Interpretation of Electrical Networks as Probability Networks (and vice versa)

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1. Introduction.

We are interested in looking at transformations between electrical resistor networks and probabilistic networks (Markovian systems).

An electrical network consists of an graph Ω with non-negative edge conductances $\Gamma = [\gamma_{ij}]$, and a set $\partial\Omega$ of nodes which are to be considered boundary nodes. We consider Ω to be a complete graph, that is, for any nodes i and j, $\widehat{ij} \in \Omega$; and when we speak of circuits and paths in Ω we really mean circuits and paths through edges of non-zero weight; and in the discussion Ω will play little rôle, since all of the connectivity information is contained in Γ. For the purposes of this paper it will be assumed that the graph is connected (in the weighted sense), although it is easy to generalize the statements made here to appropriate statements for non-connected networks.

A probability network consists of a digraph $\tilde{\Omega}$ with transition probabilities $P = [p_{ij}]$, and a set $\partial\tilde{\Omega}$ of nodes which are to be considered boundary nodes. As with the electrical network, we consider $\tilde{\Omega}$ to be complete; and again we assume that the digraph is connected.

In the case of an electrical network, we can assign voltage potentials to produce current in each edge, using Kirkhoff's and Ohm's laws; in the case of probabilistic networks, we have particles, moving about the network according to the transition probabilities. The question, then, is this: is it possible, given an electrical network Γ, to produce a probabilistic network P, with interpretations of potential and current on P that match those on Γ? Or, if the probabilistic currents and potentials are not equal to their electrical counterparts, at least related by a 1–1 correspondence?

2. Definition of p_{ji}, u_i, and i_{xy}.

The obvious approach is to take p_{ji}, the probability that a particle at node i will travel to node j, equal to $\gamma_{ij}/\sum_{k\neq i}\gamma_{ik}$, at both boundary and interior nodes; this gives $\sum_j p_{ji} = 1$, and would seem, at least on first inspection, to be reasonable: a particle at node i is more likely to travel down a path of lesser resistance than down one of greater resistance. Now we can construct the transition matrix $P = [p_{ij}]$, which is analogous to the conductivity matrix Γ of the electrical network. Lacking, so far, are interpretations of voltage potential, current, and Kirkhoff's and Ohm's laws that relate the two.

In an electrical network, a potential function $\vec{v}$ defined on Ω is said to be γ-*harmonic* on the interior of Ω, i.e. it is true that for any interior node i, it obeys Kirkhoff's law:

$$\sum_{j\neq i}(v_j - v_i)\gamma_{ij} = 0 \tag{1}$$

We take as our probabilistic voltage $u_i = v_i\sigma_i$, where $\sigma_i = \sum_{j\neq i}\gamma_{ij}$; this gives us a probabilistic Kirkhoff's law:

$$\sum_{j\neq i}\left(\frac{u_i}{\sigma_i} - \frac{u_j}{\sigma_j}\right)\gamma_{ij} = 0$$

i.e.,

$$\sum_{j\neq i}(u_i p_{ji} - u_j p_{ji}) = 0$$

or

$$\sum_{j\neq i}(u_j p_{ji}) = u_i \tag{2}$$

(Note that if $P\vec{u} = \vec{u}$, then (2) is satisfied.)

Next we need an interpretation of current. As we used Kirkhoff's law above to produce a probabilistic voltage potential, likewise now we use Ohm's law to produce a probabilistic current flow.

Ohm's law states that $i_{ij} = (v_i - v_j)\gamma_{ij}$, that the current flowing between two adjacent nodes is proportional to the voltage drop between those nodes, and that the proportion is exactly the conductance of the conductor joining them. Using our definitions of u_i and p_{ji}, we have the following:

$$i_{ij} = (v_i - v_j)\gamma_{ij} \tag{3}$$

$$i_{ij} = (\frac{u_i}{\sigma_i} - \frac{u_j}{\sigma_j})\gamma_{ij}$$

$$i_{ij} = (\frac{u_i\gamma_{ij}}{\sigma_i} - \frac{u_j\gamma_{ij}}{\sigma_j})$$

$$i_{ij} = (u_i p_{ji} - u_j p_{ij}) \tag{4}$$

which gives us a probabilistic Ohm's law.

