Recovering Sources in Rectangular Networks with Known Resistors

Margaret Chaffee, University of Washington
A.R.K. Whitley, Reed College

University of Washington 1996 Summer REU

Abstract

For a rectangular array with unit resistors and set boundary voltages, the boundary currents will uniquely determine the position of all internal current sources assuming that all sources are of the same strength. If there is only one internal current source, the resistors need only be known.

Table of Contents

1. Introduction

Motivation for this problem came from Strakhov and Brodsky's paper [1] which looks at uniquely determining the shape and density of planar gravitating objects. They find that the object can be uniquely determined when certain restrictions are placed on the shape and density of the object. We attempted to find the discrete analog of their problem using electricity instead of gravity. We set up a network of resistors with current sources arranged inside the network. We tried to uniquely determine the location of the current sources based on the currents and electric potentials on the boundary. In truth, the problem we solved is not the discrete analog. We focused on finite networks, not infinite networks and there were several other differences. Nevertheless, that is where the idea for this problem came from and we hope that the results in this paper will be useful to others who wish to look at the discrete analog of Strakhov and Brodsky's paper.

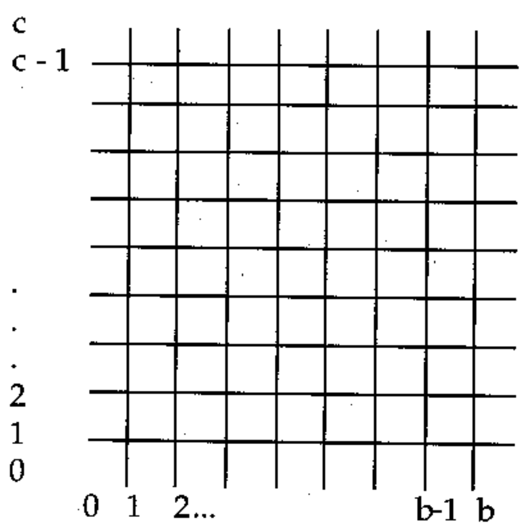


Figure 1

We consider rectangular networks of resistors in the plane as in Curtis and Morrow [2] which we shall restate for the reader's convenience. Let $\mathbf{Z}^2$ be the lattice in $\mathbf{R}^2$ consisting of the points with integer coordinates. Two lattice points p and q are *adjacent* if there is a horizontal or vertical segment of length one joining them. We construct a rectangular network Ω as follows. The *nodes* of Ω are the lattice points p = (i,j) for which $0 \leq i \leq b$ and $0 \leq j \leq c$ (b, c, i, j are all integers), with the four corner points (0,0), (0,c), (b,0), (b,c) excluded. The *resistors* are the horizontal and vertical lines that connect adjacent nodes. The *boundary* of Ω, called $\partial\Omega$, consists of the points (0,j), (b,j), (i,0) and (i,c) where $1 \leq j \leq c - 1$ and $1 \leq i \leq b - 1$. The *interior* of Ω, intΩ, consists of all the nodes in Ω that are not on the boundary. A *source* is an interior node with positive net current. A node is *harmonic* if the net current is zero.

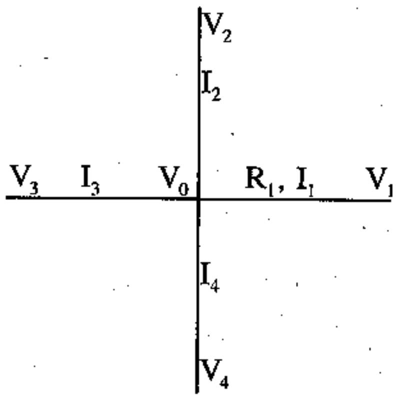
In previous papers no sources were allowed on the interior so the boundary nodes were the only nodes with a nonzero net current. It is important to keep this in mind when looking at results and definitions from other papers, like the maximum principle, to make sure the results do not depend on there being no source or sink nodes on the interior.

We consider a network Ω with some current sources in the interior and no current sources on the boundary. The resistors are known, the boundary potentials are set to zero, and the current flowing out can be measured.

The forward problem we pose is: *Given the location of the sources, what are the currents on the boundary?*

The inverse problem is: *Given the currents on the boundary can we uniquely determine the location of the current sources in the interior?*

Throughout this paper we use Ohm's Law and Kirchhoff's Law. We state them here.



Ohm's Law:

$$V_0 - V_1 = I_1R_1$$

Kirchhoff's Law (where V_0 is harmonic):

$$I_1 + I_2 + I_3 + I_4 = 0$$

Here we should remember the sign conventions for current. Current is positive if it is flowing out of a node and negative if it is flowing into a node.

2. The Forward Problem

This problem was solved in [3] and we shall restate the results here. We look at the Dirichlet problem in block form.

$$\left[\begin{array}{c|c} K' & B^T \\ \hline B & A \end{array}\right]\left[\begin{array}{c} u_b \\ \hline u_i \end{array}\right] = \left[\begin{array}{c} \Psi_b \\ \hline M \end{array}\right]$$

Writing this as two equations we have:

$$K'u_b + B^Tu_i = \Psi_b$$

$$Bu_b + Au_i = M$$

Where u_b is the vector of boundary potentials, u_i is the vector of interior potentials, and Ψ_b is vector of boundary currents. M is the vector of current sources on the interior whose entries are nonzero only at sources. The big matrix is the Kirchhoff matrix and is determined by the resistors in the network. Higginson proves that the solution is unique.

