Parallel and Series Addition of Networks

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1 Introduction

In the past, work has been done by Anderson and Duffin on series addition of conductors in an electrical network. It is my aim to look at parallel and series additions of networks and to find interesting properties from the resulting Λ matrices.

I will be looking at the division of current in networks that result from the parallel addition of previously formed electrical networks and looking at the power dissipated by the resulting network as compared to the power dissipated by the component networks. The same will be done for the series addition of networks. Also a way to calculate the inverse of the Λ matrix in terms of the effective resistances is formed.

2 Parallel Addition

Let Γ_1 and Γ_2 be the graphs of networks 1 and 2 respectively, with boundary nodes $\{p_i^{(1)}\}_1^n, \{p_i^{(2)}\}_1^m$ and with $n = m$, taking the ordering of the boundary nodes to be counterclockwise around the network. We will define the parallel addition of networks 1 and 2 to be the connecting of the boundary nodes in such a way that $p_1^{(1)}$ is connected to $p_1^{(2)}$, $p_2^{(1)}$ is connected to $p_2^{(2)}$, and so on, as in Figure 1. Parallel addition of networks is defined only when the number of boundary nodes in network 1 is equal to the number of boundary nodes of network 2. Current flowing into the boundary nodes of the newly formed network now has a choice of which way to flow. The current may flow into either network 1 or network 2, although the total current flow cannot change. We will assume that the current will divide in such a way that the power dissipated is a minimum. Also, we will assume that there are

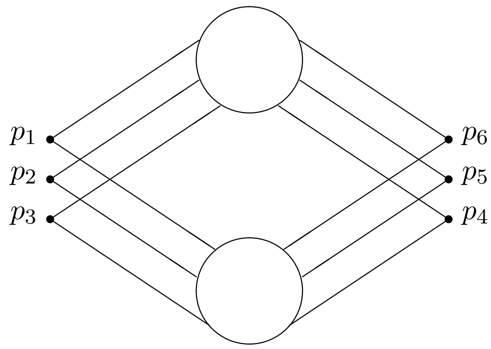

Figure 1: Parallel Addition of Networks

no sources or sinks inside either component network, so there are no sources or sinks in the network formed by their parallel addition.

Given a graph Γ, we will let Λ denote its associated Dirichlet to Neumann map. For our definition of parallel addition of resistor networks, the following is true:

Theorem 1 *Let Γ_1, Γ_2 be a pair of networks with associated Dirichlet to Neumann maps Λ_1, Λ_2 and boundary nodes $\{p_i^{(1)}\}_1^n, \{p_i^{(2)}\}_1^n$. Take Γ_P to be the graph resulting from the parallel addition of Γ_1 and Γ_2 as defined above. Then the map associated with Γ_P is:*

$$\Lambda_P = \Lambda_1 + \Lambda_2 \tag{1}$$

A Simple Example Consider two networks Γ_1 and Γ_2, each with two boundary nodes. Take the conductance between the two boundary nodes in network 1 to be a and the conductance between the two boundary nodes in network 2 to be b. Then $\Lambda_1 = \begin{pmatrix} a & -a \\ -a & a \end{pmatrix}$ and $\Lambda_2 = \begin{pmatrix} b & -b \\ -b & b \end{pmatrix}$. Taking the parallel addition of the two networks results in another network, Γ_P, with conductance $a + b$ between the two boundary nodes of this network. Therefore, $\Lambda_P = \begin{pmatrix} a+b & -a-b \\ -a-b & a+b \end{pmatrix} = \Lambda_1 + \Lambda_2$.

Proof. Let u_1 be the solution of the Dirichlet problem for Γ_1 and let u_2 be the solution of the Dirichlet problem for Γ_2. Since the current at the boundary nodes of the combined network is just the sum of the currents at the boundary nodes of the individual networks, the solution of the Dirichlet problem for Γ_P is $u_1 + u_2$. Therefore, $\Lambda_P = \Lambda_1 + \Lambda_2$.

