RECOVERING THE GEOMETRY OF CIRCULAR PLANAR RESISTOR NETWORKS FROM BOUNDARY MEASUREMENTS

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Abstract. The following is a step by step algorhithm used to recover the "unknown" geometry of a circular planar graph. The bulk of this project is devoted towards **Theorem 4.1.10**, which relates connections between two boundary nodes in circular planar graph to the rank of a submatrix in the Dirichlet-to-Neumann map denoted by Λ. Briefly, the Dirichlet-to-Neumann map is a function which relates boundary information to the interior of a circular planar resistor network. More information regarding the Dirichlet-to-Neumann map can be found in [1]. A computer program writen in *Mathematica* 3.0 accompanies this presentation in section 6, as well as a complete *Mathematica* "package" format including examples of Dirichlet-to-Neumann maps with graphic displays of the resulting circular planar graph at the end of this paper.

1. Introduction. A graph with a boundary is a triple $\Gamma = (V, E, \partial V)$, where Γ is a finite graph with V = the set of nodes, E = the set of edges where the conductivity γ acts, and ∂V = the non-empty subset of V called the boundary nodes where the current I is induced. Γ is allowed to have multiple edges (i.e., more than one edge between two nodes) or loops (i.e., an edge joining a node to itself). Within the content of this paper, we will not be looking at loops, since in previous articles, it was noted that loops can be eliminated to produce electrically equivalent graphs.

A circular planar graph is a graph with a boundary which is embedded in a disc **D** in the plane so that the boundary nodes lie on the circle C which bounds **D**, and the rest of Γ is in the interior of **D**. The boundary nodes will be labelled $v_1, \ldots, v_n$ in the (clockwise) circular order around C. A pair of sequences of boundary nodes $(A,B) = (a_1, \ldots, a_k, b_1, \ldots, b_k)$ such that the entire sequence $(a_1, \ldots, a_k, b_1, \ldots, b_k)$ is in circular order, will be called a circular pair. Note that in section 5, we will want to seperate (or divide) the circular pair (A,B) by a set of intervals denoted (I_i, I_j) such that $i \neq j$ and $i < j$. This notion will be clear later on.

A circular pair (A,B) of boundary nodes is said to be connected through Γ if there are k disjoint paths $\alpha_1, \ldots, \alpha_k$ in Γ, such that α_i starts at a_i, ends at b_i, and passes through no other boundary nodes. We say that α is a connection from A to B.

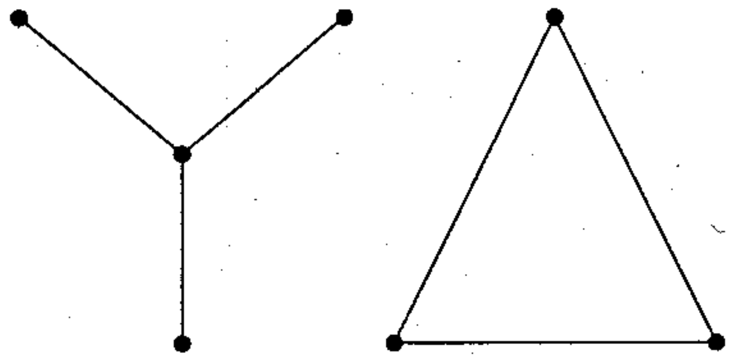
For each circular planar graph Γ, let $\pi(\Gamma)$ be the set of all circular pairs (A,B) of boundary nodes which are connected through Γ.

Recall there are two ways in which we can remove an edge from a graph Γ. First, we can delete an edge. Second, we can contract an edge to a single node. (An edge joining two boundary nodes is not allowed to be a contracted to a single node.)

We say that removing an edge breaks the connection from A to B if there is a connection from A to B through Γ, but there is not a connection from A to B after the edge is removed. A graph Γ is called **critical** if the removal of any edge breaks some connection in $\pi(\Gamma)$. *The final result of this paper is to produce a critical graph including all interior nodes and edges by simply gathering all necessary information at the boundary of the graph.* Think of a "fortune teller" predicting the shape of an object concealled within a foggy crystal ball by simply feeling the texture of its surface.

A graph Γ remains critical under Y - Δ equivalence transformations. Briefly, a Y - Δ equivalence is a geometric transformation shown below which maintains electrical equivalence since we replace three edges by three edges. For more information regarding the properties of Y - Δ equivalences in Γ, please see [1].

Y - Δ equivalence transformation in Γ



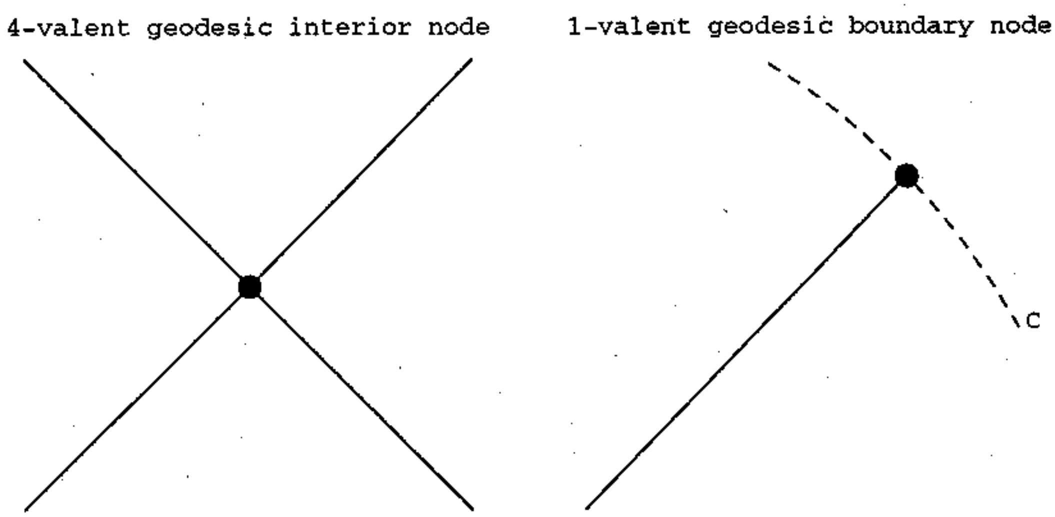
A <u>conductivity</u> on a graph Γ is a function γ which assigns to each edge $e \in E$ a positive real number $\gamma(e)$. A <u>resistor network</u> (Γ, γ) consists of a graph with a boundary together with a conductivity function γ. This paper makes no attempt to recover conductivities from boundary measurements. Therefore, we will not talk much about conductivities, except in the examples which conclude this paper. However, it should be noted that there is a linear map from boundary functions to boundary functions defined as follows. For each voltage potential $f = \{f(v_i)\}$ defined at the boundary nodes, there is a unique extension of f to all the nodes of Γ which satisfies Kirchoff's current law,

