An Inverse Problem for General Electrical Networks

Neil Burrell

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Abstract

We examine an inverse problem for electrical networks consisting of resistors, inductors and capacitors, and excited by periodic boundary voltages. We show that for a particular class of admittances, a network with that admittance can be constructed. Using these networks as edges in a more complicated network, we prove that the solution to the Dirichlet problem is unique, so long as the real part of the admittance is positive. We then describe a process for recovering the admittances of a rectangular network, and begin an investigation of a process for recovering circular planar networks using determinants.

1 Introduction

In general, the problem that we are concerned with here is the problem of recovering an electrical network, knowing only its response to periodic input. In general, we consider a graph $\Gamma = (V, E)$ with admittances $\gamma_e(\omega)$ defined on edges $e \in E$. These admittances are functions of the frequency ω at which the network is driven. For a two node network, if we apply a voltage difference $V(t) = V_0 e^{i\omega t}$ between the two nodes, the admittance γ is defined as the relationship between voltage and current, $I(t) = \gamma(\omega) \cdot V_0 e^{i\omega t}$.

A resistor has admittance $1/R$, where R is the resistance. This is because $I(t) = (1/R) \cdot V(t)$ for a resistor. A capacitor responds to an applied voltage based on the relation $I(t) = C \cdot (dV/dt)$, where C is the capacitance. Using the periodic form of our applied voltage,

$$I(t) = C\frac{dV(t)}{dt} = C\frac{d}{dt}V_0 e^{i\omega t} = iC\omega \cdot V_0 e^{i\omega t},$$

we see that the admittance for a capacitor is $iC\omega$. Similarly for an inductor, from the relationship $I(t) = (1/L) \cdot \int V(t)\ dt$, we see that the admittance is $1/(iL\omega)$, where L is the inductance.

We can combine these three components to get circuits whose admittances are more complicated functions of ω. For example, the network in Figure 1 has admittance

$$\begin{aligned}\gamma(\omega) &= \frac{1}{R_1} + \frac{1}{-iR_1^2C(\omega - i/(R_1C))} + \frac{1}{Li(\omega - R_2 i/L)}\\ &= \frac{R_1^2CL\omega^2 - R_1^2 i\omega(R_1C + R_2C) - R_1^2}{R_1^3CL\omega^2 - i\omega(R_1^2L + R_1^3R_2C) + iR_1^3R_2C}.\end{aligned}$$

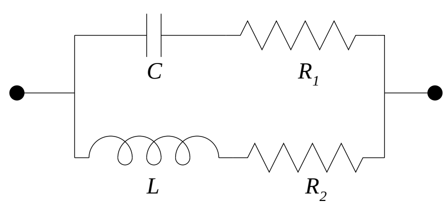


Figure 1: A simple electrical network of the type being considered

2 Uniqueness

Networks with two poles consisting of resistors, capacitors and inductors don't necessarily acquire a unique current through the network for every periodic boundary voltage. Consider the general version of Ohm's law, for periodic boundary voltage, $I(t) = \gamma(\omega) \cdot V(t)$. This relation holds for all ω where $\gamma(\omega)$ is defined. γ can have poles in the complex plane and in some cases these poles can lie on the real axis, corresponding to certain purely periodic boundary voltages. But, away from the poles of γ, the current through the network is uniquely determined by Ohm's Law.

One simple example of a circuit where a unique current is not attained is an LC circuit, where the admittance has poles at $\omega = 1/\sqrt{LC}$ and $\omega = -1/\sqrt{LC}$. In physical terms, this circuit has a resonance frequency $1/\sqrt{LC}$ and excitation by an applied voltage at that frequency will produce an unbounded current response. But, away from the poles, this network has a finite admittance. Similarly, the admittance for an inductor has a pole at $\omega = 0$, and the admittance for a capacitor has a pole at ∞.

3 Simple Networks

Our results do not consider all possible *RLC* networks. Instead, they apply to a limited class of networks that we will call *simple*.

Definition 1. *A* simple parallel network *is a two-pole electrical network that is made up of a parallel combination of specific types of serial elements. The allowed types of serial elements are resistors, capacitors, inductors, or any two of the above in series, or all three in series.*

Figure 2 shows an example of a simple parallel network containing all of the allowable types of elements.