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Lemma 1 $$(\Lambda_1+\Lambda_2)^{-1}=\Lambda_1^{-1}(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_2^{-1}$$

Proof.

$$\begin{aligned}(\Lambda_1+\Lambda_2)(\Lambda_1^{-1}(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_2^{-1}) &= (\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_2^{-1}+(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_1^{-1}\\ &= (\Lambda_1^{-1}+\Lambda_2^{-1})(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\\ &= I\end{aligned}$$

So $\Lambda_1^{-1}(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_2^{-1}=(\Lambda_1+\Lambda_2)^{-1}$.

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For notational simplicity, we will denote $\Lambda_1^{-1}(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_2^{-1}$ by $\Lambda_1^{-1}\|\Lambda_2^{-1}$.

Lemma 2 $$\Lambda_1^{-1}\|\Lambda_2^{-1}=\Lambda_2^{-1}\|\Lambda_1^{-1}$$

Proof.

$$\begin{aligned}\Lambda_1^{-1}\|\Lambda_2^{-1} &= \Lambda_1^{-1}(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_2^{-1}\\ &= (\Lambda_1^{-1}+\Lambda_2^{-1}-\Lambda_2^{-1})(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}(\Lambda_2^{-1}+\Lambda_1^{-1}-\Lambda_1^{-1})\\ &= (\Lambda_1^{-1}+\Lambda_2^{-1})(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}(\Lambda_2^{-1}+\Lambda_1^{-1})-\Lambda_2^{-1}(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}(\Lambda_2^{-1}+\Lambda_1^{-1})\\ &\quad -(\Lambda_1^{-1}+\Lambda_2^{-1})(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_1^{-1}+\Lambda_2^{-1}(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_1^{-1}\\ &= \Lambda_2^{-1}(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_1^{-1}\\ &= \Lambda_2^{-1}\|\Lambda_1^{-1}\end{aligned}$$

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Given a resistor network with associated Dirichlet to Neumann map Λ, the power dissipated by current w flowing through the boundary nodes of the network is given by $w^T\Lambda^{-1}w$.

Theorem 2 *For any x and y such that $x+y=z$, where z is the current on the boundary of the network formed by the parallel addition of Γ_1 and Γ_2,*

$$z^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z\le x^T\Lambda_1^{-1}x+y^T\Lambda_2^{-1}y \tag{2}$$

Proof. The minimum of $x^T\Lambda_1^{-1}x+y^T\Lambda_2^{-1}y$, when $x+y=z$, occurs when $x_o=(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_2^{-1}z$ and $y_o=(\Lambda_1^{-1}+\Lambda_2^{-1})^{-1}\Lambda_1^{-1}z$. So $\Lambda_1^{-1}x_o=(\Lambda_1^{-1}\|\Lambda_2^{-1})z$ and $\Lambda_2^{-1}y_o=(\Lambda_1^{-1}\|\Lambda_2^{-1})z$. Then $x_o^T\Lambda_1^{-1}x_o+y_o^T\Lambda_2^{-1}y_o=x_o^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z+y_o^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z=z^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z$. So given these values of x_o and y_o, equality holds.

Let $x = x_o + u, y = y_o - u, u \neq 0$. Then $x + y = x_o + y_o = z$. We want to show that $x^T\Lambda_1^{-1}x + y^T\Lambda_2^{-1}y > z^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z$.

$$\Lambda_1^{-1}x = \Lambda_1^{-1}x_o + \Lambda_1^{-1}u = (\Lambda_1^{-1}\|\Lambda_2^{-1})z + \Lambda_1^{-1}u$$
$$\Lambda_2^{-1}y = \Lambda_2^{-1}y_o - \Lambda_2^{-1}u = (\Lambda_1^{-1}\|\Lambda_2^{-1})z - \Lambda_2^{-1}u$$

$$\begin{aligned} x^T\Lambda_1^{-1}x + y^T\Lambda_2^{-1}y &= x^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z + x^T\Lambda_1^{-1}u + y^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z - y^T\Lambda_2^{-1}u \\ &= z^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z + x^T\Lambda_1^{-1}u - y^T\Lambda_2^{-1}u. \end{aligned}$$

In order to complete the proof, we must show that $x^T\Lambda_1^{-1}u - y^T\Lambda_2^{-1}u > 0$.