$$\Sigma_{q \in N(p)} \gamma(pq)\,(f(q) - f(p)) = 0$$

where $N(p)$ represents all neighboring nodes to p and $p \in V$, and $q \in \partial V$ or V. This function then gives a current $I = \{I(v_i)\}$ into the network at the boundary nodes. The linear map which sends f to I is called the Dirichlet-to-Neumann map and is represented by an $n \times n$ matrix denoted by Λ.

2. Medial Graphs. We will investigate the key formula of this paper, namely, *R(A) = card(A) - black(A) - max(A,B)*, where *max(A,B)* is the rank of a particular submatrix within the Λ matrix and (A,B) represents the circular pairs as defined in section 1.

A <u>medial graph</u> *M* is a circular planar graph such that its boundary nodes are 1-valent and its interior nodes are 4-valent.



The name "medial" comes from the following construction that for each circular planar graph, Γ produces a corresponding medial graph $M(\Gamma)$.

Suppose $\Gamma = (V, E, \partial V)$ is a circular planar graph with n boundary nodes. Γ is assumed to be embedded in the closed unit disk **D** so that the boundary nodes $v_1,\ldots,v_n$ occur in clockwise order around a circle $C = \partial\mathbf{D}$ and the rest of Γ is in the interior of **D**. The medial graph M(Γ) depends on the embedding. First, for each edge e of Γ, let m_e be its midpoint. Next, place $2n$ boundary points $t_1,\ldots,t_{2n}$ on C so that

$$t_1 < v_1 < t_2 < t_3 < v_2 < \ldots < t_{2n-1} < v_n < t_{2n} < t_1$$

in the clockwise circular order around C.

(1) The vertices of M(Γ) consist of the points m_e for $e \in E$, and the points t_i for $i = 1,\ldots,2n$.

(2) The edges in M(Γ) are as follows. Two vertices m_e and m_f are joined by an edge whenever e and f have a common vertex and e and f are incident to the same face in Γ. There is also one edge for each point t_j as follows. The point t_{2i} is joined by an edge to m_e where e is the edge of the form $e = v_i r$ which comes first after the arc $v_i t_{2i}$ in clockwise order around v_i. The point t_{2i-1} is joined by an edge to m_f where f is the edge of the form $f = v_i s$ which comes first after the arc $v_i t_{2i-1}$ in clockwise order around v_i.

The vertices of the form m_e of M(Γ) are 4-valent; the vertices of the form t_i are 1-valent.

An edge uv of a medial graph M has a direct extension vw if the edges uv and vw separate any other two edges incident to the vertex v. A path $u_0 u_1 \ldots u_k$ in M is called a geodesic arc if each edge $u_{i-1} u_i$ has edge $u_i u_{i+1}$ as a direct extension. A geodesic arc $u_0 u_1 \ldots u_k$ is called a geodesic if either

(1) u_0 and u_k are points on the circle C.

or

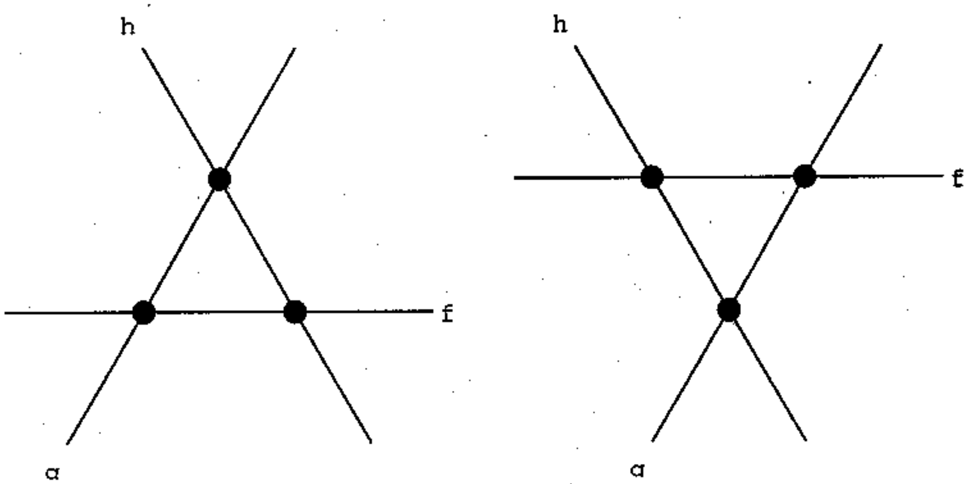
(2) $u_k = u_0$ and $u_{k-1} u_k$ has $u_0 u_1$ as a direct extension.

If each geodesic in M begins and ends on C, has no self-intersection, and if M has no lenses, we will say that M is lensless. For our purposes, we will only be looking at lensless graphs. For more information on lenses and various Lemmas associated to electrical equivalency of medial graphs with lenses, please see [2], section 4.1.2.

A triangle in M is a triple $\{f,g,h\}$ of geodesics which intersect to form a triangle with no other intersections within the configuration.

Suppose $\{f,g,h\}$ form a triangle. A motion of $\{f,g,h\}$ consists of interchanging the configuration as shown below.

