Layered Networks

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Abstract In [1], David Ingramin introduced a characterization for Dirichlet-to-Neumann maps of discrete layered networks in terms of their eigenvalues. Here we introduce an alternate characterization for Dirichlet-to-Neumann maps of layered networks with 7 radial lines and 2 layers, also in terms of their eigenvalues, which we find much simpler to evaluate, if less general. We also explicitly show that the characterization given in [1] holds for layered networks with n radial lines and 1 layer.

1 Introduction

1.1 Discrete layered networks and the eigenvalues of their Dirichlet-to-Neumann maps

Discrete layered networks are connected circular planar graphs $D(n,l)$ and $D^*(n,l)$ of the following shapes:

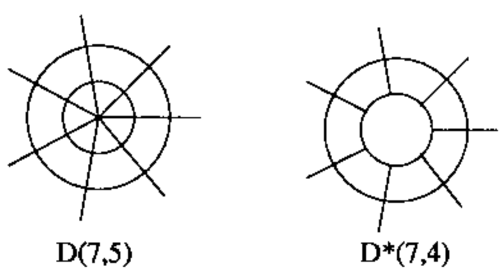

Figure 1: Discrete layered networks.

where n denotes the number of radial lines and l denotes the number of *layers*. Each *layer* of the graphs $D(n,l)$ and $D^*(n,l)$ consists of n edges with equal conductivities. We assume that the conductivity γ is constant on a given layer, so that the layered conductivity is determined by l positive numbers.

In [1], David Ingramin determined that if we let $n = 2m+1, m \in \mathbf{N}$, and

$$\omega_k^n = \omega_{-k}^n = |e^{i\frac{2\pi k}{n}} - 1|, k = -m, \ldots, 0, \ldots, m. \tag{1}$$

then for both $D(n,l)$ and $D^*(n,l)$ $\Lambda_\gamma 1 = 0$ and

$$\Lambda_\gamma e^{\pm ik\theta} = \lambda_k e^{\pm ik\theta}, k = -m, \ldots, 0, \ldots, m. \tag{2}$$

The λ_k are then the eigenvalues which uniquely determine Λ_γ. Furthermore, if we let

$$R(\lambda) = \cfrac{1}{\cfrac{1}{\delta_{\frac{l+1}{2}}} + \cfrac{1}{\xi_{\frac{l-1}{2}} + \cdots + \cfrac{1}{\cfrac{1}{\delta_3} + \cfrac{1}{\xi_2 \lambda^2 + \cfrac{1}{\cfrac{1}{\delta_2} + \cfrac{1}{\xi_1 \lambda^2 + \delta_1}}}}}} \tag{3}$$

Then the eigenvalues $\lambda_k^{(n)} of \Lambda_\gamma$ are

$$\lambda_k^{(n)} = R(\omega_k^{(n)}). \tag{4}$$

To get the similar formula for other discrete layered networks, simply make the corresponding δ_1 and/or $\frac{1}{\delta_{\frac{l+1}{2}}}$ equal to zero.

Note that by definition, $\omega_j > \omega_i$ if $|j| > |i|$.

1.2 Characterization of the Dirichlet-to-Neumann maps

Following again from [1], let Λ be an $n \times n, n = 2m+1$ discrete layered Dirichlet-to-Neumann map with non-zero eigenvalues

$$\lambda_k^{(n)}, k = 1, 2, \ldots, m,$$

and let

$$W = \left(\frac{\lambda_i^{(n)}/\omega_i^{(n)} + \lambda_j^{(n)}/\omega_j^{(n)}}{\omega_i^{(n)} + \omega_j^{(n)}} \right)_{i,j=1}^{m}. \tag{5}$$

Assuming that W is positive semi-definite, then if W is singular, there is a unique discrete layered network $D(n,l)$ or $D^*(n,l)$ with unique radially symmetric conductivity γ on it, such that

$$\Lambda(D_\gamma) = \Lambda$$

and l is equal to the size of the largest non-singular principal minor of W.

