VARIOUS COUNTING OPERATIONS ON CIRCULAR PLANAR GRAPHS

Tarn Adams

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1 Introduction

In [1], several concepts were developed. Among these there are some that represent countable objects, such as the number of 4-node Y-Δ equivalence classes or the number of graphs in a given Y-Δ equivalence class. The goal of this paper is to provide means for counting many such quantities, either by algorithm or by recursive formula.

2 Preliminary Material

For a more complete exposition of the following material, see [1]. Much of this section follows the form presented in [2] quite closely.

2.1 Graph with Boundary

Let a graph with boundary be a triple $\Gamma = (V, \partial V, E)$, where (V, E) is a finite graph with the set of nodes V and the set of edges E. Let ∂V be a nonempty subset of V called the set of boundary nodes. The interior of Γ consists of those nodes not contained in ∂V.

2.2 Circular Planar Graphs, Criticality, and Connectivity

A graph Γ is defined as circular planar if and only if Γ can be embedded in a disc D in the plane such that the boundary nodes lie on ∂D and the remainder of Γ is in the interior of D. The ∂V of such a graph will be labeled $v_1, \ldots, v_n$ in the (clockwise) circular order around ∂D. A pair of sequences of boundary nodes $(P; Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ such that the sequence $(p_1, \ldots, p_k, q_k, \ldots, q_1)$ is in circular order will be called a circular pair.

A circular pair $(P;Q) = (p_1, \ldots, p_l; q_1, \ldots, q_k)$ of boundary nodes is said to be connected through Γ if there are k disjoint paths $\alpha_1, \ldots, \alpha_k$ in Γ, such that α_i starts at p_i, ends at q_i, and passes through no other boundary nodes. We say that α is a connection from P to Q. For each circular planar graph Γ, let $\pi(\Gamma)$ be the set of all circular pairs $(P;Q)$ of boundary nodes which are connected through Γ.

An edge may be removed from a graph in two ways:

1. By deleting an edge

2. By contracting an edge to a single node, provided the edge does not connect two boundary nodes.

Removing an edge is said to break the connection from P to Q if there is a connection from P to Q through Γ before the edge is removed, but not after the edge is removed. A graph Γ is called critical if the removal of any edge breaks some connection in $\pi(\Gamma)$.

2.3 Y-Δ Transformations

Let Γ be a circular planar graph with s is a trivalent interior node of Γ with incident edges sp,sq, and sr. A Y-Δ transformation removes the vertex s, and the three incident edges. These are replaced by new edges pq,qr, and rp. A Δ-Y transformation is the opposite procedure, creating interior node s and replacing edges pq,qr, and rp by edges sp,sq, and sr. Now let Γ_1 and Γ_2 be two circular planar graphs. Γ_1 and Γ_2 are said to be Y-Δ equivalent if Γ_1 can be transformed to Γ_2 by a sequence of Y-Δ and/or Δ-Y transformations. The Y-Δ equivalence class of a graph Γ is defined to be the set of all graphs that are Y-Δ equivalent to Γ.

2.4 Medial Graphs

Again, let Γ be a circular planar graph. We may associate to Γ a medial graph $\mathcal{M}(\Gamma)$. Because Γ is circular planar, the boundary nodes $v_1, \ldots, v_n$ occur in clockwise order around a circle C and the rest of Γ is in the interior of C. For each edge e of Γ let m_e be its midpoint. Place $2n$ points $t_1, \ldots, t_{2n}$ on C such that

$$t_1 < v_1 < t_2 < t_3 < v_2 < \ldots < t_{2n-1} < v_n < t_{2n} < t_1$$

in clockwise circular order around C. The vertices of $\mathcal{M}(\Gamma)$ consists of the points m_e for $e \in E$ and the points t_i for $i = 1, \ldots, 2n$.

Now we must define the edges of $\mathcal{M}(\Gamma)$. Two vertices m_e and m_f are joined by an edge whenever e and f have a common vertex and are incident to the same face in Γ. There is also one edge for each point t_j as follows. The point t_{2i} is joined by an edge to m_e where e is the edge in Γ of the form $e = v_i r$ which comes first after arc $v_i t_{2i}$ in clockwise order around v_i. The point t_{2i-1} is joined by an edge to m_f where f is the edge in Γ of the form $f = v_i s$ which comes first after arc $v_i t_{2i-1}$ in counter-clockwise order around v_i.