A motion of f,g,h in the Medial Graph



Lemma 4.1.1. *Two circular planar graphs are Y-Δ equivalent if and only if their medial graphs are equivalent under motions.*

Proof. Each Y-Δ transformation of Γ corresponds to a motion on M(Γ). Conversely, a motion on M(Γ) corresponds to a Y-Δ transformation of Γ. ■

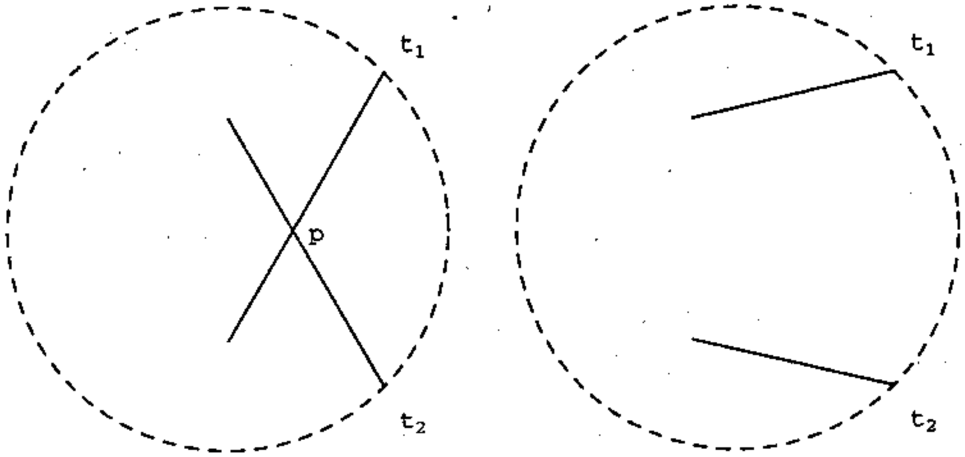
3. Z-Sequences. We begin this section with the study of the Z-sequence for a particular medial graph, M(Γ). And although we do not directly compute the Z-sequence within the computer algorithm, we mention it solely to provide a more detailed presentation of medial graphs.

Let M be a medial graph. Then M will have n geodesics each of which intersect C twice. The n geodesics intersect C in $2n$ distinct boundary points. These $2n$ points are labelled $t_1, \ldots t_{2n}$, so that

$$t_1 < t_2 < t_3 < \ldots < t_{2n-1} < t_{2n} < t_1$$

are in circular order around C. The geodesics will be labelled as follows. Let g_1 be the geodesic which begins at t_1. The remaining geodesicsare labelled $g_2, g_3, \ldots, g_n$ so that if $i < j$, then the first point of intersection of g_i with C occurs before the first point of intersection of g_j with C in clockwise order starting from t_1. For each $i = 1,2,\ldots,2n$ let z_i be the number associated with the geodesic which intersects C at t_i. In this way we obtain a sequence $z(M) = z_1, z_2, \ldots, z_{2n}$, called the Z-sequence for M. Each of the numbers from 1 to n occurs in Z-sequence for M exactly twice.

Windings and Unwindings in the Medial Graph

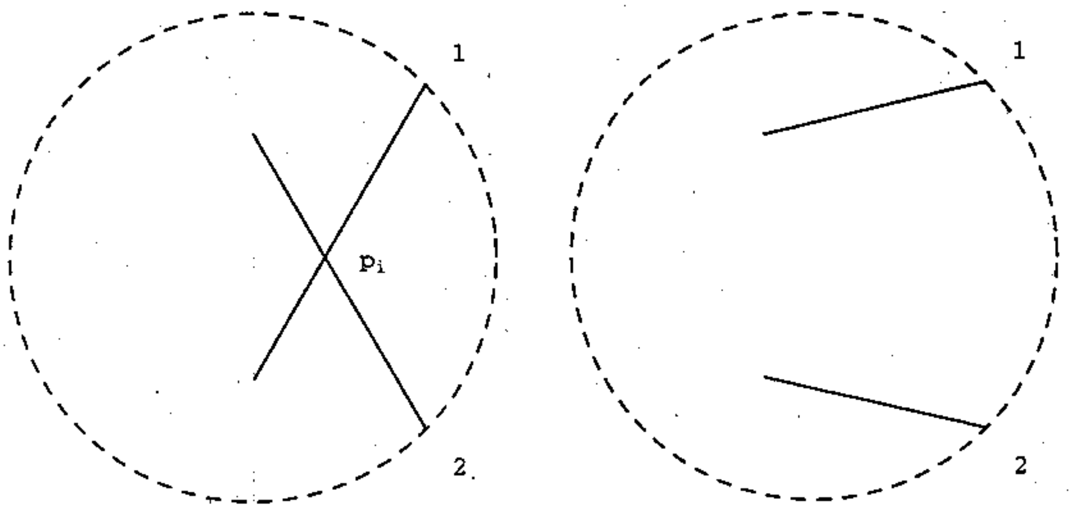


The tranformation above from left to right will be called unwinding between t_1 and t_2. The inverse of this transformation, defined if the geodesics from t_1 and t_2 are different and do not intersect in a lensless graph, will be called winding between t_1 and t_2. After winding or unwinding, the medial graph is still lensless and its Z-sequence changes by one transposition.

Lemma 4.1.6. *Two lensless medial graphs M_1 and M_2 are equivalent under motions if and only if the Z-sequence of M_1 equals the Z-sequence of M_2.*

Proof. Obviously, motions of a medial graph do not change its Z-sequence.

We show the other direction by an induction on the number of interior nodes of the medial graphs. Clearly, the lemma is true if M_1 or M_2 have no interior vertices. Now, suppose tghey have at least one. Then not all geodesics in M_1 or M_2 are parallel. WLOG we can assume that none of the geodesics of M_1 or M_2 terminate at two adjacent boundary nodes, that is there are no two equal adjacent symbols in the Z-sequence of M_1 or the Z-sequence of M_2. Therefore, WLOG we can assume that the geodesics that go through boundary nodes 1 and 2 intersect in an interior vertice p_i in M_i, $i = 1,2$. By a finite sequence of motions all other geodesics can be moved out of the triangle $1,2,p_i$. Therefore, WLOG the medial graphs look like the following figure near the boundary vertices 1 and 2.