As the outside potentials are equal to zero this equation can be further simplified to the following which is easily solved.

$$u_i = A^{-1}M$$

3. The Inverse Problem

Two different algorithms were generated to solve the inverse problem. The first works in the case of one source and leaves open a question about the generalization of the Maximum Principle. The second works for any number of unit sources in a network with unit resistors.

3.1 The First Algorithm

Consider a rectangular network with one source of current (of any size) at an interior node, all of the boundary potentials set to zero, and known resistances. The algorithm described in this section can find the location and size of the current source.

> Theorem 1.2.1: *In a rectangular network of resistors with only one source node and zero potential on the boundary, the maximum potential must occur at the source node.*

Proof: By the Maximum Principle the extrema will occur on the boundary or at a source node. As there are no sinks, all potentials must be greater than or equal to zero and the minimum is on the boundary. If the maximum occurred on the boundary then the potential at all nodes would be zero which would mean no current was flowing. The maximum must be at the source of current. □

3.1.1 The algorithm

Measure the currents on the boundary nodes. The sum of these currents is the size of the current source.

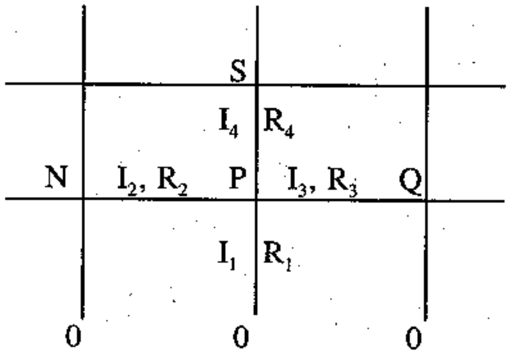


Figure 2

Figure 2 is a section of a network. The zeros are the voltages on the boundary of the network and N, P, Q, and S are interior nodes. Let us look closely at the currents around node P. R_1 and I_1 are known and can be used to calculate V_p (the voltage of P).

$$(V_p - 0) = I_1 R_1$$

Similarly, the voltages of N, Q and all other nodes that are the neighbors of the boundary can be calculated. Look at I_2 and I_3. The voltage drop and resistors are known so the currents can be calculated using Ohm's Law. Three of the currents around P have been calculated and the same can be done for every node adjacent to the boundary.

Now we compare all the voltages known so far. The node with the highest potential might be a source. All other nodes with known potentials are not current sources by Theorem 1.2.1, so the sum of the currents must be zero.

Let us use the same picture, where P does not have the highest potential. Using Kirchhoff's Law we can find I_4.

$$I_4 = -(I_1 + I_2 + I_3) \qquad (1)$$

Using Ohm's Law we can calculate the voltage of another node.

$$V_s - V_p = I_4 R_4$$

$$V_s = I_4 R_4 + V_p \qquad (2)$$

We do these calculations (1 and 2) for all nodes except the one with the highest potential.

We have calculated potentials that were previously unknown and we want to use this information to calculate even more potentials. We compare all the voltages again. If the maximum changes then we know the original maximum was not a source and we can use Kirchhoff's law to calculate the fourth current. We continue using Ohm's Law and Kirchhoff's Law to calculate voltages and currents, making sure that we never use Kirchhoff's Law on the node with the highest potential. This process is repeated until all the potentials are calculated. The current source is the node with the maximum potential.

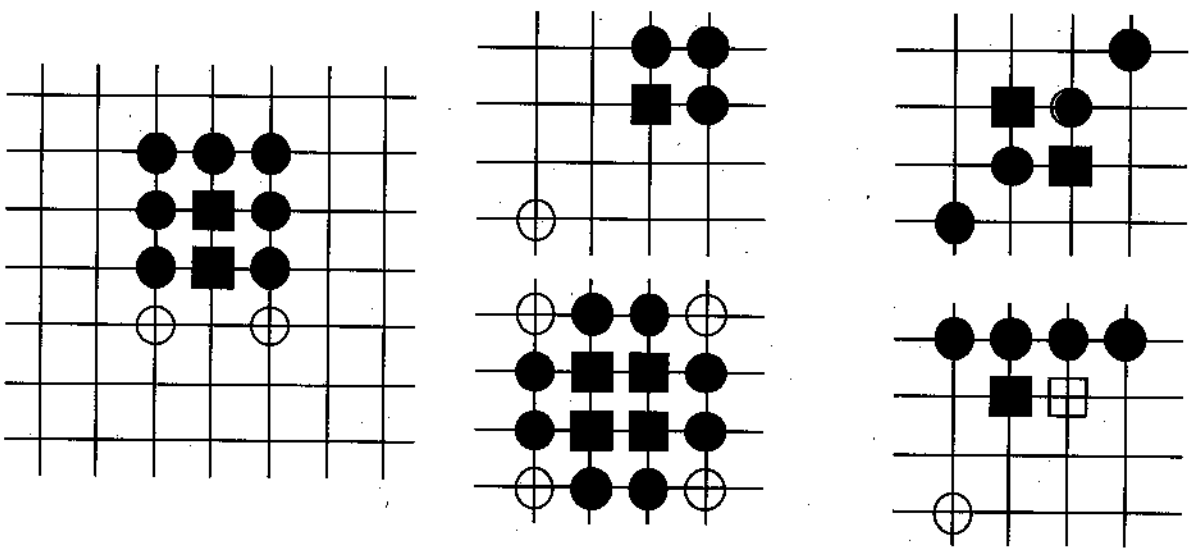
Note 1: The current source may be found before all the potentials are calculated. All four currents out of a node might be known if we know that all four neighbors are not sources. If the net current is nonzero then the node is a source.