If W is non-singular, there are unique conductivities, γ, γ' on the networks $D(n,m)$ and $D^*(n,m)$ with

$$\Lambda(D_\gamma(n,m)) = \Lambda(D^*_{\gamma'}(n,m)) = \Lambda$$

And for every $D = D(n,l)$ or $D^*(n,l)$ with $l > m$ there are infinitely many conductivities γ with

$$\Lambda(D_\gamma) = \Lambda.$$

Note that from here on out we will denote $\mu_i = \frac{\lambda_i}{\omega_i}$.

2 Λ_γ for $D(7,2)$ and $D^*(7,2)$

While the matrix W can be used to characterize all $D(n,l)$, here we introduce an alternate and, we think, simpler characterization of Λ_γ for $D(7,2)$ and $D^*(7,2)$ which avoids much of the determinental evaluation necessary when considering W. We will show that the following matrices:

$$L_1 = \begin{pmatrix} 1 & \frac{\omega_1}{\mu_1} & \omega_1^2 \\ 1 & \frac{\omega_2}{\mu_2} & \omega_2^2 \\ 1 & \frac{\omega_3}{\mu_3} & \omega_3^2 \end{pmatrix} \text{ and } L_2 = \begin{pmatrix} 1 & \mu_1\omega_1 & \omega_1^2 \\ 1 & \mu_2\omega_2 & \omega_2^2 \\ 1 & \mu_3\omega_3 & \omega_3^2 \end{pmatrix}$$

can be used to characterize Λ_γ for $D(7,2)$ and $D^*(7,2)$ respectively.

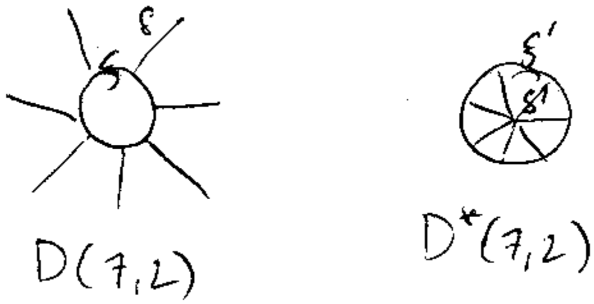

Figure 2: Discrete layered networks.

2.1 rank 2 condition on L_1 and L_2

Theorem 1.1 *If* $rk(L_1) = 2$, *then* $rk(W) = 2$.

Proof: Given that $rk(L_1) = 2$, there exists constants $\alpha, \beta > 0$ such that

$$\frac{\omega_i}{\mu_i} = \alpha + \beta\omega_i^2, i = 1, 2, 3$$

$$\Rightarrow \mu_i = \frac{\omega_i}{\alpha + \beta\omega_i^2}, i = 1, 2, 3$$

Substituting for μ_i in W yields:

$$W = \left(\; \epsilon_j(\alpha + \beta\omega_i\omega_j) \; \right)_{i,j=1}^{3}$$

where $\epsilon_j = \frac{1}{\alpha + \beta\omega_j^2}, j = 1, 2, 3.$

Letting A,B, and C denote the first, second, and third columns of W, the following linear relationship:

$$C = A[\frac{(-\alpha - \beta\omega_1^2)(\omega_2 - \omega 3)}{(\alpha + \beta\omega_3^2)(\omega_1 - \omega 2)}] - B[\frac{(\alpha + \beta\omega_2^2)(\omega_3 - \omega 1)}{(\alpha + \beta\omega_3^2)(\omega_1 - \omega 2)}]$$

shows that W does not have full rank. Now suppose $rk(W) = 1$. Then

$$det \begin{pmatrix} 1 & \frac{\alpha + \beta\omega_1\omega_2}{\alpha + \beta\omega_2^2} \\ \frac{\alpha + \beta\omega_1\omega_2}{\alpha + \beta\omega_1^2} & 1 \end{pmatrix} = 0.$$

$$\Rightarrow \frac{(\alpha + \beta\omega_1\omega_2)^2}{(\alpha + \beta\omega_1^2)(\alpha + \beta\omega_2^2)} = 1$$

$$\Rightarrow (\alpha + \beta\omega_1^2)(\alpha + \beta\omega_2^2) = (\alpha + \beta\omega_1\omega_2)^2$$

$$\Rightarrow \alpha^2 + \alpha\beta(\omega_1^2 + \omega_2^2) + \beta^2\omega_1^2\omega_2^2 = \alpha^2 + 2\alpha\beta\omega_1\omega_2 + \beta^2\omega_1^2\omega_2^2$$