The unwinding transformation above produces two new lensless medial graphs with equal Z-sequences. By the inductive statement, since these new medial graphs have fewer interior vertices, they are equivalent under motions, and therefore, so are the original graphs. ■

4. Connections and Z-sequences. Key Identity. Let Γ be a circular planar graph. A *path* β between boundary nodes a and b of Γ is either and edge (ab) or a sequence of interior nodes $p_1, \ldots, p_m$ such that

$$(ap_1), (p_1\, p_2), \ldots, (p_{m-1}\, p_m), (p_m\, b)$$

are edges of Γ.

A disjoint connection α between two disjoint k-tuples of boundary nodes $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ is a set of pairwise disjoint paths α_i between the a_i's and b_i's.

The following theorem, proved in [1], shows that the existense of disjoint connections between non-interlacing k-tuples of boundary nodes of Γ on C can be read directly from a Dirichlet-to-Neumann map Λ.

Theorem 4.1.7. (see [1]) *Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be a disjoint pair of non-interlacing boundary nodes of Γ. Then there is a disjoint connection between the a_i's and b_i's if and only if*

$$\det\{\Lambda(a_i, b_j)\} \neq 0.$$

This states that the determinante of the submatrix in Λ formed by the rows a_i and the columns b_j is not equal to zero.

We now extend the notion of disjoint connections to medial graphs, M(Γ).

A face of medial graph M is a connected component of $\mathbf{D} - M$. Due to the valences of the nodes in M one can color the faces of M in black and white so that no two faces with the same edge are of the same color (the so called two-coloring). If $M = \mathrm{M}(\Gamma)$ then one can choose the two-coloring of M so that a face is black if and only if it contains a node of Γ. Let us call this coloring induced.

The boundary nodes of M split C in into $2n$ intervals, namely, $I_1, I_2, \ldots, I_{2n}$. A two-coloring of M induces a two-coloring of the intervals.

For the remainder of this section, let c and d be two points in two distinct intervals I_k and I_j. Let $C - \{c,d\} = A \cup B$ where A and B are connected disjoint geodesic arcs. Let I and J be two black intervals on the boundary such that $I \subset A$ and $J \subset B$. A path G between I and J is a sequence of black faces $F_1, \ldots, F_m$ such that $I \in F_1$, $J \in F_m$, $F_i \cap F_{i+1} \neq \emptyset$, $F_2, \ldots, F_{m-1} \cap C = \emptyset$ and c and d are not in the closures of the F_i's.

Let I_i and J_i be two disjoint k-tuples of the black intervals, such that $I_i \subset A$ and $J_i \subset B$. A <u>disjoint connection</u> between the I_i' s and the J_i' s is a sequence of pairwise disjoint paths G_i between the I_i' s and the J_i' s.

The definitions above are chosen so that the following lemma is true.

Lemma 4.1.8. *Let Γ be a circular planar graph. Suppose M = M(Γ) is its medial graph with the induced coloring. Let* $\{a_i\} \in A$ *and* $\{b_i\} \in B$ *be two disjoint k-tuples of boundary nodes of Γ. Let* I_i *and* J_i *be corresponding black intervals. Then there is a disjoint connection between the* a_i' *s and the* b_i' *s if and only if there is a disjoint connection between the* I_i' *s and the* J_i' *s.*

Proof. From the construction of M(Γ) one has 1-1 correspondence between the nodes of Γ and black faces of M(Γ). Moreover, the interior nodes of Γ correspond to black faces that do not touch *C*. This induces the 1-1 correspondecne between disjoint connections in Γ and M(Γ). ■

We have by **Theorem 4.1.7.** that a Dirichlet-to-Neumann map Λ gives complete information about disjoint connections between non-interlacing k-tuples of boundary intervals in M(Γ). The following identity provides a link between existence of the disjoint connections and the Z-sequence of a medial graph *M*.

For the purposes of the algorithim in this paper, the following identity is more suited to explain the link between disjoint connections through the interior of Γ and the rank of a particular submatrix in Λ.

Theorem 4.1.10. *(Key Identity) Let <u>card(A)</u> be the number of boundary nodes of M in A. Let <u>Black(A)</u> be the number of black intervals in A. Let <u>Max(A,B)</u> be the size of the biggest disjoint connection between a set of black intervals in A and a set of black intervals in B = C - A. Note: Max(A,B) is the rank of a particular submatrix of Λ associated to the choice of intervals, (A,B). Let R(A) be the number of geodesics that start and terminate at A (i.e. returning geodesics for A). Then for a particular mapping Λ, the following identity holds.*

$$R(A) = card(A) - Black(A) - Max(A,B)$$

Proof. Let t_1 and t_2 be two adjacent boundary nodes of *M* so that $\{t_1, t_2\} \subset A$ or *B*. The following observation is crucial: Each of the four elements in the above equation do not change with respect of windings or unwindings of the geodesics for a particular interval *A*. By a sequence of such windings and unwindings M can be transformed to a lensless medial graph without interior nodes. For the graph without interior nodes, the identity is trivially true. By the observation, it is true for a general lensless medial graph. ■

5. Setting up the Algorithm. For the remainder of this paper, we change notation from above only slightly to suit the needs of the computer algorithm. We begin by assuming that the given Λ matrix is a matrix for a circular planar resistor network. We set up the black and white intervals along the perimeter of *C*, I_i, where $i = 1,2,...,2n$ corresponding to the number of boundary nodes, *n*, in sequential order such that I_1 begins before and next to I_2, and I_2 contains the first boundary node, $\delta\Gamma_1$. All intervals, I_i, occur clockwise around the unit circle. Even intervals contain boundary nodes (the black intervals) and odd intervals do not (the white intervals).