Note 2: Some nodes (in the corners, for instance) may be calculated from more than one neighbor. The voltage of the node is unique because it can't physically have more than one potential so the calculation is made once and it doesn't matter which neighbor you use.

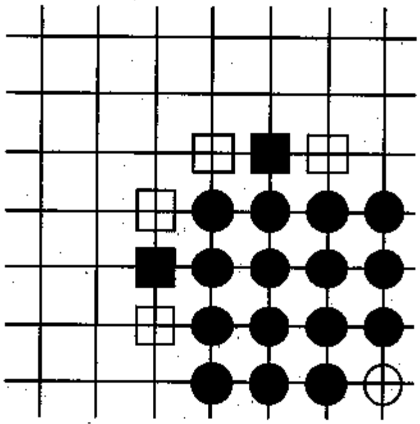
3.1.2 Generalization for more sources

For our algorithm to work when there are k sources of the same size, the k sources must have the k highest potentials. We found that this was not true in general. Here are some counter examples.

The k circles are sources and the squares are interior nodes with potentials higher than a source node. The nodes with the k highest potentials are filled in.



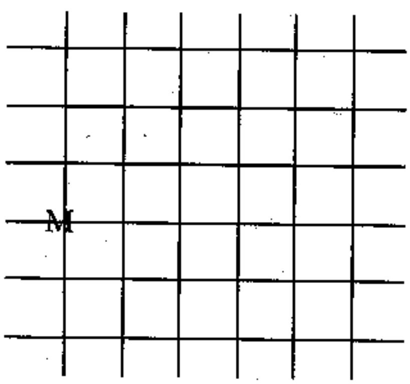
Strakhov and Brodsky had to make restrictions on the geometry and the density of the gravitational bodies in order to find a unique solution. We hoped that similar restrictions would yield situations in which our algorithm always worked. We tried defining and restricting ourselves to simply connected convex shapes. We were able to find counter-examples even with definitions so conservative that they only allowed solid rectangles.



The counter-examples for simply connected convex shapes have to do with the boundary being set to zero. This may not be a problem in the infinite case where the entire plane is a network of resistors. In the infinite case the "boundary data" is different. The differences between the voltages goes to zero as we approach infinity. The infinite case is similar to the finite case because we will look at the voltages on the boundary of a finite rectangle, large enough to contain all the sources on its interior. However, there is no reason that the voltages on the boundary of this rectangle would be the same. The infinite case and the finite case differ enough that a generalized Maximum Principle might be found in the infinite case even though we have been unsuccessful in the finite case.

3.1.3 A Conjecture

We found a simpler algorithm for finding a lone source in a network of unit resistors. Unfortunately we were not able to prove that it works. To show that it works we must prove something about the Green's function. The Green's function can be interpreted as the voltage at a node. In [4] Duffin looks at the Green's function of an infinite three dimensional network of unit resistors with a source at the. He proves that the Green's function decreases when moving from a point to a neighboring point more distant from the origin.



If this were true for a finite two dimensional network of unit resistors with a source somewhere in the interior, we could use this to find the location of the source. We would look at the boundary currents on one side of the rectangle. Notice that these currents are equal to the voltages of the neighboring interior nodes. We find the maximum voltage. We would know that the node with the maximum voltage is closer to the source then all other nodes on that side. This means that the source is in the same row as M. We would do the same for an adjacent side of the rectangle to determine what column the source is in. The source has been found.

Unfortunately Duffin's proof relies heavily on the symmetry of the network (the source is at the origin). He also uses the fact that the voltages go to zero at infinity. Even in the infinite two dimensional case the differences of the voltages go to zero at infinity, not the actual voltages. Duffin's result cannot be directly applied to our problem and it was not clear whether his approach could be used. The question remains open in both the infinite and finite two dimensional cases with one source but we suspect (without any proof) that in both cases the Green's function decreases as you get farther away from the source.

4. Another Algorithm: Exhaustive Search

The above algorithm is elegant in its simplicity, but unfortunately it is not widely applicable. Therefore, we wish to formulate an approach that will apply in general to any number of sources in a rectangular array. Before we required that the resistors in the network be known. For more than one current source, we require that all the resistors have unit resistance. This is more restrictive, however having uniform resistance is closer to the continuous case. Also we require that all the sources have unit strength.

