A CONTINUOUS INVERSE CONDUCTIVITY PROBLEM: THE RECOVERY OF CONDUCTIVITIES FROM BOUNDARY MEASUREMENTS

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Abstract. An analytic method for using boundary measurements to recover the conductivity of a circular region within the unit disc is presented. The conductivity of the circular inhomogeneity is assumed to be constant, and the background conductivity is normalized to one. The location and size of non-concentric inhomogeneities are required to apply the method.

1. Introduction. In general, the inverse conductivity problem is that of finding the internal conductivity of an object by making measurements on its boundary. Also referred to as Electrical Impedance Tomography (EIT), methods for solving this problem may be applied to areas such as geology and medical imaging.

In [2] a method was developed and implemented for imaging the conductivity of a circular plate. In experiments, their method determined the approximate location and size of circular inhomogeneities. However, it did not accurately reproduce the conductivity. Here, we will provide an analytical method for determining the conductivity of a circular inhomogeneity.

2. The General Problem. Let $\Omega \subset R^2$ be a smooth, bounded domain. If we let Ω represent an isotropic body with no sources or sinks of current, then there exists a conductivity $\gamma : R^2 \to R^+$ such that

$$\nabla \cdot \gamma(\vec{p})\nabla u(\vec{p}) = 0 \tag{1}$$

where u is the potential in Ω. Further, if f represents the potential on the boundary, $\partial\Omega$, the Dirichlet to Neumann map is defined by

$$\Lambda(f) = \gamma\frac{\partial u}{\partial n} \tag{2}$$

and represents the current density normal to $\partial\Omega$.

Here we define Ω to be the disk of radius one centered on the origin. Let $\Omega_1 \subset \Omega$ be a circular region of constant conductivity σ, and let $\Omega_0 = \Omega - \Omega_1$ have constant conductivity 1.

3. The Forward Problem. We begin the solution to the forward problem by solving a system of P.D.E. on Ω with Ω_1 centered at the origin. This requires defining separate potential functions for each region,

$$u(\vec{p}) = \begin{cases} u_0(\vec{p}) \in C^2(\Omega_0), & \vec{p} \in \Omega_0 \\ u_1(\vec{p}) \in C^2(\Omega_1), & \vec{p} \in \Omega_1 \end{cases} \tag{3}$$

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and demanding that both the potential and current density are continuous across $\partial\Omega_1$.

The next step is to use a conformal map, Ψ, on Ω, which maps a non-concentric $\partial\Omega_1$ onto a circle centered at the origin. Composing Ψ with the general solution to the concentric problem provides the general solution to the non-concentric problem.

3.1. Solution to the Concentric Problem. The following system of P.D.E. defines the concentric Dirichlet problem in terms of u_0 and u_1:

$$\Delta u_0 = 0, \Delta u_1 = 0, \tag{4}$$

$$u_0 \mid_{\partial\Omega} = f,$$

$$u_0 \mid_{\partial\Omega_1} = u_1 \mid_{\partial\Omega_1} \quad \text{and} \tag{5}$$

$$\frac{\partial u_0}{\partial r} \mid_{\partial\Omega_1} = \sigma \frac{\partial u_1}{\partial r} \mid_{\partial\Omega_1} . \tag{6}$$

Here, f is an arbitrary potential function on $\partial\Omega$. Boundary condition (5) requires the potential to be continuous at the boundary $\partial\Omega_1$ and condition (6) requires conservation of current across $\partial\Omega_1$.

Applying separation of variables to system (4) provides the following general solution:

$$u_0 = \sum_{n=1}^{\infty} r^n (A_n^0 \cos n\theta + B_n^0 \sin n\theta) + r^{-n}(D_n^0 \cos n\theta + E_n^0 \sin n\theta) \quad \text{and} \tag{7}$$

$$u_1 = \sum_{n=1}^{\infty} r^n (A_n^1 \cos n\theta + B_n^1 \sin n\theta).$$

Note that while u_0 has a r^{-n} term, this is impossible for u_1 because a singularity in the potential would result at the origin.

