THE INVERSE BOUNDARY PROBLEM FOR GENERAL PLANAR ELECTRICAL NETWORKS

DMITRIY LEYKEKHMAN

Abstract. We consider an electrical network where each edge is consists of resistor, inductor, and capacitor joined in parallel. We will make sense of Newmann-to-Dirichlet map for these networks. We will prove that any recoverable resistor network is still recoverable when capacitance and inductance are added in parallel. We will also give some examples of networks nonrecoverable as resistor networks but recoverable when inductor and capacitor are added to every edge.

1. Introduction

Let a graph with boundary be a triple $\Gamma = (V, \partial V, E)$, where (V, E) is a finite graph with the set of nodes V and the set of edges E. Let ∂V be a nonempty subset of V called the set of boundary nodes. To each edge we assign three numbers inductance L, capatance C, and resistance R joined in parallel connection. Suppose we apply an alternating voltage v on one of the boundary nodes with frequency ω and zero on the other nodes. Then at steady state according to laws of circuit theory all the nodes will have voltage with frequency ω [3]. Then I on any edge is the sum of the currents for the resistor, capacitor and inductor. From elementary circuit theory it's known that the current I going through a resistor is proportional to the voltage drop v on this edge,

$$I_R = \frac{1}{R}v$$

through a capacitor is proportional to derivative of voltage drop for this edge,

$$I_C = C\frac{\partial v}{\partial t}$$

and for an inductor the current is proportional to the antiderivative of voltage drop,

$$I_L = \frac{1}{L}\int_0^t v$$

Then the current going through a single edge is just the sum of them

$$I = I_R + I_C + I_L$$

$$I = \frac{1}{R}v + \frac{1}{L}\int_0^t v + C\frac{\partial v}{\partial t} \tag{1.1}$$

Let p be an interior node and q_j its neighbors. Suppose the voltage at node p is $v(p)$ and at nodes q_j are $v(q_j)$. Then at steady state, by Kirchhoff's law, the sum of the currents flowing out of interior node p is equal to zero.

$$\sum_{pq_j}\left(\frac{1}{R_j}(v(q_j)-v(p))+C_j\frac{\partial}{\partial t}(v(q_j)-v(p))+\frac{1}{L_j}\int_0^t(v(q_j)-v(p))\right)=0 \tag{1.2}$$

Suppose we apply a voltage in the form $e^{i\omega t}$ at one boundary node, where ω is frequency and t is time. Then the voltage is of the form $Ae^{i\omega t}$ at all interior nodes, where A is some complex function on the nodes. So we can rewrite the last equation in the following form:

$$\sum_{pq_j}\left(\frac{1}{R_j}\left(A(q_j)-A(p)\right)e^{i\omega t}+C_j\frac{\partial}{\partial t}\left(A(q_j)-A(p)\right)e^{i\omega t})+\frac{1}{L_j}\int_0^t\left(A(q_j)-A(p)\right)e^{i\omega t}\right)=0$$

$$=\sum_{pq_j}e^{i\omega t}\left(A(q_j)-A(p)\right)\left(\frac{1}{R_j}+i(C_j\omega-\frac{1}{L_j\omega})\right)=0 \tag{1.3}$$

$$\Rightarrow\sum_{pq_j}\left(A(q_j)-A(p)\right)\left(\frac{1}{R_j}+i(C_j\omega-\frac{1}{L_j\omega})\right)=0$$

We call

$$\gamma=\frac{1}{R}+i(\omega C-\frac{1}{\omega L})$$

an admittance defined on the edges, where $Re(\gamma)>0$ and $Im(\gamma)$ is some function of ω.