So suppose we are given a resistor array and are told to find the internal current sources. Note that we are given the dimensions of and the value of the resistors in the array. So we go to our lab and create a resistor array identical to the one we are given but without any current sources. Then we perform the following experiment: We first choose our favorite internal node. Setting the boundary voltages to zero, we put a unit current source on that node and record the boundary currents produced. We repeat this experiment for every internal node, recording the vector of boundary currents for each.

Note that any set of interior current sources will produce boundary currents which are a linear combination (with all coefficients one) of those from the above experiments. Also note in general we know the number of internal sources in an array; to find that number, we simply add up the boundary currents which tells us the total current flowing into the array from within. Since we are only allowing unit current sources, we then know the number of sources in the array. So when handed an array with unknown internal current sources, we calculate the number of internal sources, and find the appropriate linear combination. We thus have the locations of all the sources.

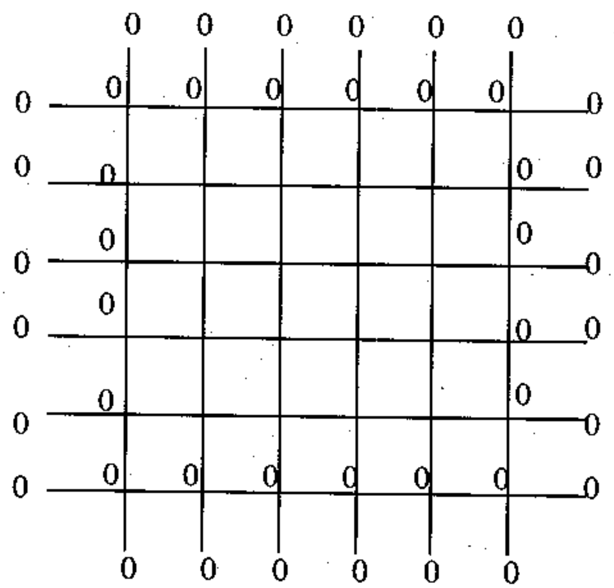
Of course, gathering the preliminary information does not require actually building an array and performing these experiments. We know the resistors, so we can write down the Kirchhoff matrix. Performing an experiment described above is just the forward problem, where M (see above) is one of the standard basis vectors. So the boundary currents from the experiments are obtained from the columns of A^{-1}.

The only problem with this approach is one of uniqueness. That is to say, we want to show that no two distinct linear combinations are the same. We can restate this in network terms: Suppose that you are given two rectangular arrays of the same shape, with unit resistors, which have the same boundary currents and thus the same number of internal sources. We need to show that the two arrays indeed have the same internal source configuration.

At this time recall that for all arrays, we set the boundary voltages to zero. Thus two arrays of the same size will trivially have the same boundary voltages. So let boundary 'data' refer to both boundary currents and voltages.

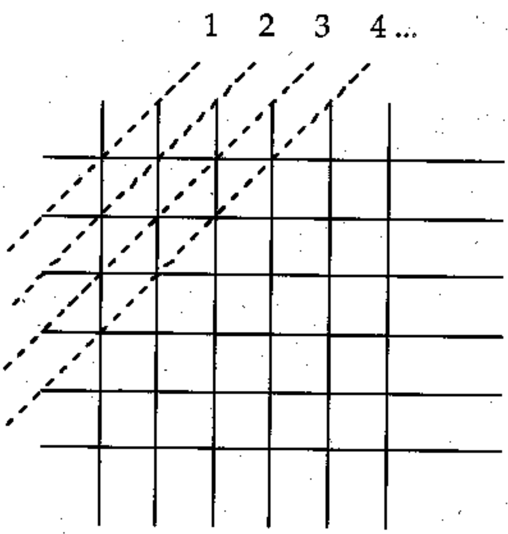
Suppose that we have two $m \times n$ resistor arrays, Γ and Θ, with unit resistors and the same boundary data. Consider the 'difference' of these two arrays. That is, consider a third array, Δ, of the same size as Γ and Θ, with unit resistors, which has unit sources where Γ has sources and unit sinks where Θ has sources. If both Γ and Θ have sources in a certain spot, Δ will have a harmonic node (net current zero) in that position. Note that Δ still has boundary voltages zero, and in addition it has boundary currents zero. Thus by Ohm's law we have that all nodes adjacent to a boundary node (on the 'first layer in') have potential zero. If we can show that this difference array has no sources or sinks, then we have shown that the two arrays , Γ and Θ, had the same source configuration. Before we prove this uniqueness, we first need to prove a small lemma.