The vertices m_e of $\mathcal{M}(\Gamma)$ are 4-valent, and the vertices of the form t_i are 1-valent. An edge uv of $\mathcal{M}(\Gamma)$ has a <u>direct extension</u> vw if the edges uv and vw separate the two other edges incident to the vertex v in $\mathcal{M}(\Gamma)$. A path $u_0 u_1 \dots u_k$ in $\mathcal{M}(\Gamma)$ is called a <u>geodesic arc</u> if each edge $u_{i-1}u_i$ has edge $u_i u_{i+1}$ as a direct extension. A geodesic arc $u_0 \dots u_k$ is called a <u>geodesic</u> if either u_0 and u_k are points on the circle C or $u_k = u_0$ and $u_{k-1}u_k$ has $u_0 u_1$ as a direct extension. $\mathcal{M}(\Gamma)$ is said to have a <u>lens</u> if two geodesics in $\mathcal{M}(\Gamma)$ intersect each other more than once. $\mathcal{M}(\Gamma)$ is said to be <u>lensless</u> is each geodesic in $\mathcal{M}(\Gamma)$ begins and ends on C, has no self-intersections, and $\mathcal{M}(\Gamma)$ has no lenses. From [1] we know that a circular planar graph is critical if and only if its medial graph is lensless.

A <u>triangle</u> in $\mathcal{M}(\Gamma)$ is a triple f, g, h of geodesics which intersect to form a triangle with no other intersections within the configuration. It can be seen through the use of illustrations that there are two ways of configuring this triangle while maintaining the clockwise ordering of its vertices. A <u>motion</u> consists of replacing one configuration by another. Suppose Γ_1 and Γ_2 are two circular planar graphs. From [1] we know that Γ_1 and Γ_2 are Y-Δ equivalent if and only if their medial graphs are equivalent under motions.

2.5 Z-Sequences

Now suppose that Γ is a critical circular planar graph embedded in a disk D. Then $\mathcal{M}(\Gamma)$ is lensless. In addition, $\mathcal{M}(\Gamma)$ will have n geodesics each of which intersects C twice. The n geodesics intersect ∂D in $2n$ distinct points. These points are labeled $t_1, \dots, t_{2n}$ such that

$$t_1 < v_1 < t_2 < t_3 < v_2 < \dots < t_{2n-1} < v_n < t_{2n} < t_1$$

in clockwise circular order around ∂D. The geodesics are labeled as follows. Let g_1 be the geodesic which begins at t_1. The remaining geodesics are labeled $g_2, g_3, \dots, g_n$ so that if $i < j$, the first point of intersection of g_i with ∂D occurs before the first point of intersection of g_j with ∂D in clockwise circular order starting from t_1. For each $i = 1, 2, \dots, 2n$, let z_i be the number associated with the geodesic which intersects C at t_i. In this way we obtain a sequence $z = z_1, z_2, \ldots, z_{2n}$ which we define as the z-sequence for $\mathcal{M}(\Gamma)$. From [1] we know that two critical circular planar graphs are Y-Δ equivalent if and only if their z-sequences are the same. If $i < j$, and if the occurrences of i and j appear in z in the order

$$\ldots i \ldots j \ldots i \ldots j \ldots$$

we say that i and j interlace in z. Otherwise, we say the i and j do not interlace in z.

3 The Counts

Now it can be seen that, using the above concepts, many countable objects can be conceived of. The following are considered (for a given number of boundary nodes n):

1. Number of Y-Δ Equivalent Graphs in a Given Class
2. Number of Possible Z-Sequences
3. Number of Meaningful Z-Sequences
4. Number of Y-Δ Equivalence Classes with One Connected Component (1)
5. Number of Y-Δ Equivalence Classes with One Connected Component (2)
6. Number of Critical Graphs

Then expression $a(n)$ will be used to signify a count based on n nodes.

3.1 Number of Y-Δ Equivalent Graphs in a Given Class

Given a z-sequence z (and its corresponding Y-Δ equivalence class):

$$a(z) = \sum_{x \in S} a(x)a(\overline{x})$$

The set S is constructed in the following way. Let K be the set of all geodesics which interlace with g_1. The first member of S,s_1, is the z-sequence formed by the arrangement of the indices of these geodesics as they appear in z, along with any others that appear between the 1's. So for the sequence $z = 12344132$, s_1 is 234432. The other members of S are constructed by permuting the second part of s_1 (32 in the example above), respecting interlace rules. The interlace rules are formed as follows. Any two geodesics which do not interlace must maintain the same ordering as they had in the second part of s_1. So, in the above example, while $s_1 = 234432$ is a valid z-sequence, $s_2 = 234423$ is not, because 2 and 3 do not interlace in the z.

The z-sequence $\overline{s_1}$ is formed in a similar fashion. Instead of using the section of z between the 1's for the first part, we use the section of z after the second 1. For the second part of $\overline{s_1}$ we use the reverse of the second part of s_1. So for the sequence $z = 12344132$, $\overline{s_1}$ is 3223. All other $\overline{s_i}$'s are formed in the same way, using the reverse of the second part of s_i to form their second part.

