Recovery and Characterization of Non-Planar Resistor Networks

Julie Rowlett

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1 Introduction

In this paper we consider non-planar conductor networks.

A *conductor* is a two-sided object which obeys Ohm's law: if potentials V_1 and V_2 are applied respectively to each end of the conductor, then the current flowing through the conductor is given by

$$I = (V_1 - V_2)\gamma$$

where γ is an intrisic property of the conductor, known as its *conductance*.

A *network* of conductors is any number of conductors, joined at the ends; such joinings are known as *nodes*. In the figures of this paper, nodes are denoted by black dots. In the networks that we consider, certain nodes form the boundary of the network, and others the interior; we restrict our ability to impose potentials or currents to the boundary only.

Because we have so restricted ourselves, Kirchhoff's current Law will hold at each of the interior nodes. Kirchhoff's Law states: the current flowing into an interior node is equal to the current flowing out of that same node. Because the interior nodes satisfy Kirchhoff's Law, there exists a unique set of interior potentials due to imposed boundary potentials (the Dirichlet problem) and a unique set of potentials due to imposed boundary currents (the Neumann problem). This uniqueness is essential to the construction of certain useful tools, such as the Dirichlet-to-Neumann map or Λ-matrix. The Dirichlet-to-Neumann map is a square matrix with dimensions given by the number of boundary nodes in the network. The entries $(\lambda_{i,j})$ represent the current flow out of the j^{th} node when the voltage at the i^{th} node is one and the voltage at all the other boundary nodes is zero. The Λ-matrix is a frequently used tool in this paper.

When we speak of a network being *recoverable* we mean to say that if we have the Λ-matrix for that network, we know the geometry of the network, and we do not know the conductivities of the network, we can manipulate the boundary voltages and utilize Λ in order to determine all the conductivities in the network. We are unable to do this with a network that is *non-recoverable*.

2 Spiky and Not-So-Spiky Tetrahedron Networks

2.1 Single Tetrahedron With and Without Spikes

The first network we consider is a regular tetrahedron. It is easily recovered from the information given by the Λ-matrix. All four nodes are boundary nodes, so the Λ-matrix is the Kirchodff matrix and it readily gives us the conductivity along all the edges.

To make this network more interesting, we add spikes to each of the corners. If we add one single spike, to any corner, the edge (spike) is not recoverable, as shown in the picture below.

In this network, if we set one of the boundary nodes with voltage zero and current zero, then all the other nodes are forced to have voltage zero and current zero. The Λ-matrix is not the Kirchoff matrix for this network, so we cannot find the conductivities in this network. It is not recoverable. Any edge that has a single spike is not recoverable.

The next network we consider is the same tetrahedron, but with two spikes at each corner.

First we recover the spikes. We assign a voltage of zero and a current of zero to nodes 3, 5, and 7. We assign a current of zero and a voltage of one to node 1. This forces the voltages on the rest of the network to be determined. Nodes 2, 4, 6 and 8 will have voltages of a, b, c, and d respectively. Nodes 9, 10, 11, and 12 will have voltages of 1, 0, 0, 0 respectively. There is only one way to connect nodes 1, 3, 5, and 7 to nodes 2, 4, 6, 8: connect 1 to 2, 3 to 4, 5 to 6, 7 to 8. Let $P = \{1,3,5,7\}$ and $Q = \{8,6,4,2\}$. Then det $\Lambda(P;Q) \neq 0$. This means that we can solve for the unknowns. We know the

