement in the introduction, this holds for networks of discrete conductors as well, so for each individual block we can apply this algorithm to get its Λ.

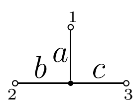
Example



$$\Lambda = \begin{pmatrix} a & -a & 0 & 0 & 0 & 0 \\ -a & a & 0 & 0 & 0 & 0 \\ 0 & 0 & b & -b & 0 & 0 \\ 0 & 0 & -b & b & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -c \\ 0 & 0 & 0 & 0 & -c & c \end{pmatrix}$$



$$\Lambda = \begin{pmatrix} a & 0 & 0 & -a \\ 0 & b & 0 & -b \\ 0 & 0 & c & -c \\ -a & -b & -b & a+b+c \end{pmatrix}$$



$$\Lambda = \begin{pmatrix} \frac{a(b+c)}{\delta} & \frac{-ab}{\delta} & \frac{-ac}{\delta} \\ \frac{-ab}{\delta} & \frac{b(a+c)}{\delta} & \frac{-bc}{\delta} \\ \frac{-ac}{\delta} & \frac{-bc}{\delta} & \frac{c(a+b)}{\delta} \end{pmatrix}$$

$$\delta = a + b + c$$

5 Multiple Operations On Networks

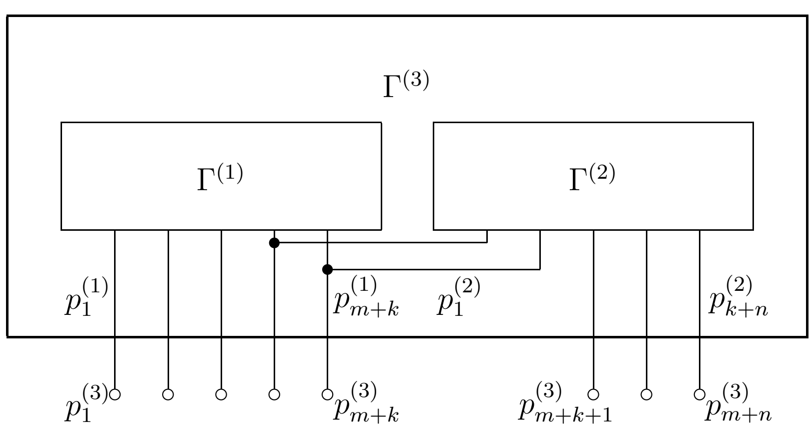


FIGURE 5

Definition 2 $\Gamma^{(3)} = \Gamma^{(1)} \circ_k \Gamma^{(2)}$ *will refer to the network with boundary nodes* $\{p_i\}_{i=1}^{m+k+n}$*, where the first* m *boundary nodes correspond to the first* m *boundary nodes of* $\Gamma^{(1)}$*, the last* n *boundary nodes correspond to the last* n *boundary nodes of* $\Gamma^{(2)}$*, and the middle* k *boundary nodes are connected to both* $\Gamma^{(1)}$ *and* $\Gamma^{(2)}$*, as in Figure 5.*

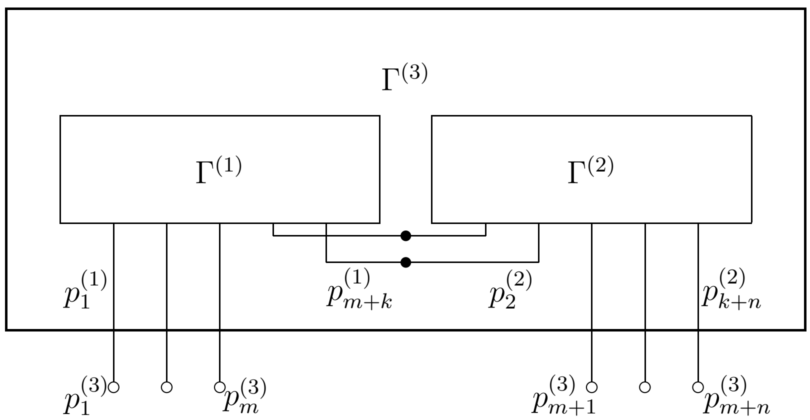


FIGURE 6

Definition 3 $\Gamma^{(3)} = \Gamma^{(1)} \bullet_k \Gamma^{(2)}$ *refers to the network obtained by first taking* $\Gamma^{(3)} = \Gamma^{(1)} \circ_k \Gamma^{(2)}$ *and then interiorizing the middle* k *nodes, as in Figure 6.*

For notational simplicity, $\circ_k$ and $\bullet_k$ may be written as $\circ$ and $\bullet$ when the number k is understood.