And since $\alpha, \beta > 0$,

$$\omega_1^2 + \omega_2^2 - 2\omega_1\omega_2 = 0$$

$$\Rightarrow (\omega_1 - \omega_2)^2 = 0$$

$$\Rightarrow \omega_1 = \omega_2$$

which contradicts the definition of ω_i. Therefore $rk(W) = 2$. □

An analogous theorem for L_2 also holds.

Theorem 1.2 *If $rk(L_2) = 2$, then $rk(W) = 2$.*

Proof: Given that $rk(L_2) = 2$, then there exists $\alpha, \beta > 0$ such that

$$\mu_i\omega_i = \alpha\omega_i^2 + \beta, i = 1, 2, 3$$

$$\Rightarrow \mu_i = \alpha\omega_i + \tfrac{\beta}{\omega_i}, i = 1, 2, 3$$

Substituting for μ_i in W yields:

$$W = \left(\alpha + \tfrac{\beta}{\omega_i^2\omega_j^2} \right)_{i,j=1}^{3}$$

Letting A,B, and C denote the first, second, and third columns of W, the following linear relationship:

$$C = A\tfrac{3\omega_1^2\omega_2\omega_3 - 2\omega_1^2\omega_2^2 - \omega_1\omega_2\omega_3^2 + \omega_1\omega_2^2\omega_3 - \omega_1^2\omega_3^2}{\omega_2^2\omega_3^2 - 2\omega_1\omega_2^2\omega_3 - \omega_1^2\omega_3^2 + 2\omega_1\omega_2\omega_3^2} - B\tfrac{\omega_1^2\omega_2\omega_3 - \omega_1\omega_2\omega_3^2 - 2\omega_1^2\omega_2^2 + 3\omega_1\omega_2^2\omega_3 - \omega_2^2\omega_3^2}{\omega_2^2\omega_3^2 - 2\omega_1\omega_2^2\omega_3 - \omega_1^2\omega_3^2 + 2\omega_1^2\omega_2\omega_3}$$

shows that W does not have full rank. Now suppose $rk(W) = 1$. Then

$$det \begin{pmatrix} \alpha + \frac{\beta}{\omega_1^2} & \alpha + \frac{\beta}{\omega_1\omega_2} \\ \alpha + \frac{\beta}{\omega_1\omega_2} & \alpha + \frac{\beta}{\omega_2^2} \end{pmatrix} = 0.$$

$$\Rightarrow \left(\alpha + \tfrac{\beta}{\omega_1^2}\right)\left(\alpha + \tfrac{\beta}{\omega_2^2}\right) = \left(\alpha + \tfrac{\beta}{\omega_1\omega_2}\right)^2$$

$$\Rightarrow \alpha^2 + \alpha\beta\left(\tfrac{1}{\omega_1^2} + \tfrac{1}{\omega_2^2}\right) + \beta^2\tfrac{1}{\omega_1^2\omega_2^2} = \alpha^2 + 2\tfrac{\alpha\beta}{\omega_1\omega_2} + \beta^2\tfrac{1}{\omega_1^2\omega_2^2}$$