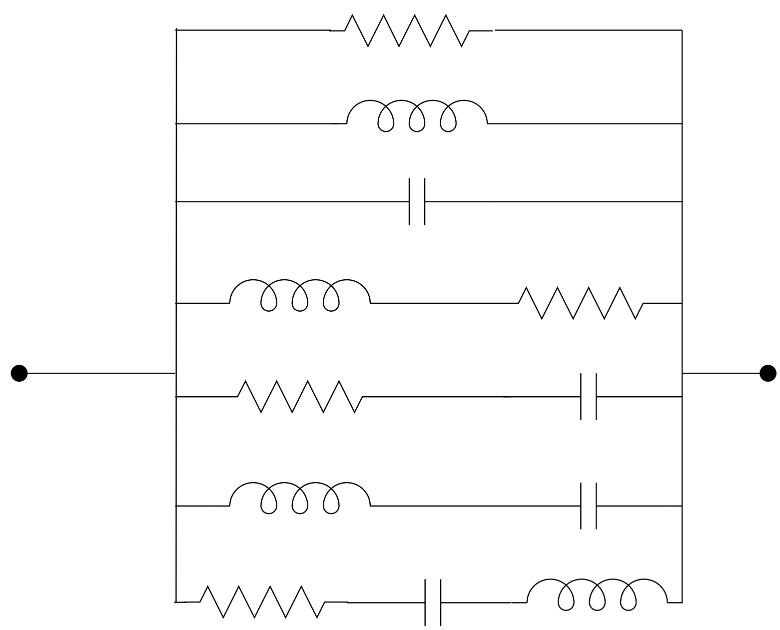


Figure 2: A Simple Parallel Network

A similar construct is possible for series networks, defining a class of networks called *simple series networks* which are made up of series combinations of 7 types of parallel elements: resistors, inductors, capacitors, or any two of those in parallel, or all three in parallel. Figure 3 shows an example of a simple series network containing all of the allowable elements. We will consider only the case of the simple parallel network, because the simple series network is its dual.

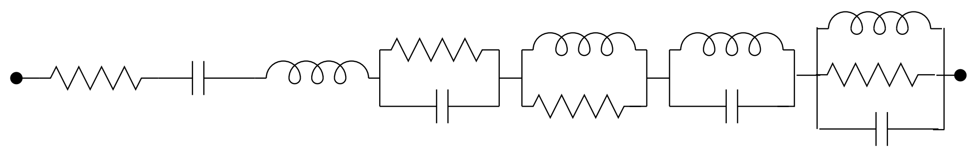


Figure 3: A Simple Series Network

The admittance of a simple parallel network is the sum of the admittances of each of its branches. By writing the admittance in its partial fraction decomposition, we can identify terms that come from specific elements.

Definition 2. *A* simple admittance *is an admittance of the form*

$$\begin{aligned}\gamma(\omega) =& a_1\omega + a_0 + \left[\frac{1}{b_1(\omega-\omega_{b_1})} + \cdots + \frac{1}{b_l(\omega-\omega_{b_l})}\right]\\ &+ \left[c_1 + \frac{1}{d_1(\omega-\omega_{d_1})} + \cdots + c_m + \frac{1}{d_m(\omega-\omega_{d_m})}\right]\\ &+ \left[\frac{1}{e_1(\omega-\omega_{e_1})} + \frac{1}{e_1(\omega+\omega_{e_1})} + \cdots + \frac{1}{e_n(\omega-\omega_{e_n})} + \frac{1}{e_n(\omega+\omega_{e_n})}\right]\\ &+ \left[\frac{1}{f_1(\omega-\omega_{f_1})} + \frac{1}{f_1'(\omega-\omega_{f_1}')} + \cdots + \frac{1}{f_p(\omega-\omega_{f_p})} + \frac{1}{f_p'(\omega-\omega_{f_p}')}\right]\end{aligned}$$

where

- $\mathrm{Re}(a_1) = 0$, $\mathrm{Im}(a_1) > 0$
- $\mathrm{Re}(a_0) > 0$, $\mathrm{Im}(a_0) = 0$
- $\mathrm{Re}(b_i) = 0$, $\mathrm{Im}(b_i) > 0$,
  $\mathrm{Re}(\omega_{b_i}) = 0$, $\mathrm{Im}(\omega_{b_i}) \geq 0$, *for all* $1 \leq i \leq l$
- $\mathrm{Re}(d_i) = 0$, $\mathrm{Im}(d_i) < 0$,
  $\mathrm{Re}(\omega_{d_i}) = 0$, $\mathrm{Im}(\omega_{d_i}) > 0$,
  $c_i = 1/(d_i\omega_{d_i})$, *for all* $1 \leq i \leq m$
- $\mathrm{Re}(e_i) = 0$, $\mathrm{Im}(e_i) > 0$,
  $\mathrm{Re}(\omega_{e_i}) \neq 0$, $\mathrm{Im}(\omega_{e_i}) = 0$, *for all* $1 \leq i \leq n$

- $f'_i = -\overline{f_i}$, $\omega'_{f_i} = -\overline{\omega_{f_i}}$,
  $\mathrm{Re}(f_i) > 0$, $\mathrm{Im}(f_i) > 0$,
  $\mathrm{Re}(\omega_{f_i}) \geq 0$, $\mathrm{Im}(\omega_{f_i}) > 0$, *for all* $1 \leq i \leq p$

Note that this admittance is the partial fraction expansion of a rational function of ω that has $l + m + n + p$ poles, all of which are simple and lie in the upper-half plane or along the real axis. Now, we can state the main theorem of the paper on recoverability of simple parallel networks.

