Recoverable sources in a (3x3) rectangular resistor network

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1. Introduction

As in Curtis and Morrow [1] , I consider a rectangular network Ω of resistors in the plane. The set of *nodes* of Ω are the lattice points in the network. The set of nodes is denoted Ω_0. An *edge* of Ω is a line segment $\sigma = pq$ which connects a pair of adjacent nodes p and q. The set of edges is denoted Ω_1. For each node p, the set of nodes which are adjacent to p will be called N(p). The *interior* int Ω_0 of Ω_0 consits of those nodes p which are connected to four edges in Ω_1. The set of *boundary* nodes $\delta\Omega_0$ consits of those nodes p that have only one node in N(p). The *interior* int Ω_0 is $\Omega_0 - \delta\Omega_0$.

A network of resistors $G = (\Omega_0,\Omega_1,\gamma)$ is a network (Ω_0,Ω_1) with a positive real function γ on Ω_1. The number γ is called the *conductance* of an edge.

For any real function u on Ω_0 we define a source function S_u on int Ω_0 ,

$$S_u(p) = \sum_{q \in N(p)} \gamma_{pq}(u(p)-u(q))$$

If a function ϕ is defined at the boundary nodes, and a function c is defined at the interior nodes, there will be a unique function u defined on the nodes of the network with $u(p) = \phi(p)$ for each boundary node p, and $S_u(p) = c(p)$ for each interior node p. The potential u determines the current $I_{u(p)}$ into each boundary node p by *Ohm's law*,

$$I(pq) = \gamma(pq)\,(u(p) - u(q))$$

where node q is the interior node adjacent to boundary node p.

The column *source vector* M contains an entry for every interior node j such that $M_j = S(j)$. An interior node j is said to obey *Kirchoffs law* if $M_j = 0$. If $M_j \neq 0$, then node j is a *source*. Entry M_j can be interpreted as the sum of the current flowing out of the interior node j. Note that M_j will be will be positive if current flows into the network from node j.

2. Forward problem

For a network of resistors $G = (\Omega_0, \Omega_1, \gamma)$ with known conductivity function γ and source vector M we want to find the *Dirichlet to Neumann map* that takes boundary potentials to boundary currents.

DEFINITION 2.1 *u*. Let u be a column of potentials where $u_{(j)}$ is the potential at node j. If we number the nodes so that boundary nodes appear first in u, we can write

$$u = [u_b \mid u_i]^T \quad ,$$

where u_b is a vector of boundary potentials, and u_i is a vector of interior potentials. Here T indicates the transpose of a matrix.

DEFINITION 2.2 Ψ. Let $\Psi = [\Psi_b \mid M]^T$,
where Ψ_b is the vector of boundary currents and M is the vector of sources at interior nodes.

The Dirichlet problem is to solve $Ku = \Psi$ where K and Ψ are known. The Kirchoff matrix $K = [k_{ij}]$, where $k_{ij} = -\gamma(ij)$ if node i and node j are adjacent, and

$$k_{ii} = \sum_{j \in N(i)} \gamma(ij)$$

We can write express the Dirichlet problem in block form as,

$$\left[\begin{array}{c|c} K' & B^T \\ \hline B & A \end{array}\right] \left[\begin{array}{c} u_b \\ \hline u_i \end{array}\right] = \left[\begin{array}{c} \Psi_b \\ \hline M \end{array}\right]$$

writing this as two equations we have,

$$K'u_b + B^T u_i = \Psi_b \tag{1}$$

$$Bu_b + Au_i = M \tag{2}$$

LEMMA 2.1 Let $G = (\Omega_0, \Omega_1, \gamma)$ be a network of resistors. *The interior potentials are uniquely determined by the boundary potential vector u_b, tand the source vector M.*

Proof. It can be shown that A is non-singular [1].

$$u_i = A^{-1}[M - Bu_b] \tag{3}$$

LEMMA 2.2 *The boundary currents are uniquely determined by the boundary potential vector u_b and the source vector M.*

Proof. This follows directly from Lemma 2.1 and equation *(2)*.

DEFINITION 2.4. Substituting *(3)* into*(2)* we get a nonlinear map T, where $Tu_b = \Psi_b$

$$[K' - B^T A^{-1} B]u_b + B^T A^{-1} M = \Psi_b$$

T is the Dirichlet to Neumann map for a network of resitors with sources.

DEFINITION 2.5. The Λ matrix is the linear Dirichlet to Neumann map for a network without sources, where $\Lambda u_b = \Psi_b$.

3. Inverse problem

Consider a network $G = (\Omega_0, \Omega_1, \gamma)$ that has a conductivity function γ and a source vector M that are unknown to us. We want to determine γ and M by measuring boundary currents due to applied boundary potentials.

DEFINITION 3.1 The column vector of the boundary currents we measure when the potential is zero at all boundary nodes will be called Φ.