And since $\alpha, \beta > 0$,

$$\tfrac{1}{\omega_1^2} + \tfrac{1}{\omega_2^2} - 2\tfrac{1}{\omega_1\omega_2} = 0$$

$$\Rightarrow \left(\tfrac{1}{\omega_1} - \tfrac{1}{\omega_2}\right)^2 = 0$$

$$\tfrac{1}{\omega_1} = \tfrac{1}{\omega_2} \Rightarrow \omega_1 = \omega_2$$

which contradicts the definition of ω_i. Therefore $rk(W) = 2$. □

2.2 Characterization of Λ_γ for $D(7,2)$ and $D^*(7,2)$

Theorem 2.1 *Let* $L_1 = \begin{pmatrix} 1 & \frac{\omega_1}{\mu_1} & \omega_1^2 \\ 1 & \frac{\omega_2}{\mu_2} & \omega_2^2 \\ 1 & \frac{\omega_3}{\mu_3} & \omega_3^2 \end{pmatrix}$. *Then* $rk(L_1) = 2$, *and* L_1 *has the following positive subdeterminants:*

$$det \begin{pmatrix} 1 & \frac{\omega_1}{\mu_1} \\ 1 & \frac{\omega_2}{\mu_2} \end{pmatrix} > 0 \text{ and } det \begin{pmatrix} \frac{\omega_2}{\mu_2} & \omega_2^2 \\ \frac{\omega_3}{\mu_3} & \omega_3^2 \end{pmatrix} > 0$$

if and only if there are unique conductivities $\delta, \xi > 0$ *on the network* $D(7,2)$ *such that*

$$\mu_i = \frac{1}{\frac{\omega_i}{\delta} + \frac{1}{\xi\omega_i}}, i = 1, 2, 3.$$

Proof of the forward direction:
Given $rk(L_2) = 2$, there exists $\alpha, \beta > 0$ such that

$$\frac{\omega_i}{\mu_i} = \alpha\omega_i^2 + \beta, i = 1, 2, 3 \tag{6}$$

$$\Rightarrow \alpha(\omega_1^2 - \omega_2^2) = \tfrac{\omega_1}{\mu_1} - \tfrac{\omega_2}{\mu_2}$$

$$\Rightarrow \alpha = \frac{\omega_2\mu_1 - \omega_1\mu_2}{\mu_1\mu_2(\omega_2^2 - \omega_1^2)} \tag{7}$$

Given $det\begin{pmatrix} 1 & \frac{\omega_1}{\mu_1} \\ 1 & \frac{\omega_2}{\mu_2} \end{pmatrix} > 0 \Rightarrow \frac{\mu_1\omega_2-\mu_2\omega_1}{\mu_1\mu_2} > 0$ and knowing that by definition $\omega_2^2 - \omega_1^2 > 0$, then from (7), we know that $\alpha > 0$.

Now consider β. If we take equation (6) for $i = 2,3$ then we get the following equation similar to (7):

$$\alpha = \frac{\mu_3\omega_2-\mu_2\omega_3}{\mu_2\mu_3(\omega_3^2-\omega_2^2)}.$$

And substituting for α in equation (6) for $i = 3$, we find:

$$\beta = \frac{\omega_3}{\mu_3} - \frac{\omega_3^2(\mu_3\omega_2-\mu_2\omega_3)}{\mu_3\mu_2(\omega_3^2-\omega_2^2)}$$

$$\Rightarrow \beta = \frac{\omega_3\omega_2(\mu_3\omega_3-\mu_2\omega_2)}{\mu_3\mu_2(\omega_3^2-\omega_2^2)}. \qquad (8)$$

Given $det\begin{pmatrix} \frac{\omega_2}{\mu_2} & \omega_2^2 \\ \frac{\omega_3}{\mu_3} & \omega_3^2 \end{pmatrix} > 0 \Rightarrow \frac{\omega_2\omega_3^2}{\mu_2} - \frac{\omega_2^2\omega_3}{\mu_3} > 0 \Rightarrow \frac{\omega_2\omega_3(\mu_3\omega_3-\mu_2\omega_2)}{\mu_2\mu_3} > 0$, and knowing that by definition $\omega_3^2 - \omega_2^2 > 0$, then from (8) we know that $\beta > 0$.