Theorem 1. *Given a simple admittance $\gamma(\omega)$, we can construct a simple parallel network with that admittance.*

4 Recovery

4.1 Components

Each of the elements allowed in a simple parallel network has an admittance of one of the types allowed in a simple admittance. Table 1 lists all of the components, and the admittances to which they correspond. As can be seen, the admittances of the six types of components correspond to the types of terms that appear in the definition of a simple admittance. It is this fact that allows us to recover a simple parallel network from a simple admittance.

4.2 From Admittance to Network

Assume that we are given a simple admittance $\gamma(\omega)$. Then, we construct a simple parallel network in the following way. First, if $a_1 \neq 0$, then there is a branch in the network consisting solely of a capacitor with capacitance $C = \mathrm{Im}(a_1)$. If $a_0 \neq 0$, then there is a branch consisting solely of a resistor, with resistance $R = 1/a_0$. If there are any terms of the form $1/\left[b(\omega - \omega_b)\right]$, then there is a branch consisting of an inductor and a resistor in series, with inductance $L = \mathrm{Im}(b)$ and resistance $R = -b\omega_b$. Any terms of the form $c + 1/\left[d(\omega - \omega_d)\right]$ tell us that there is a branch containing only a resistor and capacitor in series, with resistance $R = d\omega_d$ and capacitance $C = -\mathrm{Im}(d)/R^2$. Similarly, terms of the form $1/\left[e(\omega - \omega_e)\right] + 1/\left[e(\omega + \omega_e)\right]$ correspond to branches with an inductor and capacitor in series, with inductance $L = \frac{1}{2}\mathrm{Im}(e)$ and capacitance $1/(L\omega_e^2)$. Finally, any terms of the most complicated form $1/\left[f(\omega-\omega_f)\right]+1/\left[f'(\omega-\omega_f')\right]$ correspond to branches with all three of the components in series, and with values determined by

$$L=\frac{if\omega_f}{\omega_f-\omega_f'},\quad R=iL\cdot\frac{\omega_f+\omega_f'}{2},\quad C=\frac{-4}{L\omega_f\omega_f'}$$

A network constructed using this process for each of the terms in the given simple admittance will have that given admittance. This proves Theorem 1.

5 More Complicated Networks

Now that we have an understanding of a basic type of two pole network, we can use networks of that type as elements to build more complicated networks. Consider a connected graph with boundary $\Gamma=(V,V_B,E)$, where V_B is the set of boundary nodes. Define functions $\gamma_{ij}(\omega)$ on E such that $\gamma_{ij}(\omega)$ is the admittance of edge $ij\in E$. We also require that γ be a simple admittance for every edge.

Voltages are applied to the network at the boundary nodes, and the interior nodes obey Kirchhoff's Law, which says that the net flow of current into a node is 0. In terms of admittances and voltages, Kirchhoff's Law at a node p is

$$\sum_q \gamma_{pq}(\omega)\left[v_p(\omega)-v_q(\omega)\right]=0$$

where the sum is over all nodes q that are neighbors of p. A problem of this type, where boundary voltages are specified at all of the boundary nodes and all of the interior nodes obey Kirchhoff's Law, is known as the Dirichlet Problem for an electrical network.

5.1 Uniqueness for the Dirichlet Problem

The Dirichlet problem does not have a unique solution for a network with simple admittances. Consider the network of Figure 4(a). It is a network, where each edge is a (very) simple parallel network, and so has a simple admittance. But, as shown in Figures 4(b) and 4(c), there are two different interior voltages that give the same boundary voltages. Without a unique solution to the Dirichlet problem, the inverse problem that we are interested in does not make sense. As a result, we impose an additional condition on the admittances: that they have positive real parts.

This requirement of positivity for the real parts corresponds to the physical requirement that every simple parallel network in the larger network contain at least one resistor. This correspondence is due to the fact that the admittances for components that contain resistors have positive real parts, while the admittances for all other types of components are purely imaginary. Since the admittance of a simple parallel network is the sum of the admittances of the individual components, the admittance of a network with at least one resistor in it has positive real part.