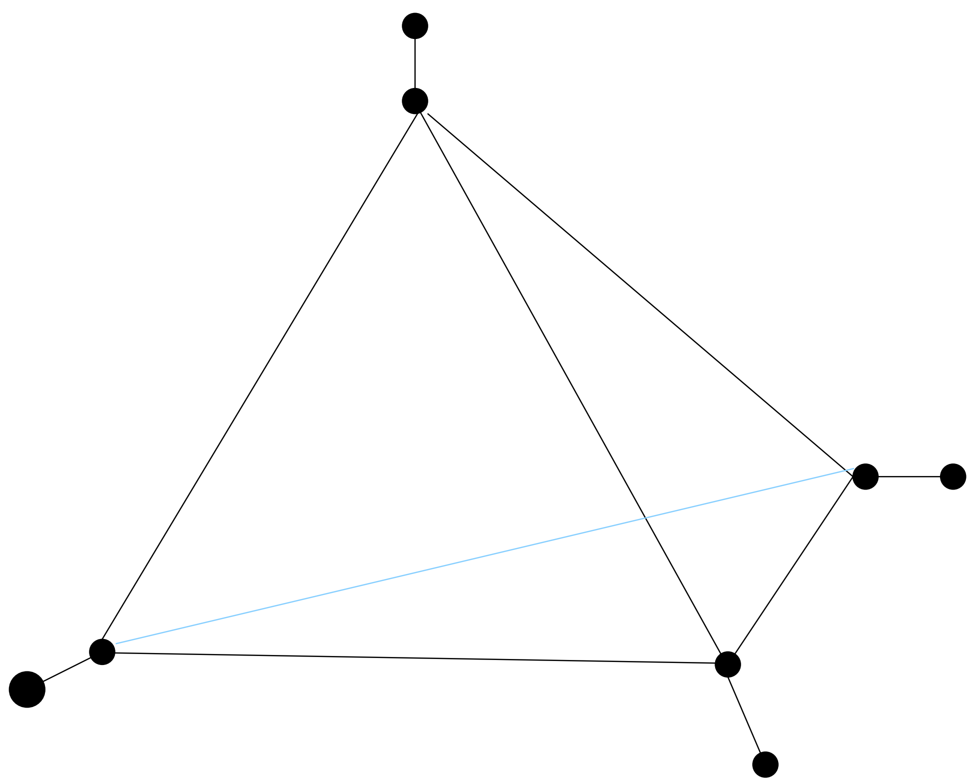

Figure 1: One Spike Tetrahedron Network

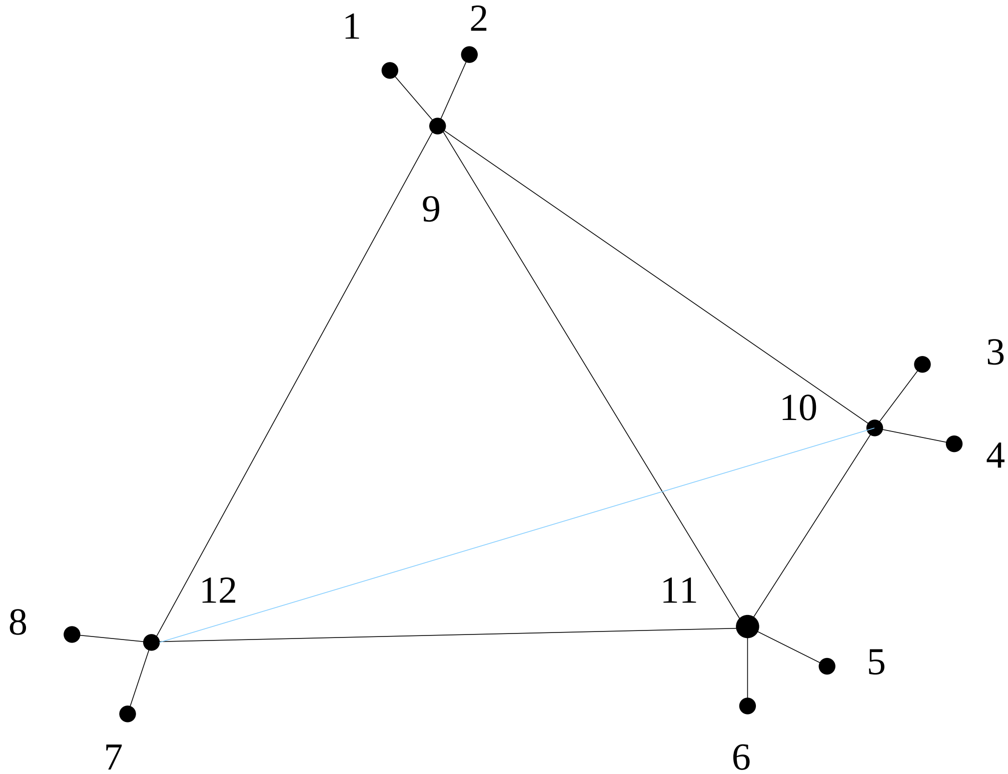

Figure 2: Two Spike Tetrahedron Network

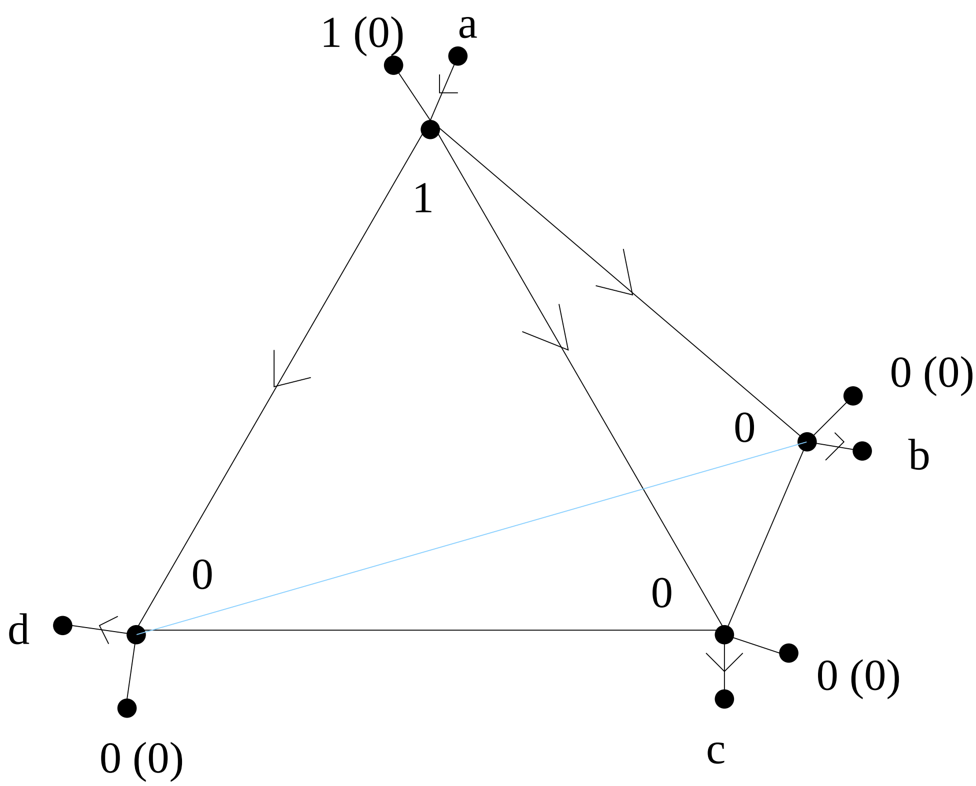

Figure 3: Voltage Pattern For Two-Spike Tetrahedron Network Recovery

current out of each boundary node from the Λ-matrix, so we can now find the conductivity along each spike, and using Kirchoff's Law, we can find the conductivity along the interior edges. We use this same pattern of voltages, changing it only slightly so that now nodes 2, 6, and 8 have voltage and current zero. We let node 4 have voltage one and current zero. This allows us to solve for the rest of the spikes and a few more edges. We rotate this pattern once more, allowing nodes 2, 4, and 8 to have voltage and current zero and node 6 to have voltage one and current zero. Now we solve for the remaining edge and we have recovered the network in its entirety.

We can recover any tetrahedron network with no spikes, with two spikes, and we can also recover a network with n spikes, where $n \geq 2$. The recovery of a network with n spikes is essentially the same as the recovery of a network with two spikes. We assign one spike from 3 of the clusters a voltage and current of zero. We assign one spike from the last cluster a voltage of one and current of zero. This forces the voltages on the interior nodes to be 0, 0, 0, and 1. We can now recover the spikes, for we can solve for the unknowns. Recovering the rest of the network is the same as in the two-spike case.

Lemma: If a network (assumed to be connected and have at least three corners – six boundary nodes if spikes are included) can be recovered with no spikes on each corner, it can also be recovered with two spikes on each corner. If a network can be recovered with two spikes on each corner, it can also be recovered without spikes on the corners.