<u>Lemma 4.1</u>: *If a rectangular resistor array, with unit resistors, has boundary currents and voltages zero, then the voltage at every node in the array must be an integer.*



Proof: As noted above, the potentials on the first layer in are zero. Now note that the currents in the boundary spikes and between nodes on the first layer in are zero because all these nodes are at the same potential. Consider a node on the first layer in. Note that three currents flowing into that node are all zero. This node may be a unit sink, unit source, or harmonic node. Thus the fourth current flowing from that node is either -1, 1, or 0. Thus all voltages on the second layer in are either 1, -1, or 0. Now suppose that we know all the nodes on the first k layers in have integer voltages. Since the voltages are integers and the resistors have unit resistance, the currents flowing between all nodes on the first k layers have integer values. Consider a point on the kth layer. Three of the four currents flowing into this node have integer values. Again since the net current flowing out of this node is either -1, 1, or 0, the fourth current must have integer value. Thus the nodes on the (k+1)st layer in have integer-valued potentials. So by induction, all nodes in the array have integer-valued potential. □

> <u>Uniqueness Theorem</u>: *If two rectangular arrays of the same size, with unit resistors, have the same boundary data, then these two arrays have the same internal source configuration.*



Proof: Consider the difference array, Δ. We want to show that there are, in fact, no sinks or sources in Δ. Note that all the potentials in Δ are zero if and only if there are no sources or sinks present.

Consider the top left-hand corner of Δ. We number diagonals as shown. Let μ be the maximum voltage attained in the difference array. We find a diagonal ∂ which has a node, say ν, with potential μ. Since μ is the highest value of any potential, ν must either be a unit source or a harmonic node. Again since μ is the highest potential, if ν is harmonic, then all its neighbors must have potential μ. Note that two neighbors of ν are on the (∂-1)st diagonal. Thus there is a node on the (∂-1)st diagonal which has the maximal voltage. Suppose that ν is a unit source. Because it has the maximal value and all voltages are integer-valued, three neighbors of ν must have potential μ. (The fourth neighbor has potential μ-1.) Thus at least one node on the (∂-1)st diagonal has potential μ. By iteration, we have that the maximum is attained on the first diagonal. The only node on the first diagonal has potential zero. Thus the highest potential in the array is zero.

By making appropriate substitutions, such as replacing 'maximum' with 'minimum', in the above argument, we have that the minimum value of the potential must also be attained on the first diagonal. Thus the minimum value must also be zero.

Thus all the potentials in the array are zero. Thus there are no sinks or sources in Δ. □

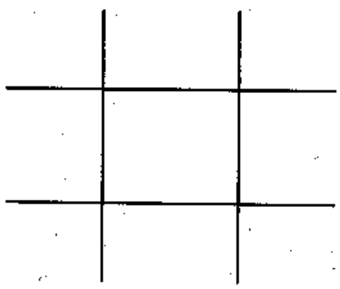
Note that we needed only that the two arrays had the same boundary voltages and not zero boundary voltages. This is because when we subtract two arrays which have the same boundary voltages, we still get zero boundary voltages in the difference array.

Before discovering the above proof, we tried many different methods to prove the uniqueness. These methods proved to be computationally involved and hard to generalize for an inductive argument. The ideas behind these methods, however, are sound, and we are hopeful that they may be instructive to some readers. So below, we shall give some proofs for small networks and then we shall sketch two unfinished proofs for the general case. For convenience, we shall say a certain size array 'has a unique solution' when two arrays of that size with unit resistors and the same boundary data have the same internal source configuration. Note that the proofs below are given for square arrays, but the techniques may be applied to rectangular networks.

<u>Theorem 4.1</u>: *A 1×1 resistor network has a unique solution.*

Proof: The only currents in this array are the boundary currents. Thus if the boundary currents are the same in two 1×1 arrays, all the currents in the arrays are the same. Thus they have the same source configurations. □

<u>Theorem 4.2</u>: *A 2×2 resistor network has a unique solution.*



Proof: This, too, is a trivial case. Suppose you have two 2×2 resistor networks with the same boundary data. Since the resistors are identical in both arrays, by Ohm's law the voltages at the four interior nodes are the same in the two networks. Thus the current flowing between the four interior nodes is also the same. Thus all the currents in the two arrays is the same in the two networks. Thus the two networks have the same source configurations. □

Note in the two proofs above, we only needed that the resistors were known and equal in the two arrays but they need not be of unit resistance. For larger arrays, we must have unit resistors. Also we did not assume anything about the current sources themselves, but again for future proofs, we need all the sources to be of unit strength.

These two cases were trivial to prove. The following cases become more cumbersome. To prove the next two cases, we first make observations about possible source configurations on the diagonals. So first we number the diagonals as shown above. We consider the first diagonal.

Lemma 4.2: If two rectangular arrays of the same size with the same resistors have the same boundary data then they must have the same source configuration on the first diagonal.

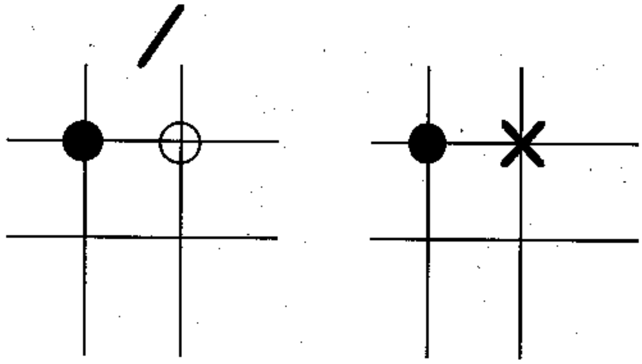
Proof: This is very similar to the proof of the 2×2 case. Since the two arrays have the same boundary data and resistors, we know that they have the same voltages on all nodes adjacent to boundary nodes. Thus they also have the same currents flowing between the nodes on the first layer in. Now note that the only node on the first diagonal is on a corner; all the four currents flowing into this node are either boundary currents or currents on the first layer in. Thus the arrays have the same source configuration at this node. □

First note that this above proof did not use the fact that we were looking at the upper left-hand corner. That is to say, this proof shows that two arrays with the same boundary data have to agree on the first diagonal coming from the top left corner and on the first diagonal coming from the top right corner and so on for the other two corners. The same is true for any diagonal.