Lemma 10 *Let* $\Gamma^{(1)}, \Gamma^{(2)}$ *be a pair of networks with associated maps* $\Lambda^{(1)}, \Lambda^{(2)}$ *and boundary nodes* $\{p_i^{(1)}\}_1^{m+k}, \{p_i^{(2)}\}_1^{k+n}$*. Write* $\Lambda^{(1)} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$ *and* $\Lambda^{(2)} = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}$*. Let* $\Gamma^{(3)} = \Gamma^{(1)} \circ_k \Gamma^{(2)}$ *and* $\Gamma^{(4)} = \Gamma^{(1)} \bullet_k \Gamma^{(2)}$*, where* A *is* m *by* m*,* C *and* P *are* k *by* k*, and* R *are* n *by* n*. Then*

$$\Lambda^{(3)} = \Lambda^{(1)} \circ_k \Lambda^{(2)} = \begin{pmatrix} A & B & 0 \\ B^t & C+P & Q \\ 0 & Q^t & R \end{pmatrix} \qquad (12)$$

and

$$\Lambda^{(4)} = \Lambda^{(1)} \bullet_k \Lambda^{(2)} = \begin{pmatrix} A & 0 \\ 0 & R \end{pmatrix} - \begin{pmatrix} B \\ Q^t \end{pmatrix} \begin{pmatrix} C+P \end{pmatrix}^{-1} \begin{pmatrix} B^t & Q \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} A - B\Delta B^t & B\Delta Q \\ Q^t\Delta B^t & R - Q^t\Delta Q \end{pmatrix}, \tag{14}$$

where

$$\Delta = \begin{pmatrix} C+P \end{pmatrix}^{-1}. \tag{15}$$

Proof. (5) follows from applying Lemmas 2 and 3 for each set of nodes in sequence. (6) follows directly from (5) and Lemma 4.

The corresponding theorems look exactly the same for block configurations. In that case we set up the matrices A, B, C, P, Q and R in the same manner as Lemma 9.

6 Derivatives Of Λ With Respect To Element Conductivities In Subnetworks

Having a closed-form expression for the operations $\circ_k$ and $\bullet_k$ acting on the Λ-matrices enables us to calculate derivatives of Λ with respect to elements of sub-networks or sub-configurations. For clarity here we use the terminology of networks. Choose any set Σ of σ's in Ω_1 that contains all the elements with which we wish to take derivatives.. Applying the construction theorem, calculate $\Lambda_{\Omega_1-\Sigma}$ and Λ_Σ. Clearly either $\Lambda = \Lambda_{\Omega_1-\Sigma} \circ \Lambda_\Sigma$ or $\Lambda = \Lambda_{\Omega_1-\Sigma} \bullet \Lambda_\Sigma$, depending on whether the subnetwork in question will be connected entirely to boundary nodes or not. In either case, we merely differentiate the equations (12) or (14), paying attention to the fact that $\frac{\partial \Lambda_{\Omega_1-\Sigma}}{\partial \gamma_i} = 0$. In the first case we get the simple expression

$$\frac{\partial(\Lambda_{\Omega_1-\Sigma} \circ \Lambda_\Sigma)}{\partial \gamma_i} = \frac{\partial}{\partial \gamma_i} \begin{pmatrix} A & B & 0 \\ B^t & C+P & Q \\ 0 & Q^t & R \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial P}{\partial \gamma_i} & \frac{\partial Q}{\partial \gamma_i} \\ 0 & \frac{\partial Q^t}{\partial \gamma_i} & \frac{\partial R}{\partial \gamma_i} \end{pmatrix} \tag{16}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial \Lambda_\Sigma}{\partial \gamma_i} \end{pmatrix} \tag{17}$$

For $\bullet$, the expression is more complicated. Before we begin, we review an elementary lemma from algebra:

Lemma 11 *Let A be a matrix function of t. When A^{-1} exists,*

$$\frac{\partial A^{-1}}{\partial t} = -A^{-1}\frac{\partial A}{\partial t}A^{-1}.$$

Proof. Let $AA^{-1} = I$. Applying the product rule gives

$$\frac{\partial AA^{-1}}{\partial t} = A\frac{\partial A^{-1}}{\partial t} + \frac{\partial A}{\partial t}A^{-1} = 0$$

Putting the right half of the sum on the right side of the equation and multiplying both sides of the equation on the left by A^{-1} gives the desired result.