3.2 Number of Possible Z-Sequences

$$a(n) = \prod_{k=1}^{n} (2k-1)$$

This count is the result of the following method of generating z-sequences. Start with the 1 node case:

$$z_{n=1} = 11$$

To generate the 2 node case, insert a new geodesic:

$$z_{n=2} = 0011$$

Propogate this new geodesic through the sequence:

$$z_{n=2} = 0011$$

$$z_{n=2} = 0101$$

$$z_{n=2} = 0110$$

Renumber the geodesics:

$$z_{n=2} = 1122$$

$$z_{n=2} = 1212$$

$$z_{n=2} = 1221$$

Repeat the process, remembering to propogate a geodesic through *all* possible z-sequences:

$$z_{n=2} = 1122:$$

$$z_{n=3} = 001122$$

$$z_{n=3} = 010122$$

$$z_{n=3} = 011022$$

$$z_{n=3} = 011202$$

$$z_{n=3} = 011220$$

$$z_{n=2} = 1212:$$

$$z_{n=3} = 001212$$

$$z_{n=3} = 010212$$

$$z_{n=3} = 012012$$

$$z_{n=3} = 012102$$

$$z_{n=3} = 012120$$

$$z_{n=2} = 1221:$$

$$z_{n=3} = 001221$$

$$z_{n=3} = 010221$$

$$z_{n=3} = 012021$$

$$z_{n=3} = 012201$$

$$z_{n=3} = 012210$$

Continue on until the n-node case is reached.

By this means of generating sequences, the validity of the count becomes apparent. There are always $2n - 1$ steps to the geodesic propogation, performed on $a(n - 1)$ sequences, with a(1)=1. So the count is correct by induction.

3.3 Number of Meaningful Z-Sequences

This count did not lend itself well to an explicit formula. The following method can be used to determine whether or not a z-sequence is meaningful, given a z-sequence z.

1. Find the connected components of a medial graph with z as its z-sequence.
2. The z-sequence is not meaningful if and only if two geodesics z_{2i-1} and z_{2i} do not belong to same connected component for $i = 1, 2, \ldots, n$.

The idea is that a z-sequence becomes meaningless when a boundary cell is connected to the circle of the circular planar graph at two or more distinct intervals. This algorithm tests for such boundary cells. Boundary cell intervals only occur between odd and even indices in the z-sequence, not between even and odd. So $z_1 z_2$ would represent a boundary cell interval, while $z_2 z_3$ would not.

The count can be determined by applying this test to each possible z-sequence, all of which can be generated by the method in the previous section.

3.4 Number of Y-Δ Equivalence Classes with One Connected Component (1)

This quantity can be obtained by using the previous method, counting only those z-sequences with one connected component.

3.5 Number of Y-Δ Equivalence Classes with One Connected Component (2)

Let $P = \{(x, y) : x + y < n; x, y \in \mathcal{N}\}$. Then

$$a(n) = (2n - 3)a(n - 1) + D(n)$$

where

$$D(n) = \sum_{(x,y)\in P} (2x - 1)a(x)(2y - 1)a(y)a(n - x - y)$$

This count can be justified in the following way. The term $(2n-3)a(n-1)$ represents those classes which can be formed by taking all connected graphs (not in the same Y-Δ equivalence class) with $n - 1$ boundary nodes and introducing one new geodesic into their medial graphs in all possible ways (without repetition). The term $D(n)$, called the n-node detritus, is similar. Given $(x, y) \in P$, x is the number of geodesics in a connected medial graph where the new geodesic will enter. y is the number of geodesics in a connected medial graph where the new geodesic will leave. There are $n-x-y-1$ remaining geodesics which are distributed in space between the entrance graph and the exit graph. There are $(2x-1)a(x)$ means for the new geodesic to enter, and $(2y-1)a(y)$ ways for it to leave. The $a(n-x-y)$ term accounts for the remaining geodesics, which will can be configured in $a(n-x-y)$ ways to form $(n-x-y)$-geodesic connected medial graphs with the new geodesic. In this way the n-node detritus accounts for those connected medial graphs not built upon $(n-)1$-node connected medial graphs. Remember that Y-Δ transformations do not come into play here, as the medial graphs constructed are representatives of a certain Y-Δ equivalence class.

3.6 Number of Critical Graphs

Let Z be the set of all n-node meaningful z-sequences. Then

$$a(n) = \sum_{z \in Z} y(z)$$

where $y(z)$ is the number of Y-Δ equivalent graphs in the class defined by z, which can be counted using the method above. The set Z can be restricted to those n-node meaningful z-sequences which represent medial graphs with one connected component if desired.

References

1. E.B. Curtis, D.Ingerman, and J.A. Morrow. *Circular Planar Graphs and Resistor Networks.*

2. D. Jerina. *Determining the Shape of a Resistor Network from Boundary Measurements.*