2. UNIQUINESS

We call a connected graph Γ with a function γ on its edges a network Γ_γ. Suppose we have a network with n nodes, m of which are boundary nodes. We number all nodes in the network in such a way that the first m nodes are boundary ones: $q_1, q_2, ..., q_m, q_{m+1}, ..., q_n$. Then we construct Kirchhoff's $n\times n$ matrix in the following way:

if $i\neq j$, then $k_{i,j}=-\sum\gamma(e)$, where the sum is taken over all edges e joining q_i to q_j (if there is no edge joining q_i to q_j, then $k_{i,j}=0$);

$k_{i,i}=\sum\gamma(e)$, where the sum is taken over all edges e with one end point at q_i and other endpoint not q_i.

From the construction of K it's clear that K is symmetric ($k_{i,j}=k_{j,i}$) and the sum of the elements of any row is zero ($\sum_i k_{i,j}=0\Rightarrow k_{i,i}=-\sum_{j\neq i}k_{i,j}$), K also has a block structure.

Suppose

$$K = \left(\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right) \tag{2.1}$$

where A is $m \times m$ submatrix representing a map for the boundary nodes. Define $\Lambda = A - B^T C^{-1} B$. This definition will be correct if C is nonsingular.

First we prove that the rank of K is $n-1$. Let x be a $n \times 1$ vector $[x_1, ..., x_n]$ such that $Kx = 0$. Then $\bar{x}^T Kx = 0$. It follows that

$$\bar{x}^T Kx = \sum_{i,j}^{n} \bar{x}_i k_{i,j} x_j \tag{2.2}$$

$$= \sum_{i \neq j}^{n} \bar{x}_i k_{i,j} x_j + \sum_{i=1}^{n} k_{i,i} |x_i|^2 = \sum_{i>j}^{n} k_{i,j} (\bar{x}_i x_j + \bar{x}_j x_i) + \sum_{i=1}^{n} k_{i,i} |x_i|^2$$

$$= \sum_{i>j}^{n} k_{i,j} (x_i - x_j)(\bar{x}_j - \bar{x}_i) = \sum_{i>j}^{n} -k_{i,j} (x_i - x_j)(\overline{x_i - x_j}) = -\sum_{i>j}^{n} k_{i,j} |x_i - x_j|^2 = 0$$

Since Γ_γ is a connected network, any node q_j can be connected to q_i and $Re(k_{i,j}) < 0$ for $i \neq j$ it follows that x must be a constant vector.

Suppose that $Cy = 0$, where $y = [y_1, ..., y_{m-n}]$ is some $(n-m) \times 1$ vector. We build a $n \times 1$ vector z that has the form $z = [0, ..., 0, y_1, ..., y_{m-n}]$. Then we have

$$\bar{z}^T Kz = \bar{y}^T Cy = 0 \tag{2.3}$$

From this it follows that z is constant vector, but z has zero entries, so y is the zero vector and C is nonsingular.

We proved only for the case when ω is real. But this result can extended for any ω in the complex plane, becuase determinant C can not be identical zero (we proved that C is non-singular for real ω) it can have only a finite number of zeros.

Suppose the voltage of the form $F_\partial = \begin{pmatrix} F_1 \\ \vdots \\ F_m \end{pmatrix} e^{i\omega t}$ is applied at the boundary nodes, and let $F_{int} = \begin{pmatrix} F_{m+1} \\ \vdots \\ F_n \end{pmatrix} e^{i\omega t}$ be the resulting potential at the interior nodes. Then at the steady state the potential inside satisfies a current version of Kirchhoff's law [3].

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} F_\partial \\ F_{int} \end{pmatrix} = \begin{pmatrix} G_\partial \\ 0 \end{pmatrix} \tag{2.4}$$

Since C is invertible, the Dirichlet problem with boundary data $F_\partial e^{i\omega t}$ has an unique solution.

$$\Rightarrow F_{int} = -C^{-1}B^T F_\partial$$

The current out of boundary nodes is given by

$$G_\partial = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} F_\partial \\ F_{int} \end{pmatrix} = \left(A - BC^{-1}B^T\right)(F_\partial) = \Lambda_\gamma(F_\partial) \tag{2.5}$$

We define the Dirichlet-to-Neumann map Λ_γ to be a linear map from the boundary potential of the form $F_\partial e^{i\omega t}$ to boundary currents $G_\partial e^{i\omega t}$.The matrix that represents Λ_γ can be calculated explicitly in terms of blocks of the K matrix.

$$\Lambda_\gamma = A - BC^{-1}B^T$$

for some γ.