4. "Physical" interpretation of potential and current

In addition to simply being appropriate numbers for probabilistic networks, these u's and i's do have some physical meaning. The potential u_i represents the number of particles at point i at any given time, and the current i_{ij} represents the (net) number of particles moving from state i to state j at any given time. Note that these quantities depend on time, but in the long run they average out to a steady state where u_i no longer depends on time, and $i_{ij} = 0$ for all i, j in the network.

5.1. Recovering Γ from P: Algorithm 1

This algorithm is due to Snell[1984].

Now suppose we were given the matrix P, *knowing that P came from an electrical network*, and were asked to find the conductances Γ.

If we had a vector, $\vec{w}$, such that $w_i = \sigma_i/\sigma$, where $\sigma = \sum_i \sigma_i$, this vector would have two interesting properties. First, $\sum_j w_j p_{ji} = w_i$, since

$$\sum_j w_j p_{ji} = \sum_j \frac{\sigma_j}{\sigma}\frac{\gamma_{ij}}{\sigma_j} = \sum_j \frac{\gamma_{ij}}{\sigma} = \frac{\sum_j \gamma_{ij}}{\sigma} = \frac{\sigma_i}{\sigma} = w_i \tag{5}$$

Second, since we have defined $w_i = \sigma_i/\sigma$, we have

$$\sum_i w_i = \sum_i \frac{\sigma_i}{\sigma} = \frac{\sum_i \sigma_i}{\sigma} = \frac{\sigma}{\sigma} = 1 \tag{6}$$

It turns out that (5) is equivalent to the statement $\vec{w}P = \vec{w}$, or $\vec{w}(I - P) = 0$; since the sum of the elements in each row of $(I - P)$ is equal to $\sum_i p_{ji} - 1 = 0$, $|I - P| = 0$, and so some $\vec{w}$ exists such that $\vec{w}(I - P) = 0$, or $\vec{w}P = \vec{w}$. It is proved in Chung[1974] that any $\vec{x}$ such that $\vec{x}P = \vec{x}$, where P has column sums equal to one, is a constant multiple of any other solution; therefore, there exists a unique $\vec{w}$ satisfying (5) and (6).

Once we have this $\vec{w}$, then, it is easy to recover Γ, at least up to a constant scale factor:

$$w_i p_{ji} = \frac{\sigma_i}{\sigma}\frac{\gamma_{ij}}{\sigma_i} = \frac{\gamma_{ij}}{\sigma}$$

and depending on our choice of σ, we have determined Γ. It is then a simple matter to recover $\vec{v}$, the electrical potential function on Ω corresponding to $\vec{u}$, the probabilistic potential function on $\tilde{\Omega}$: we simply use the definition of $\vec{u}$: $v_i = u_i/\sigma_i$. Likewise we see that the current, as we have defined it, is equal, by reading backwards from (4) to (3).

Note that none of the above requires the assumption that P come from an electrical network; the fixed vector $\vec{w}$ exists for any probability network, and from $\vec{w}$ we can come up with a set of conductances that satisfies our definition of P. The additional restriction is that the conductances must be symmetric, that is, $\gamma_{ij} = \gamma_{ji}$; and there is no guarantee that this is the case for the conductances that we find using this method. We can check whether the matrix $PI_{\vec{w}}$ is symmetric; if it is, then the entries $w_i p_{ji}$ are exactly the conductances of an electrical network that will generate P. If the matrix is not symmetric, nevertheless the definition of p_{ij} will give P back from the asymmetric Γ.

This notion of asymmetric conductances leads to the question: is the mapping one-to-one? Obviously this algorithm gives only one solution, but are there other Γ's that would also generate the same P?

This method is not well-equipped to answer this question.

5.2. Algorithm 2: Network Traversal

Suppose we were given a connected probabilistic network that is known to have come from an electrical network, and we are given the resistance of one resistor. From the construction of the p_{ji}'s we can see that for any three nodes i, j, and k in the network, $p_{ji}/p_{ki} = \gamma_{ij}/\gamma_{ik}$; so, given the conductance of any edge adjoining a node i, we can find the conductance of any other conductor adjoining the same node simply by applying the above ratio. Given one conductance, then, it will be possible to traverse the network (since it is connected) finding each conductance in turn until all are known. Note that if we are not given a conductance to start out with, we can pick any conductor ij where $p_{ji} \neq 0$ and assign it conductance 1; since in general we can only find Γ up to a scalar.