Next note it is because of this lemma that the proofs of the 1×1 and 2×2 cases are trivial. That is, note that all interior nodes in the 1×1 and 2×2 cases are on a first diagonal. So by the lemma, they must have the same source configuration at every node. Now we consider the second diagonal.

Lemma 4.3: If two arrays are of the same size, with unit resistors, and have the same boundary data, then they will have the same source configuration on the second diagonal.

Proof: Before we begin this proof, we shall state a few conventions. First we shall label all the nodes in the array: n_{ij} shall denote the node in the ith row and jth column, where n_{11} is the interior node on the first diagonal. V_{ij} shall denote the potential at n_{ij}. Likewise, I_{ij} will denote the net current out of n_{ij}, where current flowing out of a node is considered positive with respect to that node. Note the net current will either be zero or one. $I_{ij,kl}$ shall denote the current flowing from n_{ij} to n_{kl}. In this proof we shall be considering two resistor arrays. We shall differentiate one array with a prime ('); all information regarding that array shall be primed. In a graphic, a closed dot on a node indicate that the two arrays have the same source configuration at this node. An open dot indicates a unit source at that node. A cross on a node indicates that the node is harmonic.



Now suppose we have two arrays of the same size, with unit resistors and the same boundary data. Let us consider the top left-hand corner of these two arrays. We already know that two arrays agree at the node on the first diagonal. Suppose that n_{12}' is a source and n_{12} is a harmonic node. Because the boundary currents and currents on the first layer in are all the same, we have

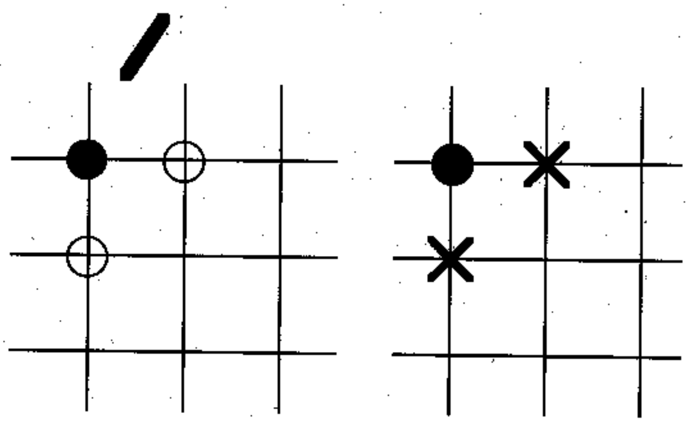
$$I'_{12,22} = I_{12,22} + 1. \tag{4.1}$$

From this we get

$$V'_{22} = V_{22} - 1. \tag{4.2}$$

Since $V'_{21} = V_{21}$ we have

$$I'_{21,22} = I_{21,22} + 1. \tag{4.3}$$



Thus we have a source at n_{21}' and a harmonic node at n_{21}. Our goal now is to show that this configuration is not possible. To do this, we look to the next diagonal. What source configurations can we put on the third diagonal with this configuration on the second diagonal. Let us consider what happens at n_{22}. We are only allowing unit current sources, so we have three possibilities:

$$I'_{22} = I_{22} + \begin{cases} 1 \\ 0 \\ -1 \end{cases} \tag{4.4}$$

So we write down the equation for I_{22}'

$$I'_{22} = 4V'_{22} - V'_{12} - V'_{21} - V'_{23} - V'_{32} \tag{4.5}$$

Using the fact that potentials on the first layer in are the same in the two arrays and equation (4.2), we have

$$I'_{22} = 4(V_{22} - 1) - V_{12} - V_{21} - V'_{23} - V'_{32} \tag{4.6}$$

Because n_{23} and n_{32} are on the second layer in, we have

$$V'_{23} = V_{23} + \begin{cases} 1 \\ 0 \\ -1 \end{cases} \tag{4.7}$$

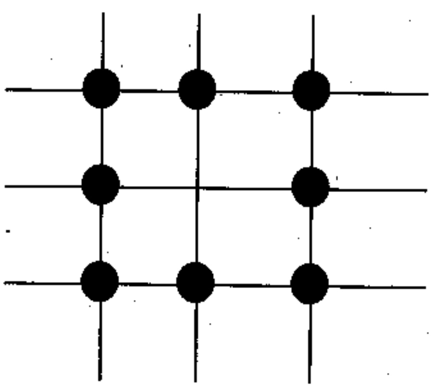
and likewise for V_{32}. So in (4.6), we have at best

$$I'_{22} = 4V_{22} - V_{12} - V_{21} - V_{23} - V_{32} - 2 = I_{22} - 2 \tag{4.8}$$

which is not possible. Thus the two arrays cannot differ on the second diagonal. □