In the next several steps we use (7) to obtain a refined general solution to our problem. Substitution of u_0 into boundary conditions (5) and (6) provides restrictions on the possible coefficients. By collecting the coefficients of the sin and cos terms, we obtain the following equations:

$$h^n A_n^0 + h^{-n} D_n^0 = h^n A_n^1, \tag{8}$$

$$h^n B_n^0 + h^{-n} E_n^0 = h^n B_n^1, \tag{9}$$

$$(h^n A_n^0 - h^{-n} D_n^0) = \sigma h^n A_n^1 \quad \text{and} \tag{10}$$

$$(h^n B_n^0 - h^{-n} E_n^0) = \sigma h^n B_n^1, \tag{11}$$

with h representing the radius of $\partial\Omega_1$. Equations (8) and (10) can be solved for A_n^1 and equated to obtain a single equation of the coefficients A_n^0 and D_n^0. Further simplification provides the following relationship :

$$\frac{(1-\sigma)h^{2n}}{1+\sigma} A_n^0 = D_n^0. \tag{12}$$

Similar operations on (9) and (11) give an equivalent relation between B_n^0 and E_n^0. Given these restrictions on the coefficients, (7) can be rewritten as:

$$u_0 = \sum_{n=1}^{\infty} (C_n^0 r^{-n} + r^n)(A_n^0 \cos n\theta + B_n^0 \sin n\theta) \quad \text{and} \tag{13}$$
$$u_1 = \sum_{n=1}^{\infty} r^n (A_n^1 \cos n\theta + B_n^1 \sin n\theta)$$

with

$$C_n^0 = \frac{h^{2n}(1-\sigma)}{(1+\sigma)}. \tag{14}$$

Further algebra on (8) through (11) yields

$$A_n^1 = A_n^0 \frac{2}{1+\sigma} \quad \text{and} \tag{15}$$
$$B_n^1 = B_n^0 \frac{2}{1+\sigma}.$$

The boundary condition on $\partial\Omega$ requires the following definitions for the remaining coefficients:

$$A_n^0 = \frac{1}{\pi(C_n^0+1)} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta \quad \text{and} \tag{16}$$
$$B_n^0 = \frac{1}{\pi(C_n^0+1)} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta.$$

3.2. The Conformal Map. Since the location and size of the object are known, we assume that Ω_1 is centered along the positive real axis. This can be accomplished by a simple angular rotation. The intersection points of Ω_1 and the positive real axis are denoted by x_1 and x_2 with $x_1 > x_2$. The conformal map, Ψ, is the reciprocal of a linear fractional transformation obtained from [1],

$$\Psi(z) = \frac{az-1}{z-a} \quad \text{and} \tag{17}$$
$$a = \frac{1 + x_1 x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 + x_2}.$$

Note that under Ψ $\partial\Omega$ is mapped onto itself, and $\partial\Omega_1$ is mapped to a circle centered on the origin. The radius, h, of the inner concentric circle of the mapped region is

$$h = \frac{x_1 - x_2}{1 - x_1 x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}. \tag{18}$$

Figure 1 shows the results of the transformation. The compression of points along the edge of the concentric circles near the positive real axis gives some indication of the