Now suppose that we are given a connected probabilistic network P, *without the knowledge that P came from an electrical network*. The same questions arise as in the end of section 5.1: how can we tell whether P is associated with a set of symmetric conductances? If not, is the set of asymmetric conductances unique?

The only way that we could fail to derive symmetric conductances from P is by running into inconsistencies: if, while traversing the network, we arrive at a conductor of known conductance, compute its value anew, and get a different answer. This is possible whenever the graph contains a circuit; and the question of whether reaching a particular edge by different paths gives the same value is the same as the question of whether going around any circuit gives the same value for the starting edge.

The condition, then, can be stated as follows: If $i_1, i_2, \ldots, i_k$ are k nodes in a probability network P which was derived from an electrical network, then

$$\prod_{j=1}^{k} \frac{p_{i_j i_{j+1}}}{p_{i_{j+1} i_j}} = 1 \tag{7}$$

Theorem 5.1. *Suppose that we are given a connected probability network P. Then a unique (up to a constant scale factor) electrical network Γ exists corresponding to P if, and only if, P satisfies (7).*

Proof. If: Take one conductor to have conductance 1. For each of its neighbors, take $\gamma_{ij} = \gamma_{ik}(p_{ji}/p_{ki}$; and do the same for all of the neighbors' neighbors, and so on. Since the network is connected, eventually all conductances will have been assigned; and because P satisfies (7), Γ is well-defined on Ω.

Only if: Suppose P fails to satisfy (7) on some cycle C. Choose an edge $A \in C$; we have then that $\gamma_A = \gamma_A \prod_{\widehat{ij} \in C} p_{ji}/p_{ij}$, from the definition of P. Since $\prod_{\widehat{ij} \in C} p_{ji}/p_{ij} \neq 1$, the only solution to this equation is $\gamma_A = 0$; but then C is not a cycle.

Theorem 5.2. *Suppose we are given a connected probability network P, where (7) fails for some cycle $C \subset P$. Then the resultant Γ is not uniquely determined by P.*

Proof. Let S be a tree spanning P. Since S contains no cycles, we can define a Γ_S that is symmetric on the tree S; and we can define an asymmetric conductance on some edge $e \in A = C \setminus S$. Let T be a tree spanning P which contains e. Then we can define a symmetric Γ_T corresponding to the tree T, and an asymmetric Γ_T on $S \backslash T$ and $A \backslash T$. But in Γ_S, the conductance $\gamma_S(e)$ is asymmetric, whereas in Γ_T, $\gamma_T(e)$ is symmetric.

Therefore $\Gamma_S \neq \Gamma_T$.

6. Calculating u and i from ϕ and P

Suppose that, instead of looking at a network from which no particle ever leaves, we want to look at networks which have a definite boundary, where we put particles (currents) and from which particles (currents) emerge. This is the point at which the boundary becomes special: the probability that a particle will leave a boundary node, once it has reached it, is zero; they "leave the network", never to return. On the other hand, we want to be able to put particles (current) in at the boundary.

The solution that I have adopted is to bifurcate the boundary nodes; this gives us a set of input nodes where we can place particles, and a set of output nodes whence we can take them. The probability that a particle at an interior node moves to an input node is zero; and the probability that a particle at an output node remains there is one. We have constructed, then, a matrix P, with the following canonical form:

$$\bar{P} = \begin{array}{c} \\ output \\ input \\ int \end{array} \begin{array}{c} \begin{array}{ccc} output & input & int \end{array} \\ \begin{pmatrix} I & S & R \\ 0 & 0 & 0 \\ 0 & T & Q \end{pmatrix} \end{array}$$

With the nodes arranged in the same way, P has the form:

$$P = \begin{array}{c} \\ output \\ input \\ int \end{array} \begin{array}{c} \begin{array}{ccc} output & input & int \end{array} \\ \begin{pmatrix} 0 & S & R \\ I & 0 & 0 \\ 0 & T & Q \end{pmatrix} \end{array}$$

which in essence identifies and insulates each pair of boundary nodes.