3. IMPLICATIONS FROM RESISTOR NETWORK

The difference between the situation when there are only resistors in a network and when there are resistors, inductors, and capacitors is great. Any network that is recoverable in the first case is still recoverable in the second, because by choosing $\omega = -i\omega_0$,where $\omega_0 > 0$ our admittance γ is always bigger than zero

$$\gamma = \frac{1}{R} + i\left(C(-i\omega_0) - \frac{1}{L(-i\omega_0)}\right) = \frac{1}{R} + C\omega_0 + \frac{1}{L\omega_0} > 0.$$

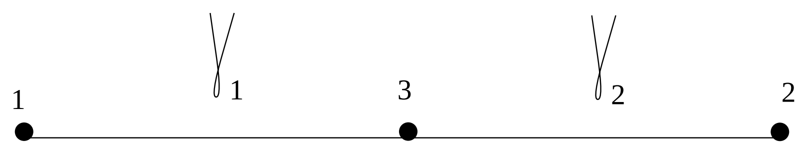
Thus we know γ for all pure imaginary ω with negative imaginary part. So we know γ for any ω.

In the last proof we didn't use the specific form of the γ. We used only the facts that γ is analytic and maps the negative imaginery part onto positive real part.

4. TWO IN SERIES

There are some cases, a resistor non-recoverable network can be recovered when inductor and capacitor are added.

We look at a graph with two edges connected in series between two boundary nodes. Assume that the first edge has conductivity γ_1, and the second γ_2.



Suppose that

$$\gamma_1(\omega) = \frac{1}{R_1} + i\left(\omega C_1 - \frac{1}{\omega L_1}\right)$$
$$\gamma_2(\omega) = \frac{1}{R_2} + i\left(\omega C_2 - \frac{1}{\omega L_2}\right)$$

and we are given a Λ matrix or in our case we know

$$\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right)^{-1} \; or \; \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right).$$

We rewrite the last expression in the simple fractions form:

$$\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) = \omega\left(\frac{1}{iC_1\omega^2 + \frac{1}{R_1}\omega - \frac{i}{L_1}} + \frac{1}{iC_2\omega^2 + \frac{1}{R_1}\omega - \frac{i}{L_1}}\right) \tag{4.1}$$

Let $\omega_1, \omega_{1'}$ be the zeros of the $\omega\gamma_1(\omega)$ and $\omega_2, \omega_{2'}$ be the roots of $\omega\gamma_2(\omega)$. Then the equation (3.1) has a unique representation in simple fractions. Let

$$D_1 = \frac{1}{R_1^2} - \frac{4C_1}{L_1}, \qquad D_2 = \frac{1}{R_2^2} - \frac{4C_2}{L_2}$$

be discriminants of $\omega\gamma_1(\omega)$ and $\omega\gamma_2(\omega)$, notice that D_1 and D_2 are real. Assume that $D_1, D_2 \neq 0$,then

(4.2)
$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = \omega\left(\frac{1}{\sqrt{D_1}}\left(\frac{1}{\omega-\omega_1} - \frac{1}{\omega-\omega_1\prime}\right) + \frac{1}{\sqrt{D_2}}\left(\frac{1}{\omega-\omega_2} - \frac{1}{\omega-\omega_2\prime}\right)\right)$$

We have several cases:

1.$D_1 \neq D_2$.