Proof: Above we have shown how to recover spikes. This recovery process will always work, for there can only be only way to connect the pairs of spikes. In the pairing of $\Lambda(P;Q)$ the only disjoint paths must go from one spike to its pair, for any other paths would meet at the vertex to which the pair of spikes is attached. There is a unique connection from P to Q, so that determinant is always non-zero and we can always solve for the unknown boundary voltages. We will have half of the spikes with unknown voltage, and half the spikes (with the exception of one whose voltage will be set to one) with zero voltage and zero current flow. We have the same number of known current flows as unknown boundary voltages, and that determinant in the Λ -matrix is non-zero so we can solve for those unknowns. We can then calculate the current flow into those boundary nodes using Λ. We know the current flow into the spike, and it can only go one place – to the corner to which the spike is attached, and we also know the voltage at that corner node, so we can calculate the conductivity along that edge. We do this with the rest of those spikes with non-zero current flow. We repeat this procedure, now assigning the nodes that formerly had unknown voltages with zero current flow and zero voltage. We solve for the conductivities along the edges of those spikes. Now we have recovered the conductivities along the edges of all the spikes. As long as the network is connected (none of the entries in Λ are zero), we can always use this technique to recover the spikes. This method of spike-recovery can always be done, regardless of whether or not the rest of the network can be recovered. It is independent of the rest of the network, the only necessity is that there be at least six boundary nodes. So, if the entire network can be recovered with spikes, it can just as easily be recovered without spikes. If a network can be recovered without spikes, then adding spikes does not change recoverability; how to recover spikes is shown above.

We can give a characterization of the Λ-matrix for a tetrahedron with two spikes on each corner.

1. Let $P = \{p_1, p_2\}$; $Q = \{q_1, q_2\}$. Let $a = \{1, 2\}$; $b = \{3, 4\}$; $c = \{5, 6\}$; $d = \{7, 8\}$. Then excluding the special case: $\{p_1, q_2\} \neq a$, b, c, or d and $\{p_2, q_1\}$ If $\{P; Q\}$ contains a, b, c, or d then $\det \Lambda(P; Q) = 0$.

   The Λ-matrix satisfies this determinental property because the only way to get from a boundary node to another boundary node is via its interior neighbor node. If a, b, c, or d is contained within $\{P; Q\}$ then, excluding the special case, the paths $p_1 \longrightarrow q_2$ and $p_2 \longrightarrow q_1$ will intersect.

2. Diagonal entries are all > 0.

   Diagonal entries represent the current flowing into the network when the i^{th} node has Voltage $= 1$, and all the other boundary nodes have Voltage $= 0$. All nodes are connected to each other, so the diagonal entries can never be zero. The potential at the other nodes is ≤ 1, so current must be flowing into the network at node i, thus $\lambda_{i,i} > 0$.

3. All off-diagonal entries are < 0.

   Every boundary node is connected to every other boundary node through the interior. $\lambda_{i,j}$ (i $\neq$ j) represents the current flow out of the i^{th} node when the potential at the j^{th} node $= 1$, and all other nodes have potential $= 0$. Current must flow in through the j^{th} node and out through the other boundary nodes, so $\lambda_{i,j}$ (i $\neq$ j) ¡ 0.

4. Let $\{p_1, q_2\} = a$, b, c, or d. Let $\{p_2, q_1\} \neq \{p_1, q_2\}$ and $\{p_2, q_1\} = a$, b, c, or d. Then $\det \Lambda(P; Q) < 0$.

   From the paper or circular planar graphs and resistor networks (E.B. Curtis, D. Ingerman, J.A. Morrow), if there is only one way to connect P to Q, then that determinant in the Λ-matrix is always one sign. By assigning conductivities and computing the Λ-matrix, that sign was determined to be negative.

5. Let $\{P; Q\}$ not contain any node more than one time. Then if $\{P; Q\}$ is not always negative by 4 or always zero by 1, then the determinant can be positive, negative or zero.

   We can assign conductivities so that the above determinants are negative. We can also assign conductivites so that they will be positive.

By the Intermediate Value Theorem, we can also assign conductivites so they will be zero.

6. Let $P = (1, 4, 6, 7)$ and $Q = (2, 3, 5, 8)$. Then $\det \Lambda(P; Q) > 0$. Also, if we let $P = (1, 3, 5)$ and $Q = (6, 4, 2)$ then $\det \Lambda(P; Q) > 0$.

   As with the case from 4, the determinant depends on the possible permutations and connections from P to Q. There is only one way to connect P to Q so the determinant can only be of one sign and never zero. By assigning conductivities and determining the Λ-matrix, the sign of those determinants is zero.

We shall now give a corollary to **Amanda's 4-Node Theorem**: The Λ-matrix of any network, planar or not, with four boundary nodes and four edges can always be made into the Λ-matrix of a planar network with no more than one row interchange and column interchange.

The network we are considering has eight boundary nodes, however, Amanda's 4-Node Theorem still applies.