$$\begin{aligned} x^T\Lambda_1^{-1}u - y^T\Lambda_2^{-1}u &= x_o^T\Lambda_1^{-1}u + u^T\Lambda_1^{-1}u - y_o^T\Lambda_2^{-1}u + u^T\Lambda_2^{-1}u \\ &= z^T\Lambda_2^{-1}(\Lambda_1^{-1} + \Lambda_2^{-1})^{-1}\Lambda_1^{-1}u + u^T\Lambda_1^{-1}u \\ &\quad -z^T\Lambda_1^{-1}(\Lambda_1^{-1} + \Lambda_2^{-1})^{-1}\Lambda_2^{-1}u + u^T\Lambda_2^{-1}u \\ &= u^T\Lambda_1^{-1}u + u^T\Lambda_2^{-1}u > 0. \end{aligned}$$

So for any x, y so that $x + y = z$, $x^T\Lambda_1^{-1}x + y^T\Lambda_2^{-1}y \geq z^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z$.

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Since we are assuming that the current flow will divide itself in such a way that the power dissipated is minimum, the current will divide itself so that $x = (\Lambda_1^{-1} + \Lambda_2^{-1})^{-1}\Lambda_2^{-1}z$ and $y = (\Lambda_1^{-1} + \Lambda_2^{-1})^{-1}\Lambda_1^{-1}z$ to minimize the power dissipated by the network formed by parallel addition.

Since we are assuming that the networks contain no sources or sinks, $\sum z_i = 0$, where z is the current flow on the boundary of the network. Let $a_1 = z^T\Lambda_1^{-1}z$ and $a_2 = z^T\Lambda_2^{-1}z$. Clearly if $z = 0$, then $a_1 = 0$ and $a_2 = 0$. We will assume $z \neq 0$ so that $a_1 > 0$ and $a_2 > 0$. Then the following is true:

Corollary

$$z^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z \leq \left(\frac{1}{a_1} + \frac{1}{a_2}\right)^{-1} \tag{3}$$

Proof. Let $x = \left(\frac{a_2}{a_1+a_2}\right)z$ and $y = \left(\frac{a_1}{a_1+a_2}\right)z$.
Then $x + y = \left(\frac{a_2}{a_1+a_2}\right)z + \left(\frac{a_1}{a_1+a_2}\right)z = z$.
So by Theorem 2, $z^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z \leq x^T\Lambda_1^{-1}x + y^T\Lambda_2^{-1}y =$
$\left(\frac{a_2}{a_1+a_2}\right)z^T\Lambda_1^{-1}z\left(\frac{a_2}{a_1+a_2}\right) + \left(\frac{a_1}{a_1+a_2}\right)z^T\Lambda_2^{-1}z\left(\frac{a_1}{a_1+a_2}\right) =$
$\left(\frac{a_2}{a_1+a_2}\right)^2 a_1 + \left(\frac{a_1}{a_1+a_2}\right)^2 a_2 = \frac{a_1a_2(a_1+a_2)}{(a_1+a_2)^2} = \frac{a_1a_2}{a_1+a_2}$.
So $z^T(\Lambda_1^{-1}\|\Lambda_2^{-1})z \leq \frac{a_1a_2}{a_1+a_2} = \left(\frac{1}{a_1} + \frac{1}{a_2}\right)^{-1}$.

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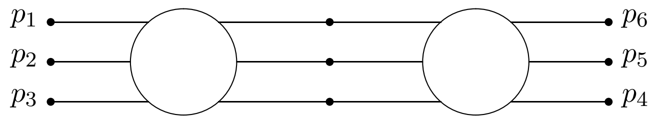