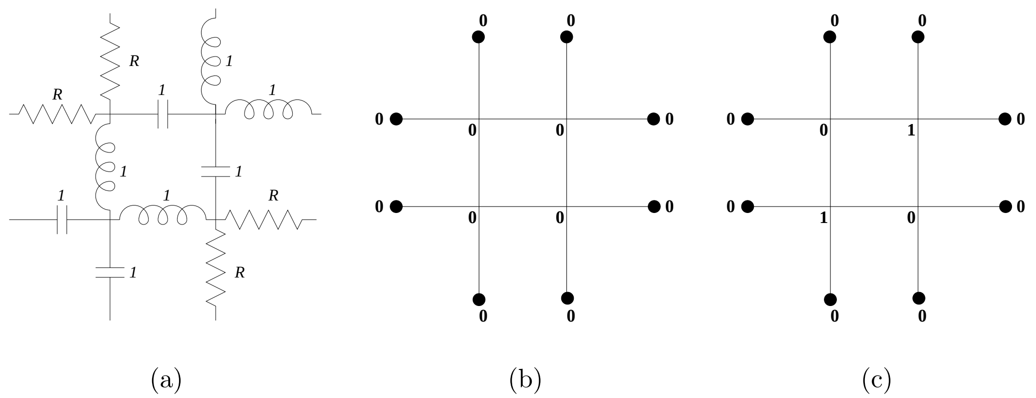


Figure 4: A network without a unique solution to the Dirichlet problem. The bold numbers represent periodic voltages, in this case, of angular frequency $\omega = 1$. Both of the sets of voltages satisfy Kirchhoff's law on the interior, and have zero boundary voltages.

Theorem 2. *Let $\Gamma = (V, V_B, E)$ be a connected graph with boundary. Let $\{\gamma_{ij}\}$ be defined on the edges of Γ such that the admittance of edge ij is given by $\gamma_{ij}(\omega)$. Suppose that the admittances are all simple and have positive real parts for all ω in some set W. Then the Dirichlet problem has a unique solution for frequencies $\omega \in W$.*

Proof. This proof is very similar to those in [4] and [1]. Consider the matrix $K = (k_{ij})$, called the *Kirchhoff matrix*, defined as follows.

$$k_{ij} = \begin{cases} -\gamma_{ij}(\omega)\,, \text{ if } i \neq j \\ \sum_l \gamma_{il}(\omega)\,, \text{ if } i = j \end{cases}$$

where the sum is over all vertices l that are adjacent to i. Note that the entries of K are functions of ω and by definition K is symmetric, and has row sums 0. K represents the map from voltages at the nodes of the networks to the currents out of each node. That is, if $\mathbf{v}e^{i\omega t}$ is a vector of voltages at all n nodes, then $(K\mathbf{v})e^{i\omega t}$ is the vector of currents out of each node.

K has a natural block decomposition based on boundary and interior nodes. If we order the nodes of Γ so that the boundary nodes are the first m nodes, and the remaining nodes are interior nodes (for a total of n nodes), then K has the block structure

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

where K is an $n \times n$ matrix, and A is an $m \times m$ matrix. Using this block structure of K, the Dirichlet problem has a natural formulation. Given an $m \times 1$ vector of boundary voltages $\mathbf{v}_\partial e^{i\omega t}$, find an $(n-m) \times 1$ vector of interior voltages, $\mathbf{v}_{int} e^{i\omega t}$ such that the following equation is satisfied:

$$K\mathbf{v} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} \mathbf{v}_\partial \\ \mathbf{v}_{int} \end{pmatrix} e^{i\omega t} = \begin{pmatrix} \mathbf{f}_\partial \\ 0 \end{pmatrix} e^{i\omega t}$$

for some $m \times 1$ vector $\mathbf{f}_\partial$.

If the matrix C is non-singular, then the Dirichlet problem has the unique solution $\mathbf{v}_{int} = -C^{-1}B^T\mathbf{v}_\partial$. To show that C is non-singular, we first show that the nullspace of K contains only constant vectors of functions. Consider a vector $\mathbf{x}$ such that $K\mathbf{x} = 0$. Then, $\bar{\mathbf{x}}^T K\mathbf{x} = 0$. Thus,

$$\begin{aligned}
\bar{\mathbf{x}}^T K\mathbf{x} &= \sum_{i,j} \bar{x}_i(\omega)k_{ij}(\omega)x_j(\omega) \\
&= \sum_{i \neq j} \bar{x}_i(\omega)k_{ij}(\omega)x_j(\omega) + \sum_{i=1}^{n} k_{ii}(\omega)|x_i(\omega)|^2 \\
&= \sum_{i<j} k_{ij}(\omega)\left[\bar{x}_i(\omega)x_j(\omega) + \bar{x}_j(\omega)x_i(\omega)\right] + \sum_{i=1}^{n} k_{ii}(\omega)|x_i(\omega)|^2 \\
&= \sum_{i<j} k_{ij}(\omega)\left[x_i(\omega) - x_j(\omega)\right]\left[\bar{x}_j(\omega) - \bar{x}_i(\omega)\right] \\
&= -\sum_{i<j} k_{ij}(\omega)|x_i(\omega) - x_j(\omega)|^2 = 0.
\end{aligned}$$