In the second case we have

$$\begin{aligned}\frac{\partial(\Lambda_{\Omega_1-\Sigma} \bullet \Lambda_\Sigma)}{\partial \gamma_i} &= \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial R}{\partial \gamma_i} \end{pmatrix} + \begin{pmatrix} B \\ Q^t \end{pmatrix} \Delta \frac{\partial P}{\partial \gamma_i} \Delta \begin{pmatrix} B^t & Q \end{pmatrix} - \\ &\begin{pmatrix} 0 \\ \frac{\partial Q^t}{\partial \gamma_i} \end{pmatrix} \Delta \begin{pmatrix} B^t & Q \end{pmatrix} - \begin{pmatrix} B \\ Q^t \end{pmatrix} \Delta \begin{pmatrix} 0 & \frac{\partial Q}{\partial \gamma_i} \end{pmatrix}, \end{aligned} \tag{18}$$

where again

$$\Delta = \begin{pmatrix} C + P \end{pmatrix}^{-1}. \tag{19}$$

The same approach works for the second derivatives:

$$\frac{\partial(\Lambda_{\Omega_1-\Sigma} \circ \Lambda_\Sigma)}{\partial \gamma_i \partial \gamma_j} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 \Lambda_\Sigma}{\partial \gamma_i \partial \gamma_j} \end{pmatrix} \tag{20}$$

and, after simplifying,

$$\begin{aligned}\frac{\partial^2(\Lambda_{\Omega_1-\Sigma} \bullet \Lambda_\Sigma)}{\partial \gamma_i \partial \gamma_j} &= \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2 R}{\partial \gamma_i \partial \gamma_j} - \frac{\partial Q^t}{\partial \gamma_i}\Delta\frac{\partial Q}{\partial \gamma_j} - \frac{\partial Q^t}{\partial \gamma_j}\Delta\frac{\partial Q}{\partial \gamma_i} \end{pmatrix} + \\ &\begin{pmatrix} 0 & B\Delta U \\ U^t \Delta B^t & Q^t \Delta U + U^t \Delta Q \end{pmatrix} - \\ &\begin{pmatrix} B \\ Q^t \end{pmatrix} \Delta \begin{pmatrix} \frac{\partial P}{\partial \gamma_i}\Delta\frac{\partial P}{\partial \gamma_j} + \frac{\partial P}{\partial \gamma_j}\Delta\frac{\partial P}{\partial \gamma_i} \end{pmatrix} \Delta \begin{pmatrix} B^t & Q \end{pmatrix} \end{aligned} \tag{21}$$

where

$$U \quad = \quad \frac{\partial P}{\partial \gamma_i}\Delta\frac{\partial Q}{\partial \gamma_j} + \frac{\partial P}{\partial \gamma_j}\Delta\frac{\partial Q}{\partial \gamma_i} - \frac{\partial^2 Q}{\partial \gamma_i \partial \gamma_j} \tag{22}$$

If Λ_Σ is from a network with no interior nodes, or $\Lambda_\Sigma = \gamma\Lambda_0$, then $\frac{\partial^2 \Lambda_\Sigma}{\partial\gamma_i\partial\gamma_j} = 0$, and the previous two equations reduce to

$$\frac{\partial(\Lambda_{\Omega_1-\Sigma} \circ \Lambda_\Sigma)}{\partial\gamma_i\partial\gamma_j} \quad = \quad 0 \tag{23}$$