We are interested in looking at $\lim_{n\to\infty} \bar{P}^n\vec{\varphi}$, where $\vec{\varphi}$ is a vector corresponding to placing a particle at one of the input nodes, i.e., $\vec{\varphi} = e_j$, for $j \in \tilde{\Omega}^\circ$. $\bar{P}^n$ has the form

$$\bar{P}^n = \begin{array}{c} \\ output \\ input \\ int \end{array} \begin{array}{c} \begin{array}{ccc} output & input & int \end{array} \\ \begin{pmatrix} I & S + R(\sum_{i=0}^{n-1} Q^i)T & R\sum_{i=0}^{n} Q^i \\ 0 & 0 & 0 \\ 0 & Q^{n-1}T & Q^n \end{pmatrix} \end{array}$$

so, because $\varphi_j = 0$ when $j \notin int(\tilde{\Omega})$, $\bar{P}^n\vec{\varphi}$ has the form

$$\bar{P}^n\vec{\varphi} = \begin{pmatrix} S + R(\sum_{i=1}^{n-1} Q^i)T \\ 0 \\ Q^{n-1}T \end{pmatrix} \vec{\varphi}$$

Since $|Q| < 1$, we can define a matrix $N = (I - Q)^{-1} = \sum_{i=1}^{n-1} Q^i$, and

$$\lim_{n\to\infty} \bar{P}^n\vec{\varphi} = \begin{pmatrix} S + RNT \\ 0 \\ 0 \end{pmatrix} \vec{\varphi}$$

I claim that the top part of this matrix, $B = S + RNT$, gives a map from current flow in at input nodes to current flow out at output nodes. The matrix $A = NT$ will give a map from current flow in at the input nodes to interior potentials.

Whether B and A actually do what I've claimed depends on whether the vector

$$\begin{pmatrix} B\vec{\varphi} \\ \vec{\varphi} \\ A\vec{\varphi} \end{pmatrix}$$

is γ-harmonic on $\tilde{\Omega}^\circ$. This, however, is obviously true, since

$$\begin{pmatrix} 0 & S & R \\ I & 0 & 0 \\ 0 & T & Q \end{pmatrix} \begin{pmatrix} S+RNT \\ I \\ NT \end{pmatrix} = \begin{pmatrix} S+RNT \\ S+RNT \\ T+QNT \end{pmatrix} = \begin{pmatrix} S+RNT \\ S+RNT \\ NT \end{pmatrix}$$

since $T + QNT = T + (QT + Q^2T + Q^3T + \ldots) = NT$. So, using these matrices B and A, we can relate (gross) input currents to (gross) output currents and interior potentials.

What we're really interested in, though, is not a neumann-to-neumann map, but a dirichlet-to-neumann map—and it follows from the definition of probabalistic Ohm's law that imposing current $\vec{\varphi}$ at the input nodes is the same as inposing potential $\vec{\varphi}$ at those nodes: since

$$i_{ij} = u_i p_{ji} - u_j p_{ji}$$

$$i_{ij} = u_i \cdot 1 - u_j \cdot 0$$

$$i_{ij} = u_i$$

Therefore, to find the (net) current flow out of the boundary nodes due to potentials imposed on the boundary, we must take the difference between the input currents $\vec{\varphi}$ and the output currents $B\vec{\varphi}$: the map, then, is given by $L = I - B$. And, due to our definition of probabilistic potential u_i, we now have that

$$L \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix} = c\Lambda$$

for some $c > 0$.

Also, since we know that Λ is symmetric, we have all the information necessary to convert L to Λ, up to the scale factor c: simply take $\sigma_i = l_{i,1}/l_{1,i}$.

Theorem 6.1. *Suppose that we are given a probabilistic dirichlet-to-neumann map L. An associated electrical dirichlet-to-neumann map exists if and only if the matrix*

$$\Lambda = L \begin{pmatrix} 1 & & & \\ & l_{21}/l_{12} & & \\ & & \ddots & \\ & & & l_{n1}/l_{1n} \end{pmatrix}$$

is symmetric.

Proof. Because all of the off-diagonal l_{ij}'s are negative, their quotients are positive; and so the sign of each entry in Λ is the same as that in L. Thus Λ is a symmetric matrix whose columns sum to 0, and which has positive diagonal entries and negative off-diagonal entries. Therefore $\Lambda \in \mathcal{K}$, the set of dirichlet-to-neumann maps for electrical networks.

While it is relatively easy to convert from L to Λ, it is not very easy to convert from Λ to L: indeed, if we could do this, then we could solve for boundary conductors from Λ, which is not always possible. The convertability, if you will, of electrical and probability networks looks like this:

$$\begin{array}{ccc} \{\gamma_{ij}\} & \rightarrow & \Lambda \\ \uparrow\downarrow & & \uparrow \\ \{p_{ij}\} & \rightarrow & L \end{array}$$

References

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