Now that we have established the perimeter of our resistor network, we take specific "cuts" or divisions in two selected intervals corresponding to $\{(I_i,I_j) \in I_{2n} \mid i \neq j, i < j\}$, where upon we use the program to find the value of R(A) for a particular division say, (I_{i_0},I_{j_0}). We find the value of R(A) for all such divisions along M(Γ) of which there are precisely $\frac{2n(2n-1)}{2}$ different choices, where once again, *n* represents the number of boundary nodes.

Now that we have established which intervals $\{(I_i,I_j) \in I_{2n} \mid i \neq j, i < j\}$ have geodesic arcs connected through the interior of M(Γ), we proceed to locate which geodesic vertices are connected to each other. We do this by first locating an interval (I_{i_0},I_{j_0}) which has a value for $R(A) \neq 0$. Since $i < j$ and $R(A) \neq 0$, we know that there must be at least one pair of geodesic vertices that are connected to each other. Let the geodesic vertex between I_{j_0} and I_{j_0-1} be denoted as the <u>interior geodesic vertex</u> to the interval I_{j_0}. Therefore, in order to find which geodesic vertex is connected to this interior vertex of I_{j_0}, we "advance" the interval I_{i_0} one by one (i.e. $i_{0+1}, i_{0+2},..., i_{0+k}$), until R(A) changes value at the k^{th} "advanced" interval. From this, we know that the geodesic vertex interior to I_{j_0} is connected to the geodesic vertex between intervals I_{k-1} and I_k.

We repeat this process until we know which geodesic vertices are connected to one another.

The final step is to draw in geodesic arcs. The result is a medial graph which, when given a two-coloring, is equivalent to a critical resistor network under Y-Δ transformations. ■

6. Recovery Algorhithm. The program below uses *Mathematica* 3.0 (or 2.2 without the special font and function formats).

Part 1: Finding the number of connections through the interior of Γ.

The user input of the Λ matrix. Below is an example of $\Lambda^{4\times 4}$.

```
A = Table[a[i, j], {i, 4}, {j, 4}];
MatrixForm[A]
```

$$\begin{pmatrix} a[1,1] & a[1,2] & a[1,3] & a[1,4] \\ a[2,1] & a[2,2] & a[2,3] & a[2,4] \\ a[3,1] & a[3,2] & a[3,3] & a[3,4] \\ a[4,1] & a[4,2] & a[4,3] & a[4,4] \end{pmatrix}$$

Using the formula from **Theorem 4.1.10**, R(A) = card(A) - Black(A) - Max(A,B), we set up the function "*intervalValues*" to find the value of R(A) for each interval $\{(I_i,I_j) \in I_{2n} \mid i \neq j,\ i < j\}$ on the perimeter of M(Γ). There are precisely $\frac{2n(2n-1)}{2}$ different choices for divisions along the perimeter of M(Γ).

```
modulo[x_, m_] := If[Mod[x, m] == 0, x, Mod[x, m]];

Attributes[modulo] = {Listable};
```

Here we define "*maxAB*" to be the Max(A,B), where the numerical value of Max(A,B) is determined by the rank of the submatrix of Λ consisting of those boundary nodes of Λ which satisfy the following criteria:

- If we take a cut along the perimeter of M(Γ) in an even interval, then we exclude the node contained within this interval (since for all boundary nodes, n, $2n \in$ within an even interval) from the submatrix whose rank is to be determined.
- All boundary nodes contained within (I_i,I_j), for example $I_i < n_1, \ldots, n_k < I_j$, these nodes will designate the sequence of rows of the submatrix. All other nodes will designate the sequence of columns.
- If any entire sequence of row entries or column entries is empty, then the rank of the submatrix formed by them will be zero.

```
maxAB[i_, j_] := If[Range[⌊j/2⌋ + 1, ⌊(i + 1 + (2 × Length[A]))/2⌋ - 1] == {},
  0, (Length[Transpose[A[[modulo[Range[⌊i/2⌋ + 1, ⌊(j + 1)/2⌋ - 1], Length[A]],
     modulo[Range[⌊j/2⌋ + 1, ⌊(i + 1 + (2 × Length[A]))/2⌋ - 1], Length[A]]]]]] -
    Length[NullSpace[A[[modulo[Range[⌊i/2⌋ + 1, ⌊(j + 1)/2⌋ - 1], Length[A]],
     modulo[Range[⌊j/2⌋ + 1, ⌊(i + 1 + (2 × Length[A]))/2⌋ - 1], Length[A]]]]])];
```

Here is the function, R(A) = card(A) - Black(A) - Max(A,B), where R(A) is applied to all "*intervals*" in order to determine the "*intervalValues*."

```
RofA[i_, j_] := (j - i) - (⌊j/2⌋ - ⌊(i+1)/2⌋ + 1) - maxAB[i, j];

Off[Transpose::nmtx];
Off[NullSpace::matrix];
Off[General::stop];

intervalValues = Table[RofA[i, j], {i, 1, 2×Length[A]},
  {j, i+1, 2×Length[A]}];
```

Below are the "*intervalValues*" determined from R(A) corresponding to the above "*intervals*," from which, we will now determine the **z-sequence.**

```
MatrixForm[intervalValues]
```

$$\begin{pmatrix} \{0,0,0,0,1,2,3\} \\ \{0,0,0,0,1,2\} \\ \{0,0,0,0,1\} \\ \{0,0,0,0\} \\ \{0,0,0\} \\ \{0,0\} \\ \{0\} \\ \{\} \end{pmatrix}$$

Part 2: Graphics.