Figure 2: Series Addition of Networks

3 Series Addition

Let Γ_1 and Γ_2 be the graphs of networks 1 and 2 respectively, with boundary nodes $\{p_i^{(1)}\}_1^n, \{p_i^{(2)}\}_1^m$. Take the ordering of the boundary nodes to be counterclockwise around network 1 and clockwise around network 2. The boundary nodes of each network are divided into two subsets, those where the current flows into the network, and those where the current flows out of the network. Let $\{q_i^{(1)}\}_1^b$ and $\{q_i^{(2)}\}_1^k$ be the boundary nodes with current flowing into the network of networks 1 and 2, respectively. Let $\{r_i^{(1)}\}_1^d$ and $\{r_i^{(2)}\}_1^j$ be the boundary nodes of networks 1 and 2 with current flowing out of the network so that $b+d=n$ and $k+j=m$. The series addition of networks 1 and 2 is defined to be the connecting and interiorizing of the second subsets of boundary nodes of the two networks in such a way that $r_1^{(1)}$ is connected in series to $q_1^{(2)}$, $r_2^{(1)}$ is connected to $q_2^{(2)}$, and so on, as in Figure 2. For series addition of networks to be defined, the boundary nodes of the two networks must be divided in such a way that $d=k$. Our definition of the series addition of networks is analagous to $\Gamma_1 \bullet \Gamma_2$ as defined in Rosema.

Theorem 3 *Let Γ_1, Γ_2 be a pair of networks with associated Dirichlet to Neumann maps Λ_1, Λ_2, and boundary nodes $\{q_i^{(1)}\}_1^b, \{r_i^{(1)}\}_1^d, \{r_i^{(2)}\}_1^j, \{q_i^{(2)}\}_1^k$, with $d=k$. Let $\Lambda_1 = \begin{pmatrix} A_1 & B_1 \\ B_1^T & C_1 \end{pmatrix}$ and $\Lambda_2 = \begin{pmatrix} A_2 & B_2 \\ B_2^T & C_2 \end{pmatrix}$. Take Γ_S to be the graph resulting from the series addition of Γ_1 and Γ_2 as defined above. Then the map associated with Γ_S is:*

$$\Lambda_S = \begin{pmatrix} A_1 - B_1(C_1+C_2)^{-1}B_1^T & -B_1(C_1+C_2)^{-1}B_2^T \\ -B_2(C_1+C_2)^{-1}B_1^T & A_2 - B_2(C_1+C_2)^{-1}B_2^T \end{pmatrix} \tag{4}$$

Proof. This follows directly from Lemma 10 in Rosema.

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Let Γ be the graph of a network with intΓ representing the set of interior nodes and $\partial\Gamma$ representing the set of boundary nodes. Let $w_j = \sum_{i\sim j} w_{ji}, w_{ij} = -w_{ji}$, where $i \sim j$ are those nodes i that are neighbors of j. Also, let $W = \{w_{ij} : w_j = 0 \text{ for } j\epsilon \text{ int}\Gamma, w_j = \psi_j \text{ for } j\epsilon\partial\Gamma, \text{ where } \sum \psi_i = 0\}$, and $Q(w, w) = \sum r_{ij} w_{ij}^2 = \sum \frac{1}{\gamma_{ij}} w_{ij}^2$.

Theorem 4 *$Q(w, w)$ is minimized for that $\{w_{ij}\}\epsilon W$ such that $w_{ij} = \gamma_{ij}(v_i - v_j)$ for some $\{v_i\}$.*

Proof. Since the Neumann problem has a solution, such $\{v_i\}$ exist. Let $w_{ij} = \gamma_{ij}(v_i - v_j)$ and let $z_{ij}\epsilon W$, such that $z_{ij} = w_{ij} + x_{ij}$. Since $z_{ij}\epsilon W$, $z_i = 0$ at each interior node. Therefore, since $z_i = w_i + x_i$ and $w_i = 0$ at each interior node, $x_i = 0$ at each interior node. Also, $z_i = \psi_i$ at each boundary node. Since $w_i = \psi_i$ at each boundary node, $x_i = 0$ at each boundary node. Thus, $x_i = 0$ for all i.
$Q(z, z) = \sum \frac{1}{\gamma_{ij}} z_{ij}^2 = \sum \frac{1}{\gamma_{ij}} (w_{ij} + x_{ij})^2 = Q(w, w) + Q(x, x) + 2 \sum \frac{1}{\gamma_{ij}} w_{ij} x_{ij}$.
But $\sum_{i,j} \frac{1}{\gamma_{ij}} w_{ij} x_{ij} = \sum_{i,j} (v_i - v_j) x_{ij} = \sum_{i,j} v_i x_{ij} - \sum_{i,j} v_j x_{ij} =$
$\sum v_i x_i + \sum v_j x_j = 0$ since $x_i = 0$ for all i. So $Q(z, z) = Q(w, w) + Q(x, x)$.
Therefore, $Q(w, w) \leq Q(z, z)$.