If $\omega_1 \neq \omega_2$ and $\omega_{1'} \neq \omega_{2'}$, or in other words, if γ_1 is not proportional to γ_2, then we can always distinguish $\omega_1, \omega_{1'}$ from $\omega_2, \omega_{2'}$ by letting ω approach ω_j, $(j = 1, 1', 2, 2')$ and see how fast $\frac{1}{\omega-\omega_j}$ goes to ∞, by the speed of $\sqrt{D_1}$ or $\sqrt{D_2}$. By the same approach we can determine D_1 and D_2. Thus we can find γ_1 and γ_2.

This method can be applied in the case that one of the discriminants is zero. One of the roots will be a double root and can always be distinguished from the other two.

If $\omega_1 = \omega_2$ and $\omega_{1'} = \omega_{2'}$, and γ_1 is proportional to γ_2 ($\gamma_1 = k\gamma_2$), then there are infinitely many solutions and we can always replace γ_1 and γ_2 by one conductor $\gamma = \frac{k+1}{k}\gamma_1$.

2.$D_1 = D_2 = D \neq 0$.

If $D < 0$ then $\sqrt{D}$ is pure imaginary and ω_1 and $\omega_{1'}$ are symmetrical with respect to the imaginary axis, so are ω_2 and $\omega_{2'}$. We can combine ω_1 with $\omega_{1'}$ and ω_2 with $\omega_{2'}$, so the solution is unique.

If $D > 0$, then we can rewrite (3.2) in the form

$$\frac{1}{\sqrt{D}}\left(\frac{1}{\omega-\omega_1} - \frac{1}{\omega-\omega_{1'}} + \frac{1}{\omega-\omega_2} - \frac{1}{\omega-\omega_{2'}}\right) \tag{4.3}$$

Notice that

$$\omega_j, \omega_{j'} = \frac{-\frac{1}{R_j} \pm \sqrt{\frac{1}{R_j^2} - \frac{4C_j}{L_j}}}{iC_j} \qquad , where\ j = 1, 2$$

thus $\omega_j, \omega_{j'}$ are pure imaginary with positive imaginary part. We know that if we approchimate ω_j from $+i\infty$ in (3.3) then two roots must have positive imaginery parts and the other two negative imaginery parts. Assume that $\omega_j = ia_j$, where $j = 1, 1', 2, 2'$. Then

$$\frac{1}{\omega - ia_j} - \frac{1}{\omega - ia_k} = \frac{a_k - a_j}{-i\omega^2 - (a_k + a_j)\omega + i(a_k a_j)}$$

Because the coefficient of ω^2 must be pure imaginary with positive imaginary part, it follows that $a_k - a_j$ must be negative. We have two cases:

a). if both $a_{1'}$ and $a_{2'}$ are greater than a_1 and a_2, then a_1 can be combined with either $a_{1'}$ or $a_{2'}$, so can a_3. Thus there are two solutions.

b). if one of the $a_{1'}$ or $a_{2'}$ is smaller or equal to a_1, or a_2, then in one of the combining we would get a wrong sign. Thus the solution is unique.

3.$D_1 = D_2 = 0$.

In this case $\omega_1 = \omega_{1'}$ and $\omega_2 = \omega_{2'}$, and

$$\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) = \omega\left(\frac{1}{iC_1(\omega-\omega_1)^2} + \frac{1}{iC_2(\omega-\omega_2)^2}\right)$$

. Then the possibilities are:

a). $\omega_1 = \omega_{1'} = \omega_2 = \omega_{2'}$, then γ_1 is proportional to γ_2, so there are infinitely many solutions.

b). $\omega_1 = \omega_{1'} \neq \omega_2 = \omega_{2'}$. In this case by letting ω approach ω_j ($j = 1, 1', 2, 2'$), we can find C_1 and C_2, so we can recover γ_1 and γ_2.

5. SIMILAR CONNECTIONS

The idea of connection of two conductors in series can be extended for any arbitrary number m.