Also, equation (6) $\Rightarrow \lambda_i = \frac{1}{\alpha+\frac{\beta}{\omega_i^2}}, i = 1,2,3$. So letting $\alpha = \frac{1}{\delta}$ and $\beta = \frac{1}{\xi}$, we find that

$$\lambda_i = \frac{1}{\frac{1}{\delta}+\frac{1}{\xi\omega_i^2}}, i = 1,2,3$$

corresponding to Λ_γ for the network $D(7,2)$ with positive conductivities δ and ξ.

Proof of the reverse direction:

Given $\delta, \xi > 0$ such that

$$\lambda_i = \frac{1}{\frac{1}{\delta}+\frac{1}{\xi\omega_i^2}}, i = 1,2,3$$

$$\Rightarrow \mu_i = \frac{\lambda_i}{\omega_i} = \frac{1}{\frac{\omega_i}{\delta}+\frac{1}{\xi\omega_i}}$$

$$\Rightarrow \frac{1}{\mu_i} = \frac{\omega_i}{\delta} + \frac{1}{\xi\omega_i}$$

$$\Rightarrow \frac{\omega_i}{\mu_i} = \frac{\omega_i^2}{\delta} + \frac{1}{\xi}, i = 1, 2, 3$$

Since $\frac{1}{\delta}$ and $\frac{1}{\xi}$ are constants, the above equation gives a linear relationship among the columns of L_1. The first two columns of L_1 are clearly not dependent, so this relationship shows that $rk(L_1 = 2$.

The above equation also gives us the following:

$$\delta = \frac{\xi\omega_1^2\mu_1}{\xi\omega_1 - \mu_1}$$

$$\Rightarrow \frac{\omega_2^2(\xi\omega_1 - \mu_1)}{\xi\omega_1^2\mu_1} = \frac{\omega_2}{\mu_2} + \frac{1}{\xi}$$

$$\Rightarrow \xi\mu_2\omega_1\omega_2^2 - \mu_1\mu_2\omega_2^2 = \xi\mu_1\omega_1^2\omega_2 - \mu_1\mu_2\omega_1^2$$

$$\Rightarrow \xi = \frac{\mu_1\mu_2(\omega_2^2 - \omega_1^2)}{\omega_1\omega_2(\mu_2\omega_2 - \mu_1\omega_1)}$$

Since we know that $\omega_2 > \omega_1$, and hence $\omega_2^2 - \omega_1^2 > 0$, then $\xi > 0$ implies:

$$\frac{\mu_1\mu_2}{\mu_2\omega_2 - \mu_1\omega_1} > 0$$

$$\Rightarrow det \begin{pmatrix} 1 & \frac{\omega_1}{\mu_1} \\ 1 & \frac{\omega_2}{\mu_2} \end{pmatrix} > 0$$

Now consider δ.

$$\frac{1}{\delta} = \frac{1}{\mu_3\omega 3} - \frac{1}{\xi\omega_3^2}$$

$$\Rightarrow \frac{1}{\delta} = \frac{1}{\mu_3\omega 3} - \frac{\omega_2\omega_3(\mu_3\omega_3 - \mu_2\omega_2)}{\omega_3^2\mu_2\mu_3(\omega_3^2 - \omega_2^2)}$$

$$\Rightarrow \frac{1}{\delta} = \frac{\mu_2\omega_3 - \mu_3\omega_2}{\mu_2\mu_3(\omega_3^2 - \omega_2^2)}$$

Since we know that $\omega_3 > \omega_2 > 0$, and hence $\omega_3^2 - \omega_2^2 > 0$, then $\delta > 0$ implies:

$$\frac{\mu_2\omega_3 - \mu_3\omega_2}{\mu_2\mu_3} > 0$$

$$\Rightarrow det \begin{pmatrix} \frac{\omega_2}{\mu_2} & \omega_2^2 \\ \frac{\omega_3}{\mu_3} & \omega_3^2 \end{pmatrix} > 0 \qquad \square$$

Now we will show that the conductivities for the dual nextwork $D^*(7,2)$ are uniquely solvable using the matrix L_2.