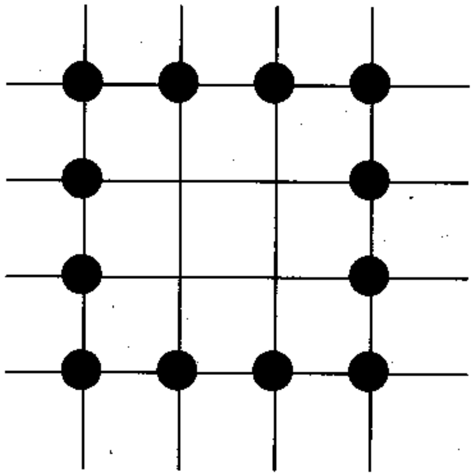
Just as the uniqueness of the 1×1 and 2×2 cases is dependent on the first diagonal, the uniqueness of the 3×3 and 4×4 cases follow easily from the previous lemma.

<u>Theorem 4.3</u>: *A 3×3 resistor network has a unique solution.*



Proof: Suppose we have two 3×3 resistor networks with the same resistors and boundary information. From above we know that these two arrays do not differ on any first or second diagonal. Thus we have that the two arrays are identical on the first layer in. So we have reduced to the 1×1 case which was know is unique. Note that we actually reduced to the 1×1 case where the boundary voltages are the same in both arrays, but they are not zero. Because we never use the fact that the voltages are zero, only that they must be the same, this is fine. □

Theorem 4.4: A 4×4 resistor array has a unique solution.



Proof: This works exactly as the last theorem. Note that all the nodes on the first layer in on a 4×4 resistor array are all on a first or second diagonal. Thus the two arrays must agree on the first layer in. Thus we have reduced to the 2×2 case. □

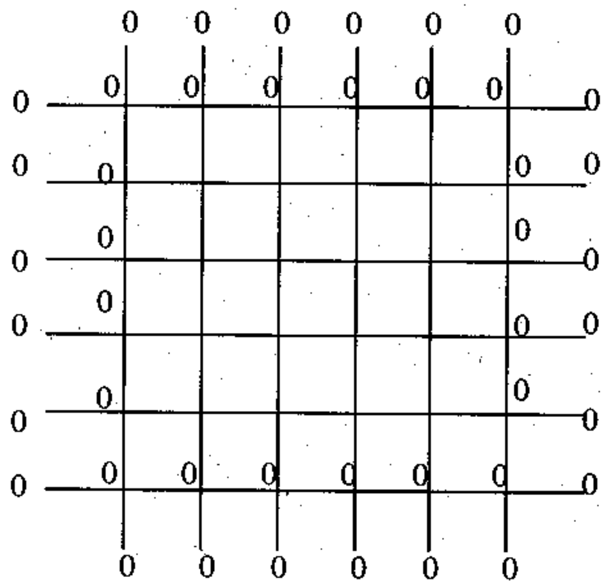
Now we present two incomplete induction proofs for the general uniqueness theorem.

Uniqueness Theorem: An $m \times m$ square resistor network has a unique solution.

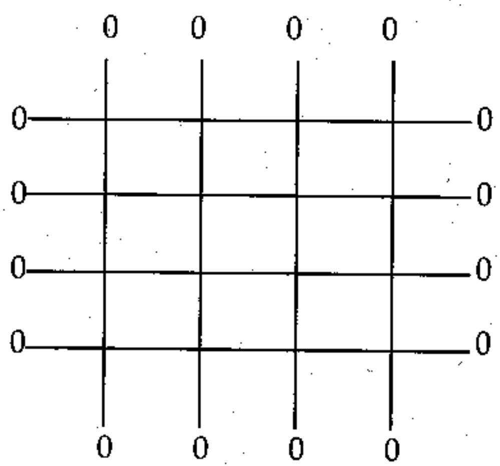
Sketch of Proofs: Suppose that we have two $m \times m$ arrays with the same, unit resistors and the same boundary information. We want to show that they have the same source configuration. There are several ways that one could prove this in general.

One possibility is an induction proof along the diagonals. Note that we have shown that the two arrays must agree on the first two diagonals. So suppose that we have shown that the two arrays cannot differ on any diagonal smaller than m. We want to show that they must agree on the mth diagonal. Note on the second diagonal, there is only one possibility for the source configurations in addition to case where the two arrays agree. On the third diagonal we get two additional possibilities. In general, there are a huge number of possible configurations to rule out. Thus taking a case-by-case approach would be too complicated, but perhaps there is a way to handle all the cases at once. Note that this proof would actually show that a rectangular network bounded on at least two adjacent sides had a unique solution.