Suppose we have conductors $\gamma_1, \dots, \gamma_m$, and suppose that $D_1, \dots, D_m \neq 0$ are discriminants and $\omega_1, \omega_{1'}, \dots, \omega_m, \omega_{m'}$ are zeros of $\omega\gamma_1, \dots, \omega\gamma_m$.Then

$$\frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_m}$$

has unique representation in simple fractions:

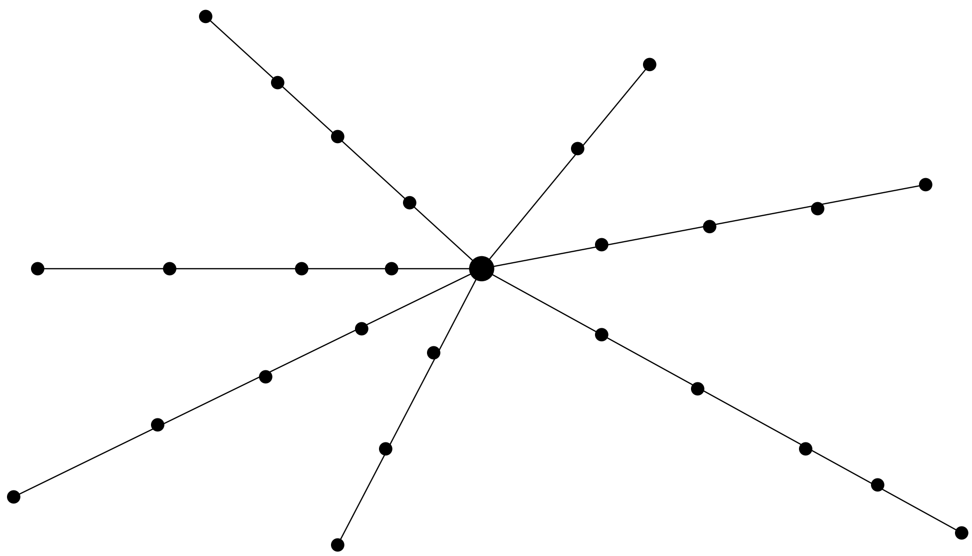
(5.1)
$$\frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_m} = \omega \left(\frac{1}{\sqrt{D_1}} \left(\frac{1}{\omega - \omega_1} - \frac{1}{\omega - \omega_{1'}} \right) + \cdots + \frac{1}{\sqrt{D_m}} \left(\frac{1}{\omega - \omega_m} - \frac{1}{\omega - \omega_{m'}} \right) \right)$$

There are many possibilities which are similar to the case when there are only two conductors joining in series. We wouldn't consider all the cases, they are rather tedious although they are can be handled by the same ideas as in the last section. The case when

$D_1 \neq \dots\dots \neq D_m$ and $\omega_1 \neq \dots \neq \omega_m$, or $\omega_{1'} \neq \dots \neq \omega_{m'}$ is recoverable because we can always distinguish ω_j from ω_k and $\omega_{j'}$ from $\omega_{k'}$, where $j, k = 1, \dots, m$ and $k \neq j$. We can also always combine ω_j with $\omega_{j'}$. By letting ω approach to $\omega - j$, we can find D_j. Knowing that we can find γ_j.

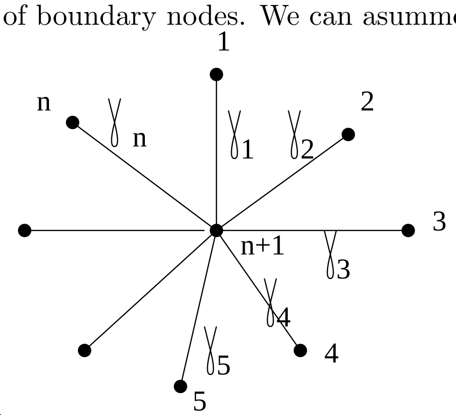
The case when $D_j = D_k$ is the same as in the last section. The cases when more than two discriminants are equal are rather messy and it's not the purpose of this paper to look at all of them.