Theorem 2.2 *Let* $L_2 = \begin{pmatrix} 1 & \mu_1\omega_1 & \omega_1^2 \\ 1 & \mu_2\omega_2 & \omega_2^2 \\ 1 & \mu_3\omega_3 & \omega_3^2 \end{pmatrix}$. *Then* $rk(L_2) = 2$ *and* L_2 *has the following positive subdeterminants:*

$$det \begin{pmatrix} 1 & \mu_1\omega_1 \\ 1 & \mu_2\omega_2 \end{pmatrix} > 0 \text{ and } det \begin{pmatrix} \mu_2\omega_2 & \omega_2^2 \\ \mu_3\omega_3 & \omega_3^2 \end{pmatrix} > 0$$

if and only if there are unique conductivities $\delta, \xi > 0$ *on the network* $D^*(7,2)$ *such that*

$$\mu_j = \xi\omega_j + \frac{\delta}{\omega_j}, j = 1,2,3.$$

Proof of the forward direction: Given $rk(L_2) = 2$, there exists $\alpha, \beta > 0$ such that

$$\mu_i\omega_i = \alpha\omega_i^2 + \beta, i = 1,2,3 \tag{9}$$

$$\Rightarrow \alpha(\omega_2^2 - \omega_1^2) = \mu_2\omega_2 - \mu_1\omega_1$$

$$\Rightarrow \alpha = \frac{\mu_2\omega_2 - \mu_1\omega_1}{(\omega_2^2 - \omega_1^2)} \tag{10}$$

Given $det \begin{pmatrix} 1 & \mu_1\omega_1 \\ 1 & \mu_2\omega_2 \end{pmatrix} > 0 \Rightarrow \mu_2\omega_2 - \mu_1\omega_1 > 0$, and knowing that by definition $\omega_2^2 - \omega_1^2 > 0$, then from (10), we know that $\alpha > 0$.

Now consider β. If we subract equation (9), $i = 2$ from the same equation for $i = 3$, we get the following equation similar to (10):

$$\alpha = \frac{\mu_3\omega_3 - \mu_2\omega_2}{\omega_3^2 - \omega_2^2}.$$

And substituting for α in equation (9) for $i = 3$, we find:

$$\beta = \mu_3\omega_3 - \frac{\omega_3^2(\mu_3\omega_3 - \mu_2\omega_2)}{(\omega_3^2 - \omega_2^2)}$$

$$\Rightarrow \beta = \frac{\mu_2\omega_2\omega_3^2 - \mu_3\omega_3\omega_2^2}{(\omega_3^2 - \omega_2^2)} \tag{11}$$

Given $det \begin{pmatrix} \mu_2\omega_2 & \omega_2^2 \\ \mu_3\omega_3 & \omega_3^2 \end{pmatrix} > 0 \Rightarrow \mu_2\omega_2\omega_3^2 - \mu_3\omega_3\omega_2^2$ and knowing that by definition, $\omega_3^2 - \omega_2^2 > 0$, then from (11), we know that $\beta > 0$.

Also, equation (9) $\Rightarrow \lambda_j = \alpha\omega_j^2 + \beta, j = 1, 2, 3$. So letting $\alpha = \xi$ and $\beta = \delta$, we find that

$$\lambda_j = \xi\omega_j^2 + \delta, j = 1, 2, 3.$$

corresponding to the network $D^*(7, 2)$ with positive conductivities δ and ξ.

Proof of the reverse direction:

Given $D^*(7, 2)$ with positive conductivities δ and ξ such that

$$\mu_j = \xi\omega_j + \frac{\delta}{\omega_j}, j = 1, 2, 3$$

$$\Rightarrow \mu_j\omega_j = \xi\omega_j^2 + \delta, j = 1, 2, 3$$

Then since δ and ξ are constants, the above equation is a linear relationship among the columns of L_2. The first two columns of L_2 clearly are not dependent, so this relationship shows that $rk(L_2) = 2$.