Corollary to Amanda's 4-Node Theorem (2): Any single tetrahedron network with 2 spikes on each corner either has the same Λ-matrix as a planar graph or by interchanging one pair of rows and columns with another pair, either a, b, c, or d – keeping the pairs together – the network will then have the same Λ-matrix as a planar network.

Proof: The 2×2 determinants due to the spikes are all either negative or zero. The 3×3 and 4×4 determinants all have the right sign. Therefore, if we interchange one pair of rows and columns, the only determinants that will possibly change are those which can have any sign. By Amanda's 4-Node Theorem, when we do this row and column interchange, those determinants will all have the right sign, and the other determinants will not have changed. Thus, we can do one row and column interchange between pairs and have the Λ-matrix for a planar network.

Remark: The recovery, Lemma, characterization of the Λ -matrix, and corollary, relating to the two-spike single tetrahedron network can be modified for an n-spike single tetrahedron network ($n \geq 2$) as well.

2.2 Two Tetrahedron Network With and Without Spikes

The first network we consider is a tetrahedron network made by glueing the faces of two tetrahedrons together. Without spikes, the recovery is quite

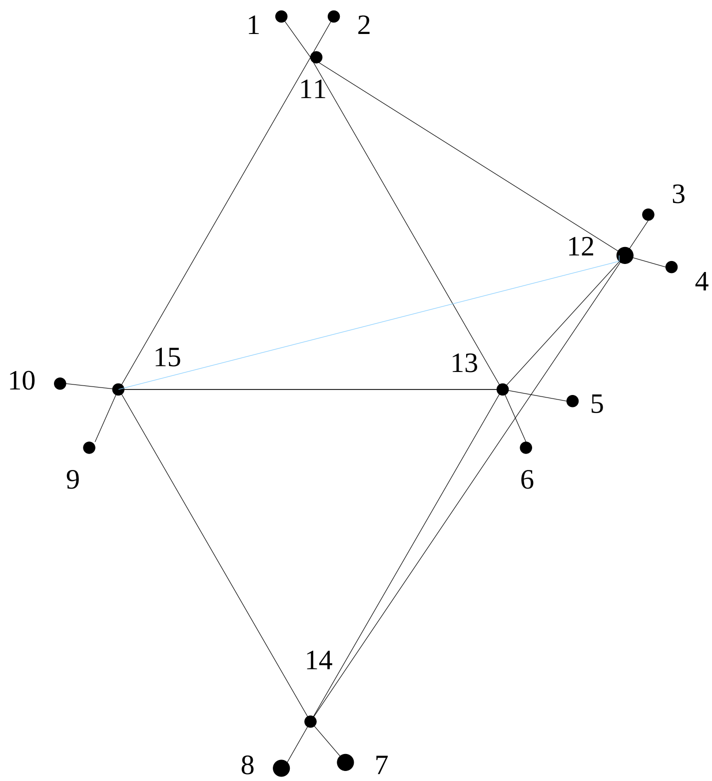

Figure 4: Two Tetrahedron Network With Two Spikes at Each Corner

simple, for the Λ-matrix is the Kirchoff matrix. When we add spikes, the network is still recoverable (as indicated by the Lemma). We give a method to recover this network:

We recover the boundary spikes using the same procedure as for the single tetrahedron: assign a voltage of zero and current of zero to nodes 2, 4, 6, 8. At node 10 we assign a voltage of one and current of one. This forces the rest of the network. Nodes 11, 12, 13, and 14 will have voltage zero, and node 15 will have voltage 1. Nodes 1, 3, 5, 7, 9 will have unknown voltages of a, b, c, d, and e, respectively. Current will flow into the network from node 9, and out of the network at nodes 1, 3, 5, and 7. We can solve for the unknowns because the determinant $\Lambda(P; Q) \neq 0$ where $P = \{2, 4, 6, 8, 10\}$ and $Q = \{1, 3, 5, 7, 9\}$. This determinant is non-zero because there is a unique way to connect P to Q. We know the current out of each boundary node, thus we can solve for the conductivity along each spike. Using Kirchhoff's