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Let Γ_S be the network formed by the series addition of Γ_1 and Γ_2. Consider a given current z on the boundary, where the current flowing into the network flows out of the network divided into the exact same values as it entered (the current flowing on the boundary of the network is $z = [x_1 \cdots x_n, -x_1 \cdots - x_n]^T$). Then the following is true:

Theorem 5

$$z^T \Lambda_S^{-1} z \leq z^T \Lambda_1^{-1} z + z^T \Lambda_2^{-1} z \tag{5}$$

Proof. Follows from Theorem 4.

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4 Inverse for the Λ Matrix

Let $\Lambda : \Re^n \longrightarrow \Re^n$ be the Dirichlet to Neumann map for a network with n boundary nodes. Since we are considering only those networks that contain no sources or sinks, $W = Im(\Lambda)$, where $W = \{w = [w_1 \cdots w_n]^T : w_1 + \cdots + w_n = 0\}$. Let e be the $n \times 1$ matrix containing all 1's. Then $ker(\Lambda) = \{te\}$. Let $R : \Re^n \longrightarrow \Re^n$ represent Λ^{-1}.

Theorem 6 *$R = \left(-\frac{(\rho_{ij})}{2}\right)$ acts as Λ^{-1}, where ρ_{ij} is the effective resistance between nodes i and j.*

Proof. From Lemma 2 in Duffin, there exists constants ρ_{ij} such that $v_i = \frac{-1}{2}\sum_{j=0}^{n} \rho_{ij} w_j + c$, where c does not depend on i. Also, $\rho_{ij} = \rho_{ji}, \rho_{jj} = 0$, and $\rho_{ij} > 0$ for $i \neq j$. It is later shown in Duffin's paper that these constants are the effective resistances for the network. So $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} \rho_{11} & \cdots & \rho_{1n} \\ \vdots & \ddots & \vdots \\ \rho_{n1} & \cdots & \rho_{nn} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} + ce$. Therefore $Rw = v - ce$. So $\Lambda R w = \Lambda(v - ce) = \Lambda v = w$ for all $w \epsilon W$ and $\Lambda R = I$ on W.

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Although Rw, where w is a current on the boundary of a network, only determines the boundary voltages v up to a constant, R may be used in place of Λ^{-1} when calculating the power dissipated by the network.

Theorem 7

$$v^T \Lambda v = w^T R w \tag{6}$$

Proof.

$$\begin{aligned} v^T \Lambda v &= v^T w \\ &= w^T v \\ &= w^T (Rw - ce) \\ &= w^T R w, \text{ since } \textstyle\sum w_i = 0. \end{aligned}$$

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Consider a network with n boundary nodes. The following is an algorithm for finding ρ_{ij}, where ρ_{ij} is the effective resistance between boundary nodes i and j.

1. Apply a voltage of 1 at boundary node i and voltage of 0 at node j. Insulate the other $n-2$ boundary nodes so that current can flow only through nodes i or j. Permute the Λ matrix so that node i is the first entry in the matrix and node j the second, followed by the other $n-2$ nodes. The Λ matrix may be divided into submatrices so that $\Lambda = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$, where A is 2×2, B is $2 \times n-2$, B^T is $n-2 \times 2$, and C is $n-2 \times n-2$.

2. Now $\Lambda v = w$, where $v = (1\ 0\ x_1 \cdots x_{n-2})^T$ and $w = (w_{ij}\ -w_{ij}\ 0 \cdots 0)^T$. Therefore, by solving the last $n-2$ equations of $\Lambda v = w$, you may find the voltages at the other boundary nodes. This amounts to solving the equation $B^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Cx = 0$, which has a unique solution since C is invertible. Therefore, the unknown boundary voltages are $x = -C^{-1}B^T \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

3. We know that $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Bx = \begin{pmatrix} w_{ij} \\ -w_{ij} \end{pmatrix}$. Substituting the known voltages for x, we get $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} - BC^{-1}B^T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} w_{ij} \\ -w_{ij} \end{pmatrix}$.