Below we define the functions for graphics. "*bNode*" and "*bullwinkle*" lists the number of boundary nodes, n, for Γ. "*rocky*" lists the intervals, $\{(I_i,I_j) \in I_{2n} \mid i \neq j, i < j\}$, for dividing up the perimeter of $M(\Gamma)$ in order to determine R(A). "*fixedValues*" determines random fixed positions of the geodesic verticies in order to prevent triple intersections of the geodesic arcs. If, when running the program, a triple intersection "appears" to have occurred, simply run "*fixedValues*" once again to select a different orientation of the geodesic verticies.

```
bNode = Table[{PointSize[0.025], Point[{Cos[2 k π/Length[A]], -Sin[2 k π/Length[A]]}]},
  {k, 0, Length[A] - 1}];

bullwinkle =
 Table[Text[δΓ_(k+1), {(1.1) Cos[2 k π/Length[A]], -(1.1) Sin[2 k π/Length[A]]}], {k, 0, Length[A] - 1}];

rocky = Table[Text[I_(k+2), {(1.3) Cos[2 k π/(2×Length[A])], -(1.3) Sin[2 k π/(2×Length[A])]}],
  {k, -1, 2×Length[A] - 2}];

geoV[n_] := Table[
  {Cos[(2 π ⌊k/2⌋ + (-1)^(k+1) × ϵ)/n], -Sin[(2 π ⌊k/2⌋ + (-1)^(k+1) × ϵ)/n]} /. ϵ → Random[Real, {0.5, 1.5}],
  {k, 0, (2×n) - 1}];

fixedValues = geoV[Length[A]];

geodesicVerticies = Thread[Point[fixedValues]];
```

Part 3: Finding specific geodesics.

Using the column vector from "intervalValues" we select rows $\{(g,h) \mid h = g+1, g \neq \emptyset\}$ corresponding to a geodesic vertex, take the difference in "intervalValues," and determine when the difference is not equal to zero. Next, we select the first element in which the statement is true. This value corresponds to a different geodesic vertex. Together, these two geodesics vertices correspond to a single geodesic curve connected through M(Γ). The process is then repeated until all geodesic curves have been located in M(Γ).

```
usedVerticies = {};

skipTest[k_] := If[(MemberQ[connection[k], True] ∧
      FreeQ[usedVerticies, Part[fixedValues, k]]), k, skipTest[k + 1]] /.
  usedVerticies → {AppendTo[usedVerticies, Part[fixedValues, k]]}

firstValues = Table[skipTest[k], {k, 1, Length[A] - 1}];

firstPairs = Table[{Part[fixedValues, Part[firstValues, k]],
    Part[fixedValues, Length[Take[connection[Part[firstValues, k]],
        Position[connection[Part[firstValues, k]], True, 1, 1][[1, 1]]]] +
      Part[firstValues, k]]},
  {k, 1 Length[A] - 1}];

geodesicPaths = Thread[Line[firstPairs]];

lastLine = Line[Complement[fixedValues, fixedValues ∩ Flatten[firstPairs, 1]]];
```

Part 4: The medial graph.

Below is the graphics output of the program showing M(Γ) together with the boundary nodes, $\delta\Gamma$, and the intervals $\{(I_i,I_j) \in I_{2n} \mid i \neq j, i < j\}$ used to divide the perimeter of M(Γ).

```
Show[Graphics[{bNode, rocky, bullwinkle,
    geodesicPaths, lastLine, {PointSize[0.015], geodesicVerticies},
    {Circle[{0, 0}, 1]}}, PlotLabel → "Medial Graph of A", AspectRatio → 1]]

End[]
```

7. Conclusion. Unfortunately, a command to find the **rank** of a matrix is unavailable in *Mathematica*. Therefore, we had to define a clumsy rank function of our own in order to make the necessary computations. The rank function involves two commands known to *Mathematica,* (namely Transpose and NullSpace) however, when the rank of an empty matrix must be computed, the program errors on trying to Transpose an empty matrix or find the NullSpace of an empty matrix. Therefore, one will note the suppression of error messages at the beginning of the program (namely, Off[*message*]). This did not cause any difficulty in the computing of the interval values however, since we take the rank of an empty matrix to be zero. Here the sacrifice is only a complete and compact program.

Another limitation is in the graphics of the resulting medial graph. It would have been convenient to "two-color" the medial graph in order to establish the black and white intervals as well as the dual graph. Nothing is sacraficed except convience. One only needs to understand the proceedure of coloring the proper regions with a pencil or pen.

Finally, the program is unable to distinguish between planar Λ matrices, non-planar Λ matrices, or random valued matrices for that matter. In effect, the program will create a critical medial graph for *any* square matrix, preferably larger than $\Lambda^{1\times1}$. The former case being an isolated boundary node. Therefore, the program does not take into account the effects of signs of determinants within the Λ matrix as discussed earlier and in [1]. Recall that the signs of the determinants of submatrices within the Λ matrix is of the form, ++,--,++,--,... . What the meaning of the medial graph associated to an arbritary matrix generated by the above program is unknown.

REFERENCES

1. E. B. Curtis, D. Ingerman, and J. A. Morrow. *Circular Planar Graphs and Resistor Networks.*

2. D. Ingerman. *Thesis Presentation*, (section 4).

EXAMPLES

The following examples were done in *Mathematica* 3.0 with the package *MedialGraph.m*. To call the package from the internal files, use the command <<MedialGraph`. Then type ?MedialGraph if you do not understand what the function MedialGraph[Λ] does. You can input any matrix (preferably a Λ matrix as described in section 1 of the writen presentation).

First I had to add my working file to the *Mathematica* $Path directory.

```
PrependTo[$Path, "My-Mac-Files"];
```

This calls for the package *MedialGraph.m*.

```
<< MedialGraph`
```

Here is the input of a Kirchoff matrix with the conductivities, $\gamma = 1$ for all edges in the graph. The resulting Λ matrix is below as well.