and

$$\begin{aligned}\frac{\partial^2(\Lambda_{\Omega_1-\Sigma} \bullet \Lambda_\Sigma)}{\partial\gamma_i\partial\gamma_j} \quad = \quad & \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\partial Q^t}{\partial \gamma_i}\Delta\frac{\partial Q}{\partial \gamma_j} - \frac{\partial Q^t}{\partial \gamma_j}\Delta\frac{\partial Q}{\partial \gamma_i} \end{pmatrix} + \\ & \begin{pmatrix} 0 & B\Delta V \\ B^t\Delta V^t & Q^t\Delta V + V^t\Delta Q \end{pmatrix} - \\ & \begin{pmatrix} B \\ Q^t \end{pmatrix} \Delta \left(\frac{\partial P}{\partial \gamma_i}\Delta\frac{\partial P}{\partial \gamma_j} + \frac{\partial P}{\partial \gamma_j}\Delta\frac{\partial P}{\partial \gamma_i} \right) \Delta \begin{pmatrix} B^t & Q \end{pmatrix} \end{aligned} \tag{24}$$

where

$$V = \frac{\partial P}{\partial \gamma_i}\Delta\frac{\partial Q}{\partial \gamma_j} + \frac{\partial P}{\partial \gamma_j}\Delta\frac{\partial Q}{\partial \gamma_i} \tag{25}$$

With the first- and second-order derivatives of Λ at hand, there is a way to determine approximate values for the γ's in the network from Λ. Recall Taylor's theorem in several variables, as applied to Λ:

$$\Lambda(\gamma + \gamma_0) = \Lambda(\gamma_0) + \sum_i \gamma_i \frac{\partial \Lambda(\gamma_0)}{\partial \gamma_i} + o(|\gamma|^2) \tag{26}$$

the absolute error being less than some constant times the maximum value of the second derivatives $\frac{\partial^2\Lambda}{\partial\gamma_i\partial\gamma_j}$ evaluated on the set $S = \{\mu : |\mu| < |\gamma|\}$. This can be viewed as a system of equations linear in γ, with a bounded error term for each equation. One should be able to obtain bounds for the possible values of γ from this, for any kind of network. Another idea would be a Newton's Method approach, using equation (26) as a local linearization of the map taking γ's to Λ. Start with some γ_0 and compute the derivatives at that point. Use a least-squares method to solve for $\gamma = \gamma_1$. At this new γ_1, compute the derivatives and solve for γ again. If an iteration of these steps converges, we will have solved the inverse problem (at least in a computational sense).

A short table of $\frac{\partial \Lambda}{\partial \gamma_i}$ is included here. The computations were done by taking the network Γ_Σ to be a single conductor. In the third case, we have used the following lemma to simplify things:

Lemma 12 *Let A be a symmetric 2 by 2 matrix. Then*

$$A^{-1} = \frac{TAT^t}{\det A} = \frac{T^t AT}{\det A},$$

where $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, *and* $T^2 = I$.

Proof. Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. It is well-known that $A^{-1} = \frac{1}{\det A} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$. Matrix multiplication shows immediately that $TA^{-1}T^t = \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$. $T^t = -T$, so $TAT^t = T^t AT$.

Configuration	*Derivative*
$1 \circ\!-\!\!-\!\!-\!\circ 2$	$\frac{\partial\Lambda}{\partial\gamma} = \begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{pmatrix}$
$1 \bullet\!-\!\!-\!\!-\!\circ 2$	$\frac{\partial\Lambda}{\partial\gamma} = \frac{1}{(c+\gamma)^2} \begin{pmatrix} BB^t & cB \\ cB^t & c^2 \end{pmatrix}$
$1 \bullet\!-\!\!-\!\!-\!\bullet 2$	$\frac{\partial\Lambda}{\partial\gamma} = \frac{B\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} C^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B^t}{\det^2\left[C+\gamma\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right]}$

7 A General Approach To The Inverse Problem

In the inverse problem, we are attempting to find the γ's for the conductors of network from information in Λ. Ideas about how to approach this problem come from a modification of the construction theorem. For the moment consider networks with no interior nodes. One can virtually read off the γ's from Λ for this type of network, because if we apply a positive potential to a single boundary node, and zero potential at all others, the currents leaving the nodes at zero potential are precisely the currents through the conductor connected between that node and the node at positive potential. A simple application of Ohm's Law then gives γ. By the same token, constructing Λ for these networks is just as easy; for a network with n boundary nodes and conductances $\gamma_{ij} = \gamma_{ji}$ between nodes p_i and p_j, the Λ is

$$\Lambda = \begin{pmatrix} \sum_{i\neq 1}\gamma_{1i} & -\gamma_{12} & \cdots & -\gamma_{1n} \\ -\gamma_{21} & \sum_{i\neq 2}\gamma_{2i} & & \vdots \\ \vdots & & \ddots & \vdots \\ -\gamma_{n1} & \cdots & \cdots & \sum_{i\neq n}\gamma_{ni} \end{pmatrix} \tag{27}$$