Since,

$$\mu_1\omega_1 = \xi\omega_1^2 + \delta \text{ and } \mu_2\omega_2 = \xi\omega_2^2 + \delta$$

then,

$$\delta = \mu_1\omega_1 - \xi\omega_1^2$$

$$\Rightarrow \mu_2\omega_2 = \xi(\omega_2^2 - \omega_1^2) + \mu_1\omega_1$$

$$\Rightarrow \xi = \frac{\mu_2\omega_2 - \mu_1\omega_1}{\omega_2^2 - \omega_1^2}.$$

Since $\xi > 0$, and $\omega_2^2 - \omega_1^2 > 0$ then,

$$\mu_2\omega_2 - \mu_1\omega_1 > 0$$

and therefore,

$$det \begin{pmatrix} 1 & \mu_1\omega_1 \\ 1 & \mu_2\omega_2 \end{pmatrix} > 0.$$

Now consider δ,

$$\delta = \mu_3\omega_3 - \frac{\omega_3^2(\mu_3\omega_3 - \mu_2\omega_2)}{\omega_3^2 - \omega_2^2}$$

$$\Rightarrow \delta = \frac{\mu_2\omega_2\omega_3^2 - \mu_3\omega_3\omega_2^2}{\omega_3^2 - \omega_2^2}$$

Since $\delta > 0$, and $\omega_3^2 - \omega_2^2 > 0$ then,

$$\mu_2\omega_2\omega_3^2 - \mu_3\omega_3\omega_2^2 > 0$$

and therefore,

$$det \begin{pmatrix} \mu_2\omega_2 & \omega_2^2 \\ \mu_3\omega_3 & \omega_3^2 \end{pmatrix} > 0. \qquad \square$$

3 $D(n,1)$ and $D^*(n,1)$

3.1 Case $n = 7$

Theorem 3.1 *Given a 3×3 matrix W as defined by (5), rk(W)=1 if and only if there is a unique positive-valued radially symmetric conductivity γ on $D(7,1)$ or $D^*(7,1)$, such that $\Lambda(D_\gamma) = \Lambda$.*

Proof of the forward direction:

Given $D(7,1)$, all of the eigenvalues $\lambda_i = \delta, i = 1,2,3$. Hence all of the $\mu_i = \frac{\delta}{\omega_i}$ and substituting for μ_i in W yields:

$$W = \begin{pmatrix} \frac{\delta}{\omega_1^2} & \frac{\delta}{\omega_1\omega_2} & \frac{\delta}{\omega_1\omega_3} \\ \frac{\delta}{\omega_1\omega_2} & \frac{\delta}{\omega_2^2} & \frac{\delta}{\omega_2\omega_3} \\ \frac{\delta}{\omega_1\omega_3} & \frac{\delta}{\omega_2\omega_3} & \frac{\delta}{\omega_3^2} \end{pmatrix}$$

Letting $\alpha_i, i = 1,2,3$ denote the columns of W then

$$\alpha_1 = \frac{\omega_2}{\omega 1}\alpha_2 = \frac{\omega_3}{\omega_1}\alpha_3$$

and thus, $rk(W) = 1$.

Similarly, given $D^*(7,1)$, each eigenvalue $\lambda_i = \xi\omega_i^2 . i = 1,2,3$. Hence all the $\mu_i = \xi\omega_i, i = 1,2,3$ and substituting for μ_i in W yields:

$$W = \begin{pmatrix} \xi & \xi & \xi \\ \xi & \xi & \xi \\ \xi & \xi & \xi \end{pmatrix}$$

which clearly has rank 1.