4. Using Ohm's Law, you get $1 = I\rho_{ij}$ or $\rho_{ij} = \frac{1}{I}$. So $A - BC^{-1}B^T$ will produce a 2×2 matrix $\begin{pmatrix} w_{ij} & -w_{ij} \\ -w_{ij} & w_{ij} \end{pmatrix}$. ρ_{ij} is just $\frac{1}{w_{ij}}$.

5. Once ρ_{ij} is found, ρ_{ji} is known. Also, we know that $\rho_{ii} = 0$, so these values need not be calculated. Once ρ_{ij} is found for every pair i, j, the matrix R may be easily calculated.

Consider a network with four boundary nodes. R, representing the inverse of the Λ matrix, will be a 4×4 symmetric matrix. Let the entries of R be denoted by r_{ij}, where $r_{ij} = \left(-\frac{(\rho_{ij})}{2}\right)$. Because of the properties of the effective resistance of a network, $r_{ii} = 0$ for all i. Thus, $R = \begin{pmatrix} 0 & r_{12} & r_{13} & r_{14} \\ r_{12} & 0 & r_{23} & r_{24} \\ r_{13} & r_{23} & 0 & r_{34} \\ r_{14} & r_{24} & r_{34} & 0 \end{pmatrix}$. This matrix may be divided into four 2×2 matrices so that $R = \begin{pmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{pmatrix}$.

The power dissipated by this network by a current $w = \begin{pmatrix} z \\ -z \end{pmatrix} = (x_1\ x_2\ \ -x_1\ \ -x_2)^T$ is:

$$
\begin{aligned}
w^T R w &= (z^T \;\; -z^T) \begin{pmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{pmatrix} \begin{pmatrix} z \\ -z \end{pmatrix} \\
&= (z^T R_1 - z^T R_2^T \;\; z^T R_2 - z^T R_3) \begin{pmatrix} z \\ -z \end{pmatrix} \\
&= z^T R_1 z - z^T R_2^T - z^T R_2 z + z^T R_3 z \\
&= z^T Q z, \text{ where}
\end{aligned}
$$

$$
Q = R_1 - R_2^T - R_2 + R_3 = \begin{pmatrix} -2r_{13} & r_{12} - r_{23} - r_{14} + r_{34} \\ r_{12} - r_{23} - r_{14} + r_{34} & -2r_{24} \end{pmatrix}.
$$

For a network with 6 boundary nodes with current flowing out exactly the same way as it came in, the power dissipated by the network can be represented by $z^T Q z$, where $Q = R_1 - R_2^T - R_2 + R_3 =$

$$
\begin{pmatrix} -2r_{14} & r_{12} - r_{24} - r_{15} + r_{45} & r_{13} - r_{34} - r_{16} + r_{46} \\ r_{12} - r_{15} - r_{24} + r_{45} & -2r_{25} & r_{23} - r_{35} - r_{26} + r_{56} \\ r_{13} - r_{34} - r_{16} + r_{46} & r_{23} - r_{26} - r_{35} + r_{56} & -2r_{36} \end{pmatrix}.
$$

The entries of Q are the transfer resistances as defined in Duffin. For the network with 6 boundary nodes, the i, j entry of Q is the transfer resistance between node pairs i, j and $i+3, j+3$. The transfer resistance between boundary node pairs a, b and c, d is ρ_t, where $v_c - v_d = J\rho_t$ when a current J enters the network at a and leaves at b, all other boundary nodes being insulated.

$$
\rho_t = \frac{\rho_{ad} + \rho_{bc} - \rho_{ac} - \rho_{bd}}{2}
$$

It follows from direct calculation that, in general, for a network with n boundary nodes with current flowing out exactly the same way as it came in, the power dissipated by the network can be represented by $z^T Q z$, where Q is the $\frac{n}{2} \times \frac{n}{2}$ matrix where the i, j entry of Q is the transfer resistance between nodes i, j and nodes $i + \frac{n}{2}, j + \frac{n}{2}$.

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