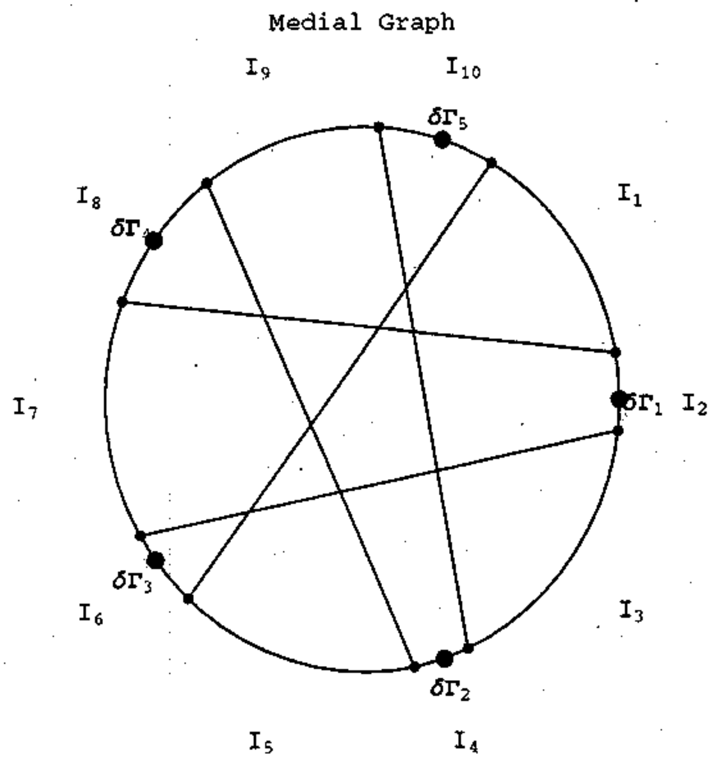
$$K = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & -1 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 \\ -1 & 0 & 0 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & 0 & 0 & -1 & 3 \end{pmatrix};$$

```
A = K[[{1, 2, 3, 4, 5}, {1, 2, 3, 4, 5}]];
B = K[[{1, 2, 3, 4, 5}, {6, 7}]];
BT = Transpose[B];
P = K[[{6, 7}, {6, 7}]];
PI = Inverse[P];
Λ = A - B . PI . BT;
MatrixForm[Λ]
```

$$\begin{pmatrix} \frac{30}{11} & -\frac{12}{11} & -\frac{1}{11} & -\frac{3}{11} & -\frac{14}{11} \\ -\frac{12}{11} & \frac{18}{11} & -\frac{4}{11} & -\frac{1}{11} & -\frac{1}{11} \\ -\frac{1}{11} & -\frac{4}{11} & \frac{7}{11} & -\frac{1}{11} & -\frac{1}{11} \\ -\frac{3}{11} & -\frac{1}{11} & -\frac{1}{11} & \frac{8}{11} & -\frac{3}{11} \\ -\frac{14}{11} & -\frac{1}{11} & -\frac{1}{11} & -\frac{3}{11} & \frac{19}{11} \end{pmatrix}$$

As the package promisses, here is the resulting Medial Graph for Λ. All that remains is to two-color the graph and connect corresponding faces for the Y - Δ equivalent graph, Γ.

```
MedialGraph[Λ]
```



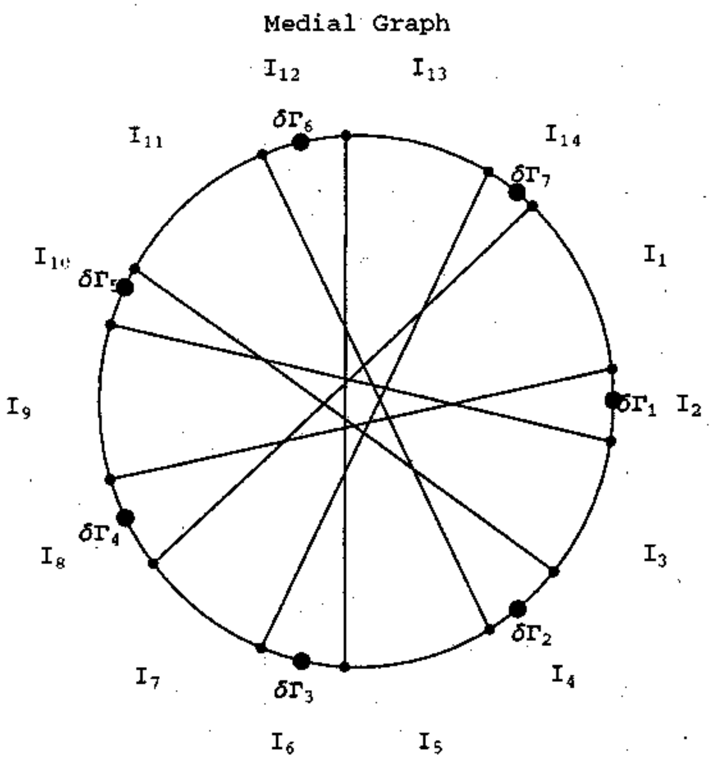
Here's a different Kirchoff matrix with $\gamma =$ to a set of **prime conductivities** with the resulting Medial Graph. Note that since both the Kirchoff and Λ matrices are large, we omit the display of both.

```
FullForm[K]
```

```
List[List[3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3, 0, 0, 0, 0],
 List[0, 12, 0, 0, 0, 0, 0, 0, 0, 0, 0, -5, -7, 0, 0, 0],
 List[0, 0, 24, 0, 0, 0, 0, 0, 0, 0, 0, 0, -11, 0, 0, -13],
 List[0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3],
 List[0, 0, 0, 0, 12, 0, 0, 0, 0, 0, 0, 0, 0, 0, -7, -5],
 List[0, 0, 0, 0, 0, 24, 0, -13, 0, 0, 0, 0, 0, 0, -11, 0],
 List[0, 0, 0, 0, 0, 0, 7, -7, 0, 0, 0, 0, 0, 0, 0, 0],
 List[0, 0, 0, 0, 0, -13, -7, 23, -3, 0, 0, 0, 0, 0, 0, 0],
 List[0, 0, 0, 0, 0, 0, 0, -3, 13, -5, 0, 0, 0, 0, -5, 0],
 List[0, 0, 0, 0, 0, 0, 0, 0, -5, 15, -7, 0, 0, -3, 0, 0],
 List[0, 0, 0, 0, 0, 0, 0, 0, 0, -7, 31, -11, -13, 0, 0, 0],
 List[-3, -5, 0, 0, 0, 0, 0, 0, 0, 0, -11, 19, 0, 0, 0, 0],
 List[0, -7, -11, 0, 0, 0, 0, 0, 0, 0, -13, 0, 42, -11, 0, 0],
 List[0, 0, 0, 0, 0, 0, 0, 0, 0, -3, 0, 0, -11, 34, -7, -13],
 List[0, 0, 0, 0, -7, -11, 0, 0, -5, 0, 0, 0, 0, -7, 30, 0],
 List[0, 0, -13, -3, -5, 0, 0, 0, 0, 0, 0, 0, 0, -13, 0, 34]]
```