Since we have assumed that $\mathrm{Re}\big(\gamma_{ij}(\omega)\big) > 0$ for all $\omega \in W$ and for every edge $ij \in E$, $\mathrm{Re}\big(k_{ij}(\omega)\big) < 0$ for $i \neq j$ and for all $\omega \in W$. Since the terms are all non-positive, they must all be zero, so we conclude that $|x_i(\omega) - x_j(\omega)|^2 = 0$ if node i is a neighbor of node j, for all $\omega \in W$. Because Γ is a connected graph, there exists a path between any two vertices. So, for every pair of vertices i and j, $x_i \equiv x_j$ on W. Thus, we conclude that $\mathbf{x}$ is a constant vector of functions.

Now, assume that there is a vector $\mathbf{y}$ of functions for which $C\mathbf{y} = 0$ for all $\omega \in W$. Then, form the vector $\mathbf{z} = [0, \ldots, 0, y_1(\omega), \ldots, y_{n-m}(\omega)]$. So,

$$\bar{\mathbf{y}}^T C\mathbf{y} = \bar{\mathbf{z}}^T K\mathbf{z} = 0$$

which implies that $\mathbf{z}$ is a constant vector of functions. But, $\mathbf{z}$ has entries which are the zero function, so $\mathbf{z}$ is the constant vector of zero functions, which implies that $\mathbf{y}$ is also the vector of zero functions. Since $C\mathbf{y} = 0 \Leftrightarrow \mathbf{y} = 0$, we conclude that C is non-singular for all $\omega \in W$. Having shown that C is non-singular, we have proved that the Dirichlet problem has a unique solution for frequencies $\omega \in W$. □

6 The Inverse Problem

Now that we have shown uniqueness for the solution of the Dirichlet problem, we can define the Dirichlet-to-Neumann map, $\Lambda = A - BC^{-1}B^T$ as the map from boundary voltages to boundary currents. The inverse problem we are interested in is, given the Dirichlet-to-Neumann map for a network with admittances γ, find a network with those admittances. The solution to this problem is not unique. Figure 5 shows two different networks that have the same admittance,

$$\gamma(\omega) = \frac{\omega - i/2}{\omega - i}$$

Two different networks can have the same admittance only when an RC and an RL branch are both present in one of the networks. This network will then be equivalent to one with either the RL or the RC branch replaced by a purely resistive branch, and the component values adjusted. This is the only situation in which the poles of the admittance lie in the same place for two different types of components (in this case on the positive imaginary axis), and so this is the only way in which two different networks can have the same admittance.

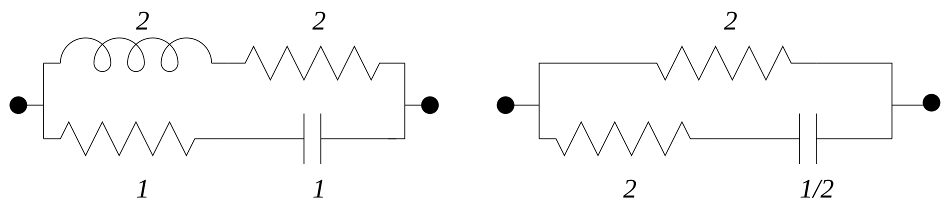


Figure 5: Two networks with the same admittance.

Since the solution to the inverse problem is not unique, we can only promise to construct a network that has the given Dirichlet-to-Neumann map. So, we concern ourselves with the recovery of the admittances of a network from its Dirichlet-to-Neumann map. If the admittances that we recover are simple, then by Theorem 1 we can construct a network with those admittances, whose elements are themselves simple parallel networks.

7 Recovering Rectangular Networks

We consider the process of recovering admittances for a rectangular network. This process is almost identical to that described in [2], with the exception of the proof of uniqueness, and the form of the voltages involved. Define a rectangular network Γ as follows. The nodes are the lattice points $p = (i, j)$ for $0 \leq i \leq n+1$ and $0 \leq j \leq n+1$ with the four corner points $(0,0), (0, n+1), (n+1, n+1)$ and $(n+1, 0)$ excluded. The interior nodes are the points $p = (i, j)$ for $1 \leq i \leq n$ and $1 \leq j \leq n$ and all of the other points are boundary nodes. Number the nodes clockwise around the boundary, starting from the upper right, as shown in Figure 6. Denote the North, East, South and West faces of the network by N, E, S, and W, respectively.

First, we have to show that a certain problem, involving both boundary currents and boundary voltages has a unique solution. We will show that the associated homogeneous problem has only the trivial solution. Consider the following problem for a rectangular network: Set the boundary voltages to be 0 on the N, W, and S faces. Also, require that the boundary current be 0 on the W face.