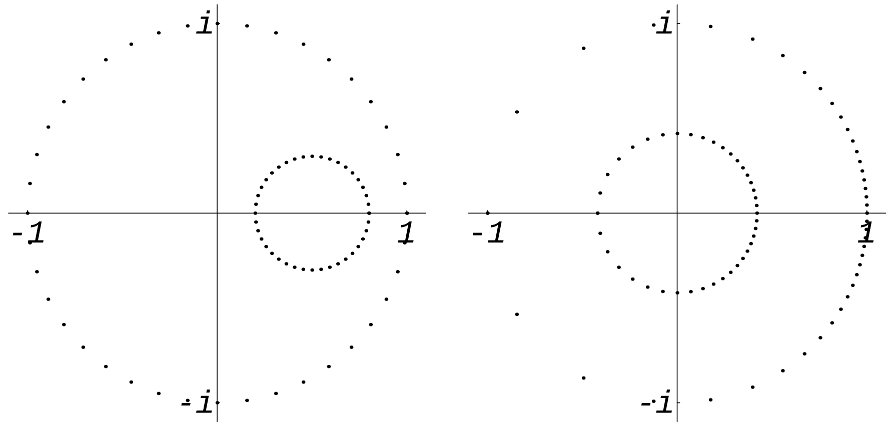

FIG. 1. *The diagram on the left shows points placed at equal angles on non-concentric disks in the complex plane. The diagram on the right shows the transformation of these points under* Ψ*.*

impact the transformation has on Dirichlet boundary functions. Neumann boundary conditions have a more complicated transformation process which will be discussed later. One aspect of the transformation, not evident in figure 1, is that 1 maps to -1 and -1 to 1.

The benefit of using a conformal map is that harmonic functions in the concentric reference frame pull back to harmonic functions in the non-concentric system. Therefore, we may compose the solution in the concentric case with Ψ to form the non-concentric solution. Before we can do this, we must rewrite (17) as

$$\Psi(re^{i\theta}) = \rho(r,\theta)e^{i\phi(r,\theta)}, \tag{19}$$
$$\rho(r,\theta) = \sqrt{\frac{1+ar^2-2ar\cos\theta}{a^2+r^2-2ar\cos\theta}} \quad \text{and}$$
$$\phi(r,\theta) = \arctan\left(\frac{(1-a^2)r\sin\theta}{a+ar^2-(1+a^2)r\cos\theta}\right).$$

This polar form maps $(r,\theta) \xrightarrow{\Psi} (\rho,\phi)$, where (r,θ) are in the non-concentric reference frame and (ρ,ϕ) are in the concentric. Functions are pulled back by composition with the transformation in the following manner. Suppose we have a Dirichlet boundary function $f(r,\theta)$ on $\partial\Omega$, then the corresponding boundary function in the concentric reference frame is $g(\rho,\phi)$ given by

$$f(r,\theta) \xrightarrow{\circ\psi^{-1}} g(\rho,\phi). \tag{20}$$

3.3. General Solution to the Non-concentric Problem. Now the concentric solution can be composed with the polar transformation yielding the following solution

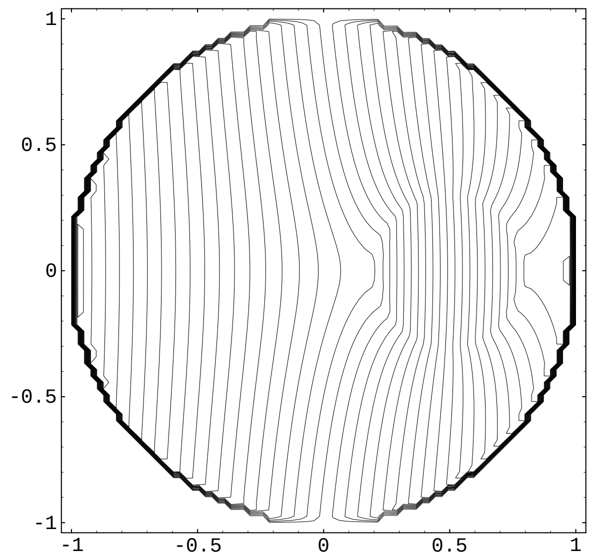

FIG. 2. *A graph of the equipotentials for an object with conductivity $\sigma = 0.005$.*

for the non-concentric problem:

$$u_0 = \sum_{n=1}^{\infty} (C_n^0 \rho^{-n} + \rho^n)(A_n^0 \cos n\phi + B_n^0 \sin n\phi), \tag{21}$$