Proof of the backward direction:

Now suppose we are given a matrix

$$W = \begin{pmatrix} \frac{\mu_1}{\omega_1} & \frac{\mu_1+\mu_2}{\omega_1+\omega_2} & \frac{\mu_1+\mu_3}{\omega_1+\omega 3} \\ \frac{\mu_1+\mu_2}{\omega_1+\omega_2} & \frac{\mu_2}{\omega_2} & \frac{\mu_2+\mu_3}{\omega_2+\omega_3} \\ \frac{\mu_1+\mu_3}{\omega_1+\omega 3} & \frac{\mu_2+\mu_3}{\omega_2+\omega_3} & \frac{\mu_3}{\omega_3} \end{pmatrix}$$

such that $rk(W) = 1$. Then

$$det \begin{pmatrix} \frac{\mu_i}{\omega_i} & \frac{\mu_i+\mu_j}{\omega_i+\omega_j} \\ \frac{\mu_i+\mu_j}{\omega_i+\omega_j} & \frac{\mu_j}{\omega_j} \end{pmatrix} = 0, 1 \leq i < j \leq 3$$

yielding the following equations:

$$\begin{aligned} \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} = \frac{\mu_1}{\mu_2} + \frac{\mu_2}{\mu_1} \\ \frac{\omega_2}{\omega_3} + \frac{\omega_3}{\omega_2} = \frac{\mu_2}{\mu_3} + \frac{\mu_3}{\mu_2} \\ \frac{\omega_3}{\omega_1} + \frac{\omega_1}{\omega_3} = \frac{\mu_3}{\mu_1} + \frac{\mu_1}{\mu_3} \end{aligned}$$

If we let

$$\begin{aligned} x = \frac{\omega_1}{\omega_2}, a = \frac{\lambda_1}{\lambda_2} \\ y = \frac{\omega_2}{\omega_3}, b = \frac{\lambda_2}{\lambda_3} \\ z = \frac{\omega_3}{\omega_1}, c = \frac{\lambda_3}{\lambda_1}, \end{aligned}$$

then we get the following quadratic equations:

$$\begin{aligned} x + \frac{1}{x} = \frac{1}{a}x + a\frac{1}{x} \\ y + \frac{1}{y} = \frac{1}{b}y + b\frac{1}{y} \\ z + \frac{1}{z} = \frac{1}{c}z + c\frac{1}{z} \end{aligned}$$

with the following solutions:

$$a = x^2, 1 \qquad b = y^2, 1 \qquad c = z^2, 1$$

If $a = x^2$, $b = y^2$, and $c = z^2$ then it follows that

$$\xi = \frac{\lambda_1}{\omega_1^2} = \frac{\lambda_2}{\omega_2^2} = \frac{\lambda_3}{\omega_3^2} > 0$$

and

$$\lambda_i = \xi\omega_i^2, i = 1, 2, 3$$

corresponding to the layered network $D^*(7,1)$.

If $a = b = c = 1$ then it follows that $\delta = \lambda_i$, $i = 1, 2, 3$ corresponding to the layered network $D(7,1)$.

Suppose $a = b = 1$ and $c = z^2$. Then

$$W = \begin{pmatrix} \begin{pmatrix} \frac{1}{\mu_1} \\ \frac{1}{\mu_2} \\ \frac{1}{\mu_3} \end{pmatrix} \begin{pmatrix} \frac{1}{\omega_1} \\ \frac{1}{\omega_1} \\ \frac{1}{\omega_3} \end{pmatrix} & \begin{pmatrix} \frac{1}{\mu_1} \\ \frac{1}{\mu_2} \\ \frac{1}{\mu_3} \end{pmatrix} \begin{pmatrix} \frac{1}{\omega_2} \\ \frac{1}{\omega_2} \\ \frac{1}{\omega_2} \end{pmatrix} & \begin{pmatrix} \frac{1}{\mu_1} \\ \frac{1}{\mu_2} \\ \frac{1}{\mu_3} \end{pmatrix} \begin{pmatrix} \frac{1}{\omega_1} \\ \frac{1}{\omega_3} \\ \frac{1}{\omega_3} \end{pmatrix} \end{pmatrix}$$

which clearly does not have $rk(W) = 1$. The same is true for all other combinations of solutions for a, b, c. □