Knowing how to recover m conductors joined in series allows us to recover the case of star shape connection like this:



where m rays are coming from one point, and each ray has n admittances in sries (n and m are arbitrary).

This star is also recoverable in the most cases. We numerate all nodes boundary nodes first and the central the last $1, 2, \ldots, n, n+1$. Let rays be $1, 2, \ldots, n$ by numbers of boundary nodes. We can asumme that each ray j



has one admittance γ_j.

Then the Kirchhoff's matrix is

$$\begin{pmatrix} \gamma_1 & 0 & \cdots & -\gamma_1 \\ 0 & \gamma_2 & \cdots & -\gamma_2 \\ \cdots & \cdots & \cdots & \cdots \\ -\gamma_1 & -\gamma_2 & \cdots & \Sigma \end{pmatrix}, \qquad where \qquad \Sigma = \gamma_1 + \gamma_2 + \ldots + \gamma_n.$$

Thus Λ matrix for this network is

$$\Lambda = \frac{1}{\Sigma} \begin{pmatrix} S_1 & -\gamma_1\gamma_2 & -\gamma_1\gamma_3 & \cdots & -\gamma_1\gamma_n \\ & S_2 & -\gamma_2\gamma_3 & \cdots & -\gamma_2\gamma_n \\ & & S_3 & \cdots & -\gamma_3\gamma_n \\ & & & \cdots & \cdots \\ & & & & S_n \end{pmatrix} \tag{5.2}$$

$$where\ S_k = \sum_{j,\, j\neq k}^{n} \gamma_k\gamma_j$$

We know that the Λ matrix is symmetric and determined by its above diagonal elements. Suppose

$$a_{1,2} = \frac{\gamma_1\gamma_2}{\Sigma}, \ldots, a_{i,j} = \frac{\gamma_i\gamma_j}{\Sigma}.$$

Then

$$\frac{\gamma_1\gamma_2\gamma_3}{\Sigma} = a_{1,2}a_{1,3} + a_{1,2}a_{2,3} + a_{1,3}a_{2,3} + a_{1,2}\sum_{j>3} a_{3,j}.$$

Thus

$$\gamma_1 = \frac{a_{1,2}a_{1,3} + a_{1,2}a_{2,3} + a_{1,3}a_{2,3} + a_{1,2}\sum_{j>3} a_{3,j}}{a_{2,3}},$$

$$\gamma_2 = \frac{a_{1,2}a_{1,3} + a_{1,2}a_{2,3} + a_{1,3}a_{2,3} + a_{1,2}\sum_{j>3} a_{3,j}}{a_{1,3}},$$

and

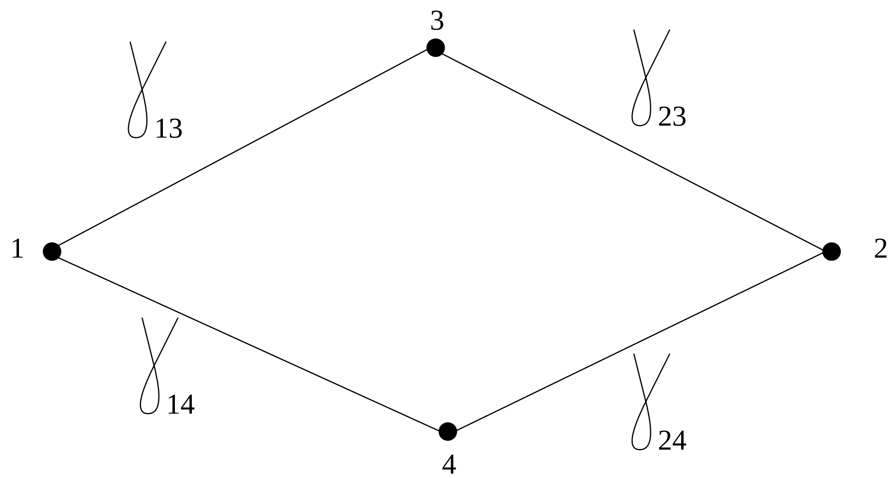
$$\gamma_3 = \frac{a_{1,2}a_{1,3} + a_{1,2}a_{2,3} + a_{1,3}a_{2,3} + a_{1,2}\sum_{j>3} a_{3,j}}{a_{1,2}}.$$

Similary we can find all γ_j, $j = 1, 2, \ldots, n$. Knowing this sum everything can be reduced to the case of m admittances joined in series.