A straightforward approach to constructing Λ for networks would be to first write Λ_i for constituent networks with no interior nodes, and then connect these larger pieces together with the operations $\circ_k$ and $\bullet_k$.

At this point, the method of solution is clear: Decompose the network into subnetworks, where by themselves, these subnetworks are of the special type just mentioned. Given a Λ, we would have something similar to

$$\Lambda = \Lambda_1 \circ_{k_1} \Lambda_2 \bullet_{k_2} \Lambda_3 \circ_{k_3} \cdots \bullet_{k_n} \Lambda_n, \tag{28}$$

where the individual Λ_i's are unknown. We would like to invert the operations $\circ_k$ and $\bullet_k$, so that if we can solve for the last Λ in equation (29), we can essentially "break it off". Piece by piece, we will have decomposed Λ into the Λ_i's for the subnetworks, where each individual γ is plain to see.

The jumping-off point for this approach is the pair of equations (12) and (14). The two questions we ask are:

Given Λ and $\Lambda^{(2)}$, and knowing that $\Lambda = \Lambda^{(1)} \circ_k \Lambda^{(2)}$, can we find $\Lambda^{(1)}$?

and

Given Λ and $\Lambda^{(2)}$, and knowing that $\Lambda = \Lambda^{(1)} \bullet_k \Lambda^{(2)}$, can we find $\Lambda^{(1)}$?

Looking at equation (12), the answer to the first question is obvious. If we know the dimensions of all the matrices and their subblocks (which we should, since we know the geometry of the situation and consequently the number of boundary nodes involved), as well as the matrices P, Q, and R, we can simply subtract off those matrices and read off the remaining A, B, and C . The situation is not quite as easy for $\bullet_k$. There are situations where the resultant $\Lambda = \Lambda^{(1)} \bullet_k \Lambda^{(2)}$ is not unique. If we make some assumptions about $\Lambda^{(2)}$, we can narrow this down to the cases where we do have uniqueness.

Theorem 2 *Let $\Lambda = \begin{pmatrix} X & Y \\ Y^t & Z \end{pmatrix} = \Lambda^{(1)} \bullet_k \Lambda^{(2)}$, and let $\Lambda^{(2)} = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}$. If the submatrix Q^t is nonsingular, $\Lambda^{(1)}$ may be solved for.*

Proof. First, we construct a special matrix. Let Q be the matrix of columns $q_i \ldots q_n$. Choose the numbers $\tau(1) \ldots \tau(m)$ such that the square matrix $\tilde{Q}$ with columns $q_{\tau(1)} \ldots q_{\tau(m)}$ is invertible. Let $\tilde{q}_j^{-1}$ be the jth row of $\tilde{Q}^{-1}$. Take Q^* to be the matrix of the same size as Q^t, where the rows of q_i^* of Q^* are $q_{\tau(i)}^* = \tilde{q}_i^{-1}$, and $q_i^* = 0$ for all other rows. Then $QQ^* = \tilde{Q}\tilde{Q}^{-1} = I$, where I has the same size as Δ.

Now, we proceed block-by-block. We know the matrices P, Q, R, X, Y and Z. $Z = R - Q^t \Delta Q$, so $\Delta = -(Q^*)^t(R - Z)Q^*$. $\Delta = (C + P)^{-1}$, and by construction Δ is invertible. Thus we have solved for C:

$$C = (-(Q^*)^t(R - Z)Q^*)^{-1} - P \tag{29}$$

The other two blocks are much simpler. We from $Y = -B\Delta Q$ that

$$B = -YQ^*\Delta^{-1}. \tag{30}$$

Finally, $X = A - B\Delta B^t$ gives

$$A = X + B\Delta B^t = X + YQ^*\Delta^{-1}(Q^*)^t Y^t. \tag{31}$$

If the submatrix Q^t is singular, there are many solutions to the equation. Starting from the equation $Q^t \Delta Q = R - Z$, we may calculate some possible values for Δ, and from those some possible matrices A, B and C that satisfy equation (15). Apply elementary row operations to Q^t on the left until Q^t is in row-echelon form. Letting K^t be the product of those operations, $K^t Q^t = \begin{pmatrix} I & M \end{pmatrix}$. We break Δ up into pieces Δ_{ij} so that

$$K^t Q^t \Delta Q K = \begin{pmatrix} I & M \end{pmatrix} \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \begin{pmatrix} I \\ M^t \end{pmatrix}. \tag{32}$$

Expanding this out, we solve for Δ_{11} as a variable dependent on the other Δ_{ij}'s:

$$\Delta_{11} = K^t(R - Z)K - \Delta_{12}M - M^t\Delta_{21} - M^t\Delta_{22}M. \tag{33}$$

Everything already said in this section could just as well be applied to finite blocks, though with blocks perhaps it is easier to follow this process by "subtracting" only one block at a time. In either case, what we are ultimately looking for are certain conditions on the boundary nodes or faces such that we may know all the potentials around some subnetwork or block. Application of Ohm's law will then give the solution. The vital questions left are:

For which network (or configuration) geometries do there exist special boundary functions that allow us to solve for each Λ_i, step by step?

and

For which network (or configuration) geometries is the solution of the inverse problem unique?

Research on these problems was found to be particularly difficult, and no results are included here, though the answers are known for certain specific geometries.

8 Conclusion

This paper brings to light many structures applicable to a whole variety of specific problems regarding conductive networks and configurations. A problem I found quite intriguing was the following: *Let* Λ *be a Dirichlet-Neumann map with rational entries. Find an algorithm that gives a network for* Ω *such that* $\Lambda = \Lambda_\Omega$, *where* Λ_Ω *is the map for the network* Ω *with conductivities* $\gamma = 1$ *on all the edges.* This problem is quite different than other problems previously studied in the REU program. Personally, I think more research ought to be done on this geometric level, especially since, as was remarked at the end of the previous section, this would give complete solutions to the inverse problem for any geometry of network or configuration.

Having closed-form expressions for the derivatives of Λ also opens up new avenues of research. Studies of the manifold $M_\Lambda = \{\Lambda(\gamma) | \gamma_i \in \mathbf{C}^+\}$ suggest that this manifold may have interesting properties, presumably due to the solvability of the inverse problem or perhaps other conditions that come up during construction of Λ. It may be that requirements on its curvature can tell us what sort of convergence we hope to get from the Newton's Method approach described in §6. For geometries with nonunique solutions, can the sets $\{\gamma = (\gamma_1 \ldots \gamma_n) | \Lambda(\gamma) = \Lambda_0\}$ be described readily? Symmetries of the manifold related to the geometry of the network might be interesting.

This summer, some research was done on "mixed problems", where potentials were allowed on some of the boundary nodes, and currents on the others. Instead of Λ, we would have a mixed map of both Dirichlet-to-Neumann and Neumann-to-Dirichlet data. It may be the case that a constructional approach similar to the one given here could be applied. The same classifications of solvable inverse geometries could be studied, and more practical applications might be sought.

There is a continuous case of blocks also. In this case, γ is a function whose range has positive real part, defined on some region such that potential

ϕ on the boundary induces a potential u inside the region satisfying the equation

$$\nabla \cdot (\gamma \nabla u) = 0. \tag{34}$$

The currents on the boundary are the normal derivatives $\frac{\partial u}{\partial \mathbf{n}}$. The map Λ again takes boundary potentials to boundary currents, but in the continuous case the representation of Λ is not simple. By linearity of Λ, if we partition the boundary of the region into n faces, we can think of Λ as an n by n matrix of Λ-operators Λ_{ij}, where a given Λ_{ij} gives the current function on the ith face due to a potential ϕ_j on the jth face. Taking ϕ to be an n-vector of functions ϕ_i, we again have $\Lambda\phi = I_\phi(\mathbf{F})$. I have put a moderate amount of thought into this approach, but there appear to be some fundamental departures from the finite situation. The work in this area is very difficult, but perhaps this paper will provide some motivation.