$$u_1 = \sum_{n=1}^{\infty} \rho^n (A_n^1 \cos n\phi + B_n^1 \sin n\phi),$$

$$C_n^0 = \frac{h^{2n}(1-\sigma)}{(1+\sigma)},$$

$$A_n^0 = \frac{1}{\pi(C_n^0 + 1)} \int_{-\pi}^{\pi} g(\phi) \cos n\phi d\phi,$$

$$B_n^0 = \frac{1}{\pi(C_n^0 + 1)} \int_{-\pi}^{\pi} g(\phi) \sin n\phi d\phi,$$

$$A_n^1 = A_n^0 \frac{2}{1+\sigma} \quad \text{and}$$

$$B_n^1 = B_n^0 \frac{2}{1+\sigma}.$$

The equations for u_0 and u_1 are equations (13) composed with Ψ. This pulls the concentric function into the non-concentric system. Since the integrals used to compute A_n^0 and B_n^0 involve the boundary potential in the concentric system, we use $g(\rho, \phi)$ as defined in (20) with $\rho = 1$.

Figure 2 is a graph of the equipotential lines of the non-concentric solution with a boundary potential of $f = \cos\theta$.

4. The Inverse Problem. The objective of the inverse problem is to obtain σ from measurements of potentials and currents on $\partial\Omega$. In theory the Dirichlet to Neumann map provides all of the information for this calculation. A known boundary

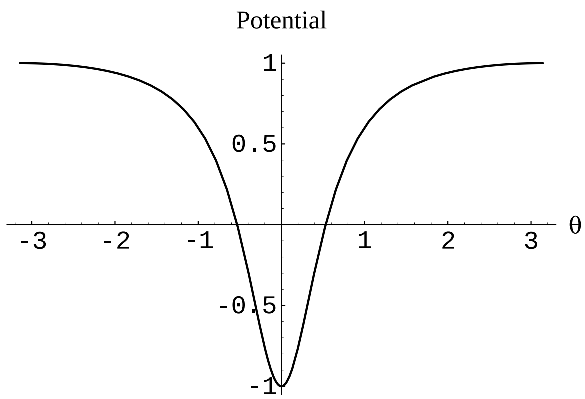

FIG. 3. *Graph of* $\cos(\phi(1,\theta))$ *for* $x_1 = 0.8$ *and* $x_2 = 0.2$.

potential can be applied to $\partial\Omega$ and the Dirichlet to Neumann map for either the concentric or non-concentric problem can be evaluated using equation (2). Measured values of currents could further be substituted into (2) and the result would be an equation in σ.

Recovering the conductivity, however, is difficult for arbitrary boundary potentials. Since σ is found in the coefficient C_n^0, it generally is part of a complicated infinite series and not easily recovered. We will show in this section how carefully chosen boundary potentials can be used to truncate the series at one term.

4.1. Recovery of σ for the Concentric Problem. The key to solving the inverse problem for the concentric case is to find an eigenvalue equation in the Dirichlet to Neumann map. That is, we want to find a function f and a value λ such that

$$\Lambda_C(f) = \lambda f, \tag{22}$$

where Λ_C is the Dirichlet to Neumann map for the concentric problem and corresponds to the current density, $I(\theta)$, on $\partial\Omega$. Since taking the derivative of u_0 (13) with respect to r does not effect the $A_n^0 \cos n\theta$ or $B_n^0 \sin n\theta$ terms, it is readily apparent that $f = \cos\theta$ and $f = \sin\theta$ are eigenfunctions of the map. Moreover, the eigenvalue is exclusively a function of C_1^0,

$$\lambda = \frac{1 - C_1^0}{1 + C_1^0}. \tag{23}$$

Recall, C_1^0 (14) is a function of σ and the known radius of the inner circle, h. Therefore, by placing $f = \cos\theta$ or $f = \sin\theta$ on $\partial\Omega$, (22) reduces to

$$\sigma = \frac{I(\theta)\,(h^2+1) - f(\theta)\,(h^2-1)}{I(\theta)\,(h^2-1) + f(\theta)\,(h^2+1)}, \tag{24}$$

where $I(\theta)$ are measurements of the current density on the boundary.