The answer that this star is recoverable follows from section three and the fact that this star network is recoverable in the case of resistor network.

6. Example of nonrecoverable network

From the last examples it may seem that networks with admittances as conductivities can always be recovered in the most cases. It is not true. Example of network such that can be



where $1, 2$ are boundary nodes.

In this cases the idea of two condactors joined in parallel works. The Λ matrix of this network is 2×2 and determinited by one of diagonal element, say a_{12} that equal to:

$$a_{12} = \left(\frac{1}{\gamma_{23}} + \frac{1}{\gamma_{13}}\right)^{-1} + \left(\frac{1}{\gamma_{14}} + \frac{1}{\gamma_{24}}\right)^{-1} \tag{6.1}$$

We can rewrite (5.1) in the form

$$a_{12} = \frac{1}{\omega}\left(\frac{\gamma_{13}\gamma_{23}(\gamma_{14} + \gamma_{24}) + \gamma_{14}\gamma_{24}(\gamma_{13} + \gamma_{23})}{(\gamma_{13} + \gamma_{23})(\gamma_{24} + \gamma_{14})}\right) \tag{6.2}$$

Thus a_{12} is a rational function of ω: polynomial of degree 6 in numerator and a polynomial of degree 4 in denominator.

Suppose that $\omega_1, \ldots, \omega_6$ are roots of numerator, and $\omega_{1'}, \ldots, \omega_{4'}$ are roots of denominator. Let a be a leading term for the numerator and b for the dominator. Then we can rewrite (5.2) in the following form:

$$a_{12} = \frac{a}{b\omega}\left(\frac{(\omega - \omega_1)\ldots(\omega - \omega_6)}{\omega - \omega_{1'})\ldots(\omega - \omega_{4'})}\right) \tag{6.3}$$

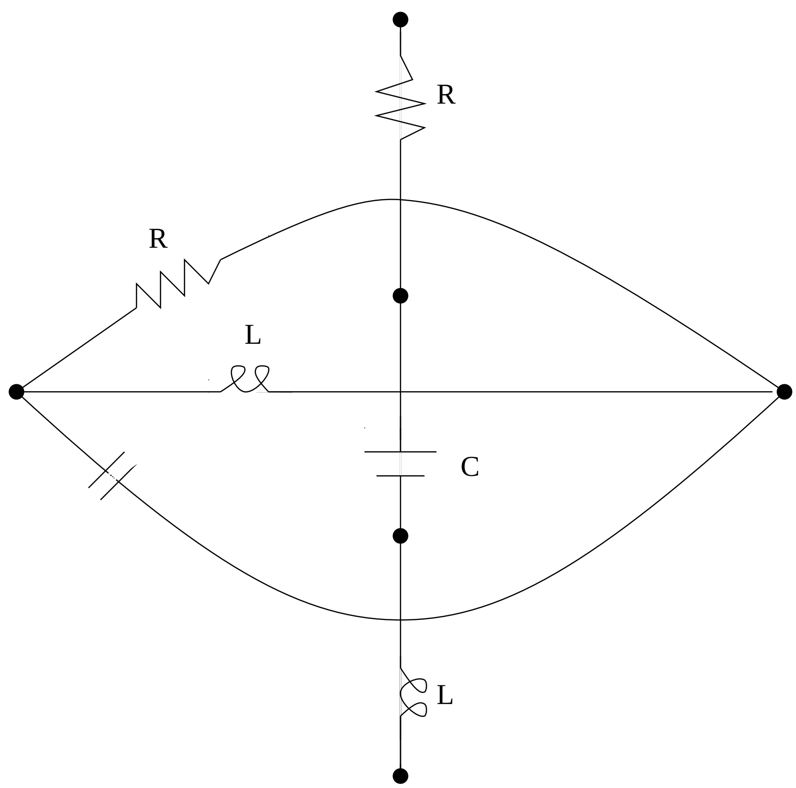
From expression (6.3) we have 11 pieces of information: 6 zeros, 4 poles, and the ratio $\frac{a}{b}$. On the other hand we have 12 unknown parametrs. So we don't have unough information for recovering all parametrs.