Lemma 1. *This problem has only one solution for the voltages on the E face: they are all 0.*

Proof. Kirchhoff's Law is a five point formula in the rectangular case. It in-

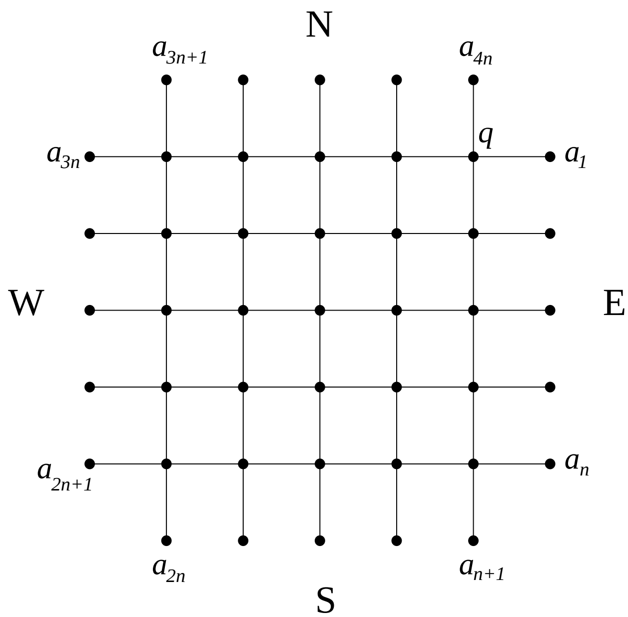


Figure 6: A rectangular network with n nodes on a side

volves the voltages at one interior point and its four neighbors. The positivity of the real parts of the admittances guarantees that all of the coefficients are non-zero, and so the voltages at any four of the points uniquely determine the voltage at the fifth point. Since no current flows into or out of the W face, the voltage must be 0 along the entire first column of interior nodes. Using Kirchhoff's Law at the interior nodes and the values along the N and S faces, we can work our way across the network from W to E, determining that all of the voltages are 0. This process continues, until we reach the E face, where all of the voltages must be 0. $\square$

Consider the mixed problem with 0 boundary voltages and currents on the W face and 0 voltages on the S face. On the N face, set the voltage at a_{4n-k+1} to be $e^{i\omega t}$, and zero everywhere else on the N face. Let the voltages on the E face be $\alpha_1(\omega), \ldots, \alpha_n(\omega)$.

Theorem 3. *The mixed problem has a unique solution for $\alpha_1(\omega), \ldots, \alpha_n(\omega)$, and $\alpha_{k+1}, \ldots, \alpha_n$ are all zero.*

Proof. Consider the difference of two solutions to the mixed problem. This difference satisfies the homogeneous problem of Lemma 1. Thus, the difference between any two solutions of the mixed problem is identically zero, and so the solution of the mixed problem is unique. $\alpha_{k+1}, \ldots, \alpha_n$ are all zero by Kirchhoff's Law, from the data specified. $\square$

The first step in recovering the admittances is the recovery of the admittances for the edges on the corners, in this case, $\gamma_{a_{4n},q}(\omega)$ and $\gamma_{a_1,q}(\omega)$. By Theorem 3, there is a unique function $\alpha(\omega)$, which solves the mixed problem with zero boundary voltages, except for $e^{i\omega t}$ at a_{4n} and $\alpha(\omega)e^{i\omega t}$ at a_1 and zero boundary currents on the W face. This function α can be computed as $-\lambda_{3n,4n}(\omega)/\lambda_{3n,1}(\omega)$. Using that boundary potential, the voltage at q is zero, which allows us to calculate the admittances of $a_{4n}q$ and a_1q, since we know both the voltage drops by construction, and the currents into the network from the Dirichlet-to-Neumann map. This process gives us the conductances

$$\gamma_{a_{4n},q}(\omega) = \lambda_{4n,4n}(\omega) - \frac{\lambda_{3n,4n}(\omega)}{\lambda_{3n,1}(\omega)}\lambda_{4n,1}(\omega)$$
$$\gamma_{a_1,q}(\omega) = -\frac{\lambda_{1,4n}(\omega)\lambda_{3n,1}(\omega)}{\lambda_{3n,4n}(\omega)} + \lambda_{1,1}(\omega)$$

To show that we can compute all of the admittances in the entire network, assume that we know all of the admittances of all of the edges above the staircase joining node a_{4n-k+1} to node a_k. The admittances to be computed are those marked with an $\times$ in Figure 7. Consider the mixed problem with