Note, an application of a second eigenfunction, such as $f = \cos 2\theta$, will provide a second eigenvalue equation and enough information to recover both σ and h. For

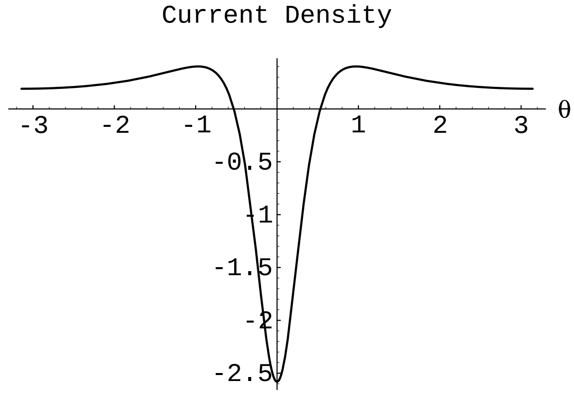

FIG. 4. *A graph of the current density on $\partial\Omega$. The boundary potential in this case is $f = \cos(\phi(1,\theta))$, $x_1 = 0.8$, and $x_2 = 0.2$.*

instance, let $f_1 = \cos\theta$ and $f_2 = \cos 2\theta$ be potential functions placed on $\partial\Omega$ in separate experiments. If $I_1(\theta)$ and $I_2(\theta)$ represent the respective current densities, then

$$h = \sqrt{\frac{(I_1(\theta) + f_1(\theta))(I_2(\theta) - f_2(\theta))}{(I_1(\theta) - f_1(\theta))(I_2(\theta) + f_2(\theta))}}. \tag{25}$$

4.2. recovery of σ for the Non-concentric Problem. An eigenvalue equation for the Dirichlet to Neumann Map is difficult to find in the non-concentric problem because the derivative of u_0 with respect to r contains $\frac{\partial\rho}{\partial r}$ and $\frac{\partial\phi}{\partial r}$ terms. However, $\frac{\partial\phi}{\partial r}$ is zero on $\partial\Omega$ so the map retains only the $\frac{\partial\rho}{\partial r}$ term. This allows us to find an analytic solution for σ with certain boundary functions.

We pursue a similar strategy to the one used in the concentric solution. By selecting a Dirichlet boundary condition of $f = \cos\phi$, the general solution (21) to the non-concentric problem truncates to

$$u_0 = \frac{C_1^0\rho^{-1} + \rho}{C_1^0 + 1}\cos\phi. \tag{26}$$

Differentiating this with respect to r provides the relationship

$$\Lambda_{NC}(f) = \lambda\frac{\partial\rho}{\partial r}f, \tag{27}$$
$$\frac{\partial\rho}{\partial r} = \frac{a^2 - 1}{a^2 + 1 - 2a\cos\theta}.$$

Here, Λ_{NC} represents the Dirichlet to Neumann map for the non-concentric problem and λ is the eigenvalue for the concentric problem. Figures 3 and 4 show the relationship between an induced boundary potential of $f = \cos(\phi(1,\theta))$ and the resulting current density on $\partial\Omega$. Similar to the concentric case, (27) can be used to derive an expression for σ in terms of experimental values of the current density:

$$\sigma = \frac{I(\theta)\,(h^2+1) - \frac{\partial\rho}{\partial r}f(\theta)\,(h^2-1)}{I(\theta)\,(h^2-1) + \frac{\partial\rho}{\partial r}f(\theta)\,(h^2+1)}. \tag{28}$$

5. Conclusion. We have derived analytic equations for recovering σ in terms of experimentally measured values on the boundary. For the special case of concentric inhomogeneities, σ can be recovered by applying simple trigonometric boundary potentials, such as $f = \cos\theta$. In fact, these boundary functions may be used regardless of the size of the inhomogeneity and can even be used to recover the radius. An important aspect of the non-concentric solution is that the boundary potential required is a function of the location and size of the inner region. This dependence is the primary obstacle to imaging the object analytically.

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