7. DUAL NETWORKS

The concept of duality for the resistor network has natural extension to the case when resistor R,capacitor C, and inductor L are joined in parallel connection. Using dual quantities in the table (7.1)

$$\begin{array}{cccccccc} Kirchhoff's\,Voltage\,Law & C & R & L & v & I & z & -series \\ Kirchhoff's\,Current\,Law & L & \frac{1}{R} & C & I & v & \gamma & -parallel \end{array} \tag{7.1}$$

we get that resistor, inductor, and capacitor joined in series is the dual to resistor, capacitor, and inductor joined in parallel.

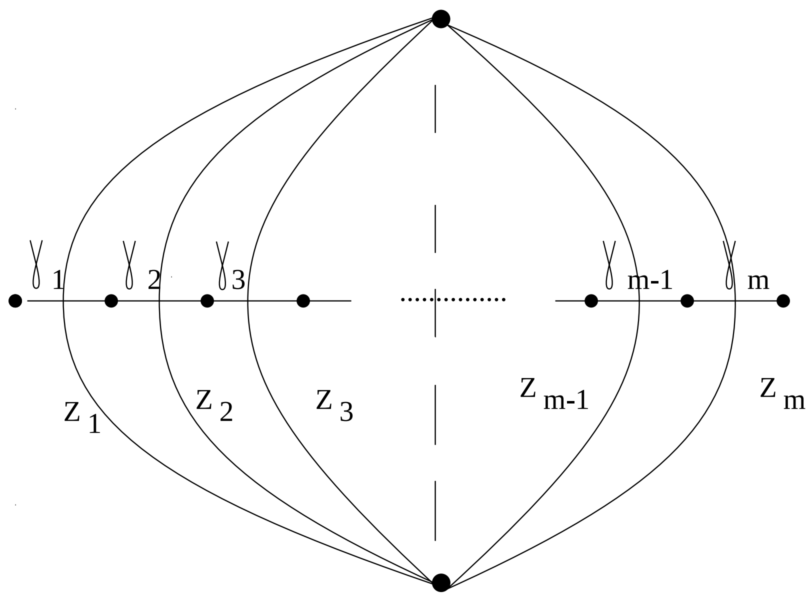


If instead of writting the integro-differential equation for the C, R, L joined in parallel connection we write the integro-differential equation for L, R, C joined in series, and instead of using Kirchhoff's Current Law we use Kirchhoff's Voltage Law, we get an impedance z in the form

$$z = R + i(L\omega - \frac{1}{C\omega})$$

on edges, where each edge is R, L, C joined in series.

When we recover one network, we automatically recover the dual one. Thus for example knowing the result for n admittances joined in series, we automatically know the result for n impendances joined in parallel.



References

[1] E. Curtis, D. Ingerman, J. Morrow, *Circular planar graphs and resistor networks*, submitted.

[2] E. Curtis, E. Mooers, J. Morrow, *Finding conductors in circular networks from boundary measurments*, Math. Modelling and Numerical Mathematical Modelling and Numerical Analysis 28 (1994), 781-813.

[3] Smith, *Circular devices and systems*

E-mail address: `dmitriy@math.washington.edu`