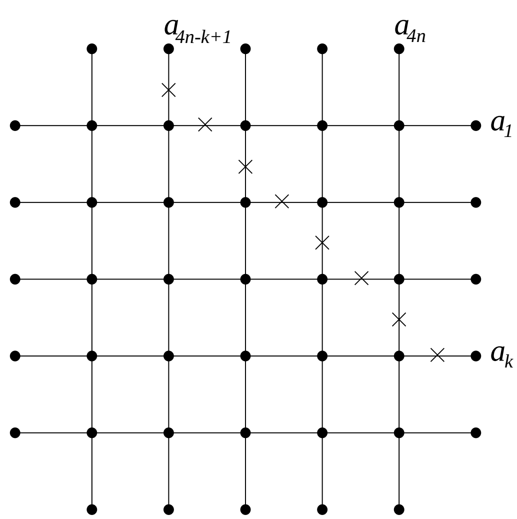


Figure 7: Inductive step for computing admittances

voltage and current 0 on the W face, voltage 0 on the S face, voltage 0 on the N face, except at a_{4n-k+1} where it is $e^{i\omega t}$. The solution for $\alpha_1(\omega), \ldots, \alpha_k(\omega)$ is unique by Theorem 3. Using this boundary voltage and the admittances that we have assumed to know, we can compute the voltages at all of the nodes in the network. Now that we know all of those voltages, we can use Kirchhoff's Law to compute the admittance of each of the conductors in the staircase. By induction, we can find the admittance of every conductor above the main diagonal. By a similar process, we can recover the admittances of all of the conductors below the main diagonal, and we have computed all of the admittances in the network.

8 Connections and Determinants

In this section, we restate a result from the theory of resistor networks, that appeared in [3]. While the same result is true, the conclusions that can be drawn in the case of general electrical networks are more limited. The conclusions drawn from this result are key to a method of recovery for circular planar graphs mentioned in [3].

Suppose $\Gamma = (V, V_B, E)$ is a connected graph with boundary. Let $I = V - V_B$ be the set of interior nodes. A *path* between two boundary nodes p and q is a sequence of edges $pr_1, r_1r_2, \ldots, r_mq$, where all of the r_j are distinct interior nodes. A *connection* between two sets of boundary nodes $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ is a set $\alpha = (\alpha_1, \ldots, \alpha_k)$ of disjoint paths. Let $\mathcal{C}(P; Q)$ be the set of all possible connections from P to Q. For every α in $\mathcal{C}(P; Q)$, define the following three objects:

- τ_α, the permutation of the vertices $(q_1, \ldots, q_k)$ that results at the endpoints of $(\alpha_1, \ldots, \alpha_k)$;
- E_α, the set of edges present in the connection α;
- J_α, the set of interior nodes which are not endpoints of any of the edges in E_α.

Theorem 4. *Let (Γ, γ) be a connected electrical network, with admittances*

$\gamma = \{\gamma_e(\omega)\}$. *Let P and Q be disjoint sets of k boundary nodes. Then,*

$$\det \Lambda(P;Q) \cdot \det K(I,I) =$$

$$(-1)^k \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \left\{ \sum_{\substack{\alpha \in \mathcal{C}(P;Q) \\ \tau_\alpha = \tau}} \left[\det K(J_\alpha, J_\alpha) \cdot \prod_{e \in E_\alpha} \gamma_e(\omega) \right] \right\}$$

Proof. The proof of Lemma 4.1 in [3] is also valid in the case of admittances. □

In the case of planar networks, there is only one permutation τ possible in the above formula (the identity permutation) and so we get the following formula for the sub-determinant of Λ corresponding to the sets of vertices P and Q:

$$\det \Lambda(P;Q) = \frac{(-1)^k}{\det K(I;I)} \left\{ \sum_{\alpha \in \mathcal{C}(P;Q)} \left[\det K(J_\alpha, J_\alpha) \cdot \prod_{e \in E_\alpha} \gamma_e(\omega) \right] \right\}.$$

In the case of resistor networks, since all of the admittances are positive real numbers, the only way that the determinant corresponding to two sets of vertices can be zero is if there is no connection between them. This is not true if we relax the conditions on the admittances. Specifically, if we allow the admittances to be complex numbers, then there are sets of vertices in some graphs that have zero determinants, even though there is a connection between them. Even if we limit the complex numbers to have positive real parts, there are still cases where connections exist, but the corresponding determinant is 0. An example is shown in Figure 8. It is a planar graph with complex resistances. The determinant of the connection from node 1 to node 4 is

$$\begin{aligned} \det \Lambda(1;4) &= \frac{-1}{\det K(I;I)} \left\{ \sum_{\alpha \in \mathcal{C}(1;4)} \left[\det K(J_\alpha, J_\alpha) \cdot \prod_{e \in E_\alpha} \gamma_e(\omega) \right] \right\} \\ &= \frac{-1}{\det K(I;I)} \big[\gamma_5\gamma_7(\gamma_1 + \gamma_2 + \gamma_3) + \gamma_1\gamma_2(\gamma_4 + \gamma_5 + \gamma_6 + \gamma_7) \big]. \end{aligned}$$