The Kirchoff matrix above is converted to a Λ matrix. The resulting Medial Graph is below. Note how larger matrices cause a greater probability of triple intersections within the interior of the medial graph. Recall that triple intersections are not allowed, and in fact, they do not exist in the graph below due to the random assignments of the geodesic vertices.

```
MedialGraph[A]
```



The *Mathematica* Package

```
(* :Title:  MedialGraph.m *)

(* :Context:  MedialGraph` *)

(* :Author:  Chris Staskewicz *)

(* :Summary:
 This package was developed for the REU program at the University of Washington,
 Seattle involving resistor networks.  This
  package makes use of the Theorm 4.1 .10 proven by David Ingerman,
 which states that connections of resistors through the interior of a
  resistor network is based upon specific ranks of the A matrix. *)

(* :Mathematica Version:  3.0 or 2.2 *)

(* :Limitations:
 Since no rank function is built into Mathematica, we had to create our own rank
  function which evaluates submatricies of the A matrix.  Unforntuately, we must
  evaluate the empty matrix at certain steps of this program, where upon, Mathematica
  returns error messages about Transpose and NullSpace functions.  Therefore,
 we had to supress these error messages for a smooth running program.  See
  the first few lines under the `Private` context. *)
```

```
(* :Thanks to:  Jim Morrow, David Ingerman, and Jack Lee *)

BeginPackage["MedialGraph`"]

MedialGraph::usage = "MedialGraph[A_] takes a Λ matrix imput and plots
   the resulting medial graph with boundary nodes and selected intervals."

Begin["`Private`"]

MedialGraph[A_] :=
 Module[{g, h, i, j, k, m, n, x, e, intervalValues, bNode, bullwinkle, rocky, fixedValues,

   geodesicVerticies, usedVerticies, firstValues, firstPairs, geodesicPaths, lastLine}, (

Off[Transpose::nmtx];

Off[NullSpace::matrix];

Off[General::stop];

modulo[x_, m_] := If[Mod[x, m] == 0, x, Mod[x, m]];

Attributes[modulo] = {Listable};

RofA[i_, j_] :=
    (j - i) - (⌊j/2⌋ - ⌊(i + 1)/2⌋ + 1) - (If[Range[⌊j/2⌋ + 1, ⌊(i + 1 + (2 × Length[A]))/2⌋ - 1] == {},
      0, (Length[Transpose[A[[modulo[Range[⌊i/2⌋ + 1, ⌊(j + 1)/2⌋ - 1], Length[A]],
     modulo[Range[⌊j/2⌋ + 1, ⌊(i + 1 + (2 × Length[A]))/2⌋ - 1], Length[A]]]]]] -
    Length[NullSpace[A[[modulo[Range[⌊i/2⌋ + 1, ⌊(j + 1)/2⌋ - 1], Length[A]],
      modulo[Range[⌊j/2⌋ + 1, ⌊(i + 1 + (2 × Length[A]))/2⌋ - 1], Length[A]]]]]])]);

intervalValues = Table[RofA[i, j], {i, 1, 2 × Length[A]},
  {j, i + 1, 2 × Length[A]}];

bNode = Table[{PointSize[0.025],
     Point[{Cos[2 k π / Length[A]], -Sin[2 k π / Length[A]]}]}, {k, 0, Length[A] - 1}];

bullwinkle = Table[
    Text[δΓ_{k+1}, {(1.1) Cos[2 k π / Length[A]], -(1.1) Sin[2 k π / Length[A]]}], {k, 0, Length[A] - 1}];

rocky = Table[Text[I_{k+2},
     {(1.3) Cos[2 k π / (2 × Length[A])], -(1.3) Sin[2 k π / (2 × Length[A])]}], {k, -1, 2 × Length[A] - 2}];
```

```
geoV[n_] := Table[{Cos[(2 π ⌊k/2⌋ + (-1)^(k+1) × ε)/n], -Sin[(2 π ⌊k/2⌋ + (-1)^(k+1) × ε)/n]} /.
    ε → Random[Real, {0.5, 1.5}], {k, 0, (2 × n) - 1}];

fixedValues = geoV[Length[A]];

geodesicVerticies = Thread[Point[fixedValues]];

connection[g_] := Table[Part[intervalValues, g, h] - Part[intervalValues, g + 1, h - 1] ≠ 0,
    {h, 2, Length[intervalValues] - g}];

usedVerticies = {};

skipTest[k_] := If[(MemberQ[connection[k], True] ∧
        FreeQ[usedVerticies, Part[fixedValues, k]]), k, skipTest[k + 1]] /.
    usedVerticies → {AppendTo[usedVerticies, Part[fixedValues, k]]}

firstValues = Table[skipTest[k], {k, 1, Length[A] - 1}];

firstPairs = Table[{Part[fixedValues, Part[firstValues, k]],
     Part[fixedValues, Length[Take[connection[Part[firstValues, k]],
         Position[connection[Part[firstValues, k]], True, 1, 1][[1, 1]]]] +
       Part[firstValues, k]]},
    {k, 1 Length[A] - 1}];

geodesicPaths = Thread[Line[firstPairs]];

lastLine = Line[Complement[fixedValues, fixedValues ⋂ Flatten[firstPairs, 1]]];

Show[Graphics[{bNode, rocky, bullwinkle,
     geodesicPaths, lastLine, {PointSize[0.015], geodesicVerticies},

  {Circle[{0, 0}, 1]}}, AspectRatio -> 1, PlotLabel -> "Medial Graph"]]]

End[]

Protect[MedialGraph]

EndPackage[ ]
```