LEMMA 3.1 *The sum of the components of Φ is equal to the negative sum of the components of the source vector M.*

Proof. Define current as positive if it is leaving a node. Because all interior nodes that are not sources obey Kirchoffs law, the sum of the current out of the sources plus the sum of the current out of the boundary nodes must be zero.

I will now describe how to get the Dirichlet to Neumann map T for a network $G = (\Omega_0, \Omega_1, \gamma)$ that contains sources. We start by placing a voltage of 1 at boundary node j, zero at all other boundary nodes and measuring the resulting current. We continue applying this pattern of potentials, allowing node j to be each boundary node. We order the boundary nodes so that the matrix formed by the column vectors of boundary potentials is the identity matrix. Let the corresponding column vectors of measured boundary currents be written as the matrix θ. Thus θ_{ij} is the current at node boundary node i due to a potential of 1 at node j and a potential of 0 at all other boundary nodes.

We can find the lambda matrix of the network by subtracting from each column of θ the vector Φ. For example to find Λ_{ij},

$$\Lambda_{ij} = \theta_{ij} - \Phi_i$$

Now we have the lambda matrix that is equivalent to the Dirichlet to Neumann map for the same network G with the same conductivity function γ but with source current vector 0. We can use the lambda matrix to recover the conductivities by the method described in Curtis and Morrow [1].

4. Finding the sources

For a resistor network $G = (\Omega_0, \Omega_1, \gamma)$ that does not have any interior sources, we know there is a unique Dirichlet to Neumann map for each conductivity funciton γ. The Dirichlet to Neumann map may not be unique if the network is allowed to have sources. Suppose that some subset of sources in a network are known. I define the entire set of sources recoverable if there is not a second set of sources with the same known subset and Dirichlet to Neumann map.

For the rest of this section I consider the (3x3) network shown in figure 1. I want to identify which sets of sources are recoverable for this network. For the vectors referred to in this section it is assumed that the indexing of the vector is consistent with the numbering scheme in figure 1. For example M_1 is the first entry in the source vector M so it corresponds to the magnitude of the source at the lowest numbered interior node, which is node 13.

Figure 1

Let L be the null space of $B^T A^{-1}$.

LEMMA 4.1 *The dimension of L is 1.*

Proof. B^T is a (12x9) matrix with exactly one nonzero entry in each row and rank of 8. Also, A^{-1} is a non-singular square matrix, so the rank of B^T and $B^T A^{-1}$ are the same. $B^T A^{-1}$ is a (12x9) matrix so Rank + Dim(L) = 9. This means Dim(L) = 1.

Refering to figure 1, let a be the conductance of the edge connecting nodes 14 and 21, let b be the conductance of the edge connecting nodes 16 and 21, let c be the conductance of the edge connecting nodes 18 and 21, and let d be the conductance of the edge connecting nodes 20 and 21. Also, f is equal to the sum of a,b,c,d.

LEMMA 4.2 *L is spanned by the set of vectors* ($w [0 , -a/f , 0 , -b/f , 0 , -c/f , 0, -d/f , 1]^T$), *where w is a real constant.*

Proof. Finding a vector in L is equivalent to finding a source vector M such that boundary currents are zero when boundary potentials are zero. $M = [0 , -a/f , 0 , -b/f , 0 , -c/f , 0, -d/f , 1]^T$ is one source vector that would satisfy this condition. Because Dim(L) = 1, all other vectors in L are constant multiples of this vector M.

LEMMA 4.3 *Let* $G = (\Omega_0,\Omega_1,\gamma)$ *be a network that contains sourcest and has a known Dirichlet to Nuemann map T. Then the set of all source vectors M that will provide the same map T can be described in terms of any one source vector R that will give T and any vector x belonging to L.* $\{M: M = R + x\}$

Proof. Recall the Dirichlet to Nuemann map T written in terms of blocks in the Kirchoff matrix and the source vector M. $[K' - B^TA^{-1}B]u_b + B^TA^{-1}M = \Psi_b$ The lambda matrix is fixed by the conductivity function. Thus, if we want to change the source vector M and keep the map T constant we must preserve the product involving M. Say there is a source vector Q that provides the same map T. Then it will always be possible to write Q as the sum of M and another vector P. For T constant, $B^TA^{-1}M = B^TA^{-1}(M + P) = B^TA^{-1}M + B^TA^{-1}P$. This can only be true if P is in L.

THEOREM 4.2 *For the three by three network in figure 1, any set of sources can be recovered if the value of the source at one of the nodes {14,16,18,20,21} is known.*

Proof. Assume the value of the source at node i in {14,16,18,20,21} is known. Define E as the source vector that places the known value at node i and an arbitrary set of other values at the other interior nodes. E along with a fixed conductivity function γ will give us the Dirichlet to Neumann map T. The set of all other possible source vectors that would give the same map T is $\{H: H = E + x, x \in L\}$. But every x has a nonzero entry at i. Thus the Dirichlet to Neumann map for a source vector E that contains a known value of one of {14,16,18,20,21} is unique.