This determinant is zero for many choices of $\gamma_1, \ldots, \gamma_7$. For example, if

$$\gamma_1 = 1 + 2i, \qquad \gamma_2 = 1/2 + 2i, \qquad \gamma_3 = 251\frac{1}{2} - 5i, \qquad \gamma_4 = 250$$
$$\gamma_5 = 2 - i, \qquad \gamma_6 = 250, \qquad \gamma_7 = 4 - i$$

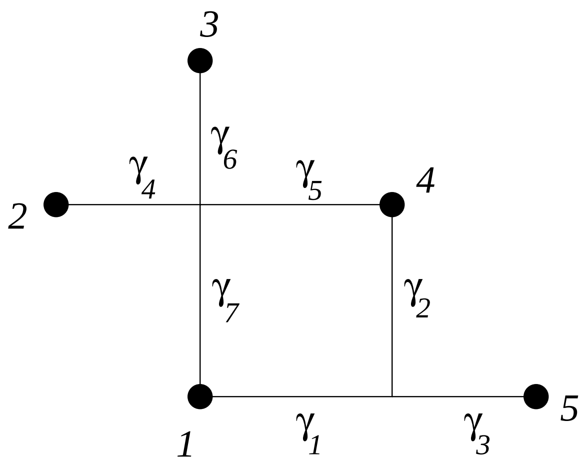


Figure 8: A graph with a connection whose corresponding determinant is 0 for certain admittances.

then $\det \Lambda(1;4) = 0$. This counterexample shows that it is not true for complex-valued admittances with a positive real part that the determinant corresponding to a connection is never zero. If we consider the case of simple admittances, the above example shows us that the determinant corresponding to a connection can have zeroes for real values of ω. On the other hand, it is not known if that determinant function can be identically zero if there is a connection.

9 Further Work

As mentioned, it may be true that the determinant in Λ corresponding to two sets of vertices can only be identically zero as a function of ω if there is no connection between them, with some appropriate conditions on the admittances. Another unanswered question lies in the recoverability of critical circular planar networks. The above conjecture, if it were true, would provide a major step towards the possibility of applying the recovery process of [3] to critical circular planar networks. Also, there are other classes of RLC electrical networks that could be investigated, whose admittances might have similar properties. For example, a network that was a combination of simple parallel and simple serial networks might be recovered by some recursive process for decomposing its admittance.

References

[1] Curtis, Edward B. and James A. Morrow. “Determining the Resistors in a Network.” SIAM J. Appl. Math. **50**:3. pp. 918–930.

[2] Curtis, Edward B. and James A. Morrow. “The Dirichlet to Neumann Map for a Resistor Network.” SIAM J. Appl. Math. **51**:4. pp. 1011–1029.

[3] Curtis, Edward B., David Ingerman and James A. Morrow. “Circular Planar Graphs and Resistor Networks.”

[4] Leykekhman, Dmitriy. “The Inverse Boundary Problem for General Planar Electrical Networks.”

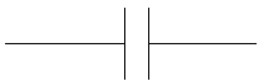
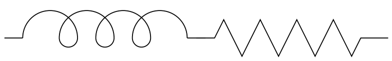
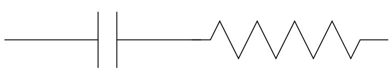
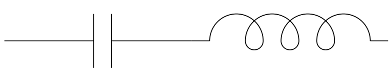
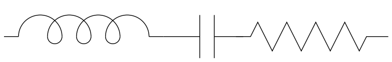
Component	Admittance	Values
 C	$a_1\omega$	$C = \mathrm{Im}(a_1)$
 R	a_0	$R = 1/a_0$
 L R	$\frac{1}{b(\omega-\omega_b)}$	$L = \mathrm{Im}(b)$, $R = -b\omega_b$
 C R	$c + \frac{1}{d(\omega-\omega_d)}$	$C = -\mathrm{Im}(d)/R^2$, $R = d\omega_d$
 C L	$\frac{1}{e(\omega-\omega_e)} + \frac{1}{e(\omega+\omega_e)}$	$L = \frac{1}{2}\mathrm{Im}(e)$, $C = 1/(L\omega_e^2)$
 L C R	$\frac{1}{f(\omega-\omega_f)} + \frac{1}{f'(\omega-\omega'_f)}$	$L = \frac{if\omega_f}{\omega_f-\omega'_f}$, $R = iL\frac{\omega_f+\omega'_f}{2}$, $C = \frac{-4}{L\omega_f\omega'_f}$

Table 1: The components of a simple parallel network, their admittances, and the component values in terms of the admittances.