We could also induct on the size of the array. Note that we have shown that the 1×1 and 2×2 cases are unique. Suppose that the conjecture is true for all cases smaller than the mxm case. Now we check for the mxm case itself.



reduces to



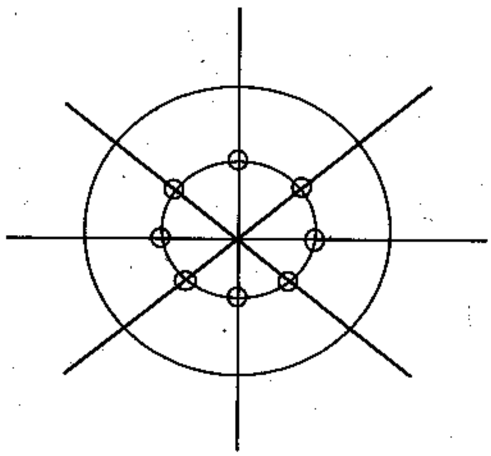
Suppose that we have two $m \times m$ resistor arrays with unit resistors and the same boundary information. Consider the difference array, Δ. Recall from above that Δ still has boundary voltages zero, and in addition it has boundary currents zero. Thus we have that all nodes on the first layer in have potential zero. So no current flows on the boundary spikes nor between nodes on the first layer in. Thus we can ignore those edges in our graph. We have reduced to the m-2 case. Note, however, that now there might be sources or sinks on the boundary of the difference array, which we never allowed before. So we must show that there cannot be any sources or sinks on the boundary nodes.

Consider n_{01}. We want to show that this is a harmonic node. So suppose that n_{01} is a source. That means $I_{01,11} = 1$. Because the boundary voltages are zero, we have $V_{11} = -1$. Thus $I_{10,11} = 1$. Note n_{11} can be a unit source, unit sink or a harmonic node. So at the very least we have either $I_{11,12} = 1$ or $I_{11,21} = 1$. Thus in the best case we have either $V_{12} = -2$ or $V_{21} = -2$, both of which force a boundary node to be a current source of two. Thus n_{01} cannot be a current source. By negating all the numbers, this also shows that n_{01} cannot be a sink. So we have that all boundary nodes on the corners must be harmonic.

Now we do another induction proof to show that all nodes on the boundary are harmonic. Again, an argument that applies in general was not found. But with that inductive step, this proof would be complete. □

5. Conclusion

Let us pause for a moment to consider what we have shown here. We have seen in a particular network, by taking some boundary measurements, we can uniquely determine the internal sources present, demanding only that the sources be uniform in strength. This is a somewhat astonishing result. Recall in the continuous analog, this result is just not true. Given the field produced. by some configuration outside a sphere, assuming that the configuration has uniform charge density, one cannot determine the shape of that configuration. Brodsky and Strakhov had to constrict themselves to lemniscates to get the uniqueness in the continuous case. We did not have to so constrict our view. Ah, but we *have* made some pretty rigid assumptions about our system; we have placed our restrictions on the network itself and not on the placement of the sources.



First we require that the network be rectangular. Consider the 'circular' array. That is, consider a network that is composed of at least two circles and any number of spokes. Suppose that we put unit current sources on the inner-most circle, as shown. Assuming that the resistors and sources have unit strength, this gives the same boundary currents as the case where we have current sources on any other circle. Parenthetically, if there were only one current source in the picture, we could use the first algorithm to locate it. So our general uniqueness proof is highly dependent on the geometry of the rectangular array. That is, we can prove the uniqueness for the rectangular array because it is highly asymmetric. One could ask for which other arrays can the uniqueness be proven. Perhaps this result is true for networks in which all nodes are of degree four. This question is completely open.

In addition to only considering rectangular arrays, we assumed that all the resistors and all the sources were of unit strength. As we stated above, having unit resistance is in keeping with the continuous case. In the standard continuous set-up, we have some source configuration confined to some sphere. We usually assume that this configuration is in empty space which has uniform, unit conductance. Thus in the discrete analog of this, we use uniform, unit resistors in our array (recall resistance is the inverse of conductance). Conversely, having non-uniform resistors in our array is analogous to placing our charge configuration not in empty space, but in an infinite, non-uniform conductor. Additionally, allowing only uniform current sources is equivalent to demanding that the charge configuration in the continuous realm have uniform charge density. This, too, is a standard necessary assumption; it is easy to formulate examples even in the rectangular array where our uniqueness does not hold when we allow different strengths of current sources.

So in fact our assumptions on the resistors and sources were not extreme in any sense. One still wonders if there are situations where we can relax these conditions and still have our desired result. Indeed in the case of one source and in the 1×1 and 2×2 cases, we need neither the size of the source nor uniform resistance. Are these the only cases? So in the end, we do have a rather interesting result with some room for further research.

References

[1] V. N. Strakhov and M. A. Brodsky, "On the Uniqueness of the Inverse Logarithmic Potential Problem", SIAM Journal of Applied Mathematics, Vol. 46, No. 2, April 1986, pp324-344.

[2] E. B. Curtis and J. A. Morrow, "Determining the Resistors in a Network", SIAM Journal of Applied Mathematics, Vol. 50 (1990), pp 918-930.

[3] Higginson, "Recoverable Sources in a (3x3) Rectangular Resistor Network", August 1993.

[4] R.J. Duffin, "Discrete Potential Theory", Duke Math. J. Vol 20, (1953),pp233-251.