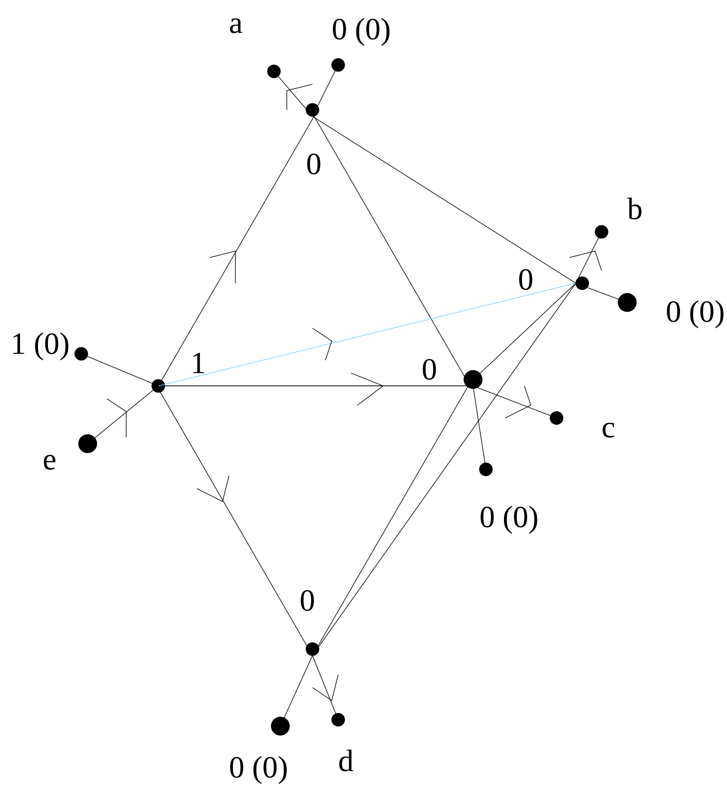

Figure 5: Voltage Pattern for Recovery of Two Tetrahedron Network

law, we can compute the conductivity along all these interior edges as well. We can change our pattern, so that nodes 1, 3, 5, 7, and 9 have zero current flow and first assign a voltage of one to node 1, and then to node 3, and finally to node 7. By following the same procedure given above we can recover the remaining conductivities. Now we have recovered the whole network.

The Λ-matrix of the two-tetrahedron network with two spikes has essentially the same characteristics as the single tetrahedron network with spikes. The above characterization applies to this network also. One additional characteristic: The 5×5 determinant $\Lambda(P;Q) < 0$ when $P = \{1, 3, 5, 7, 9\}$ and $Q = \{10, 8, 6, 4, 2\}$. The reason is the same as for 5, above.

There is no parallel to Amanda's 4-Node Theorem for networks (planar and not) with five boundary nodes and five edges. Her work with the Λ-

matrix of those networks shows that there are some Λ-matrices for such networks that can never be made to look like the Λ-matrix of a planar network (Amanda's 5-Node Theorem).

2.3 Some Non-Recoverable Tetrahedron Networks

The next network we consider is made with two sets of four tetrahedrons glued together and then the two sets glued together.

This network is significantly more difficult to recover than the other networks we have worked with. The Λ-matrix is NOT the Kirchhoff matrix. If can recover the above network (with no spikes), then by the Lemma, the same network with spikes is also recoverable. Here is the recovery process we attempt:

To recover the conductivity along the outside edges – from node 1 to nodes 2, 3, 4, 5, and 6 and likewise from node 4 to nodes 2, 3, 5, and 6 – we need to get a current flow from one of nodes 1-6 to another node and only that node. We must be able to first compute the voltages at both nodes. Then we can measure the current flow in/out of that node and use the equation: Current = Voltage Drop * Conductivity. Let's choose node 1 to make this attempt. We let node one have voltage = a. We want current to flow only along from node 1 to one other node, let's choose node 2. Our choice of nodes really doesn't matter, because of the symmetry of the network. We then need nodes 3, 5, 6, and 7 to have the same voltage as node 1, a, so that no current will flow there. However, we want node 2 to have a different voltage, we'll call this voltage b. Current flows from a to b. In order to have the central node have voltage a, we must have one of the nodes adjacent to it have voltage a and current flow zero. We cannot have nodes 3 or 6 have zero current flow, for they already must have voltage a and are connected to node 2 which has voltage b, so current must flow from those nodes to node 2. We cannot let node 4 have zero current and voltage a, for it is directly connected to node 2 also and there would be a voltage drop causing current to flow from node 4 out to node 2. We must choose node 5. This causes node 4 to have a voltage of a also. However, if we consider what we have done, current must flow from node 7 to node 2, for current flows from a to b. However, this is an interior node and must obey Kirchhoff's Law. In this situation it does not. Current leaves the node but an equal amount has no place to enter from. We cannot recover this network. By the Lemma, we cannot recover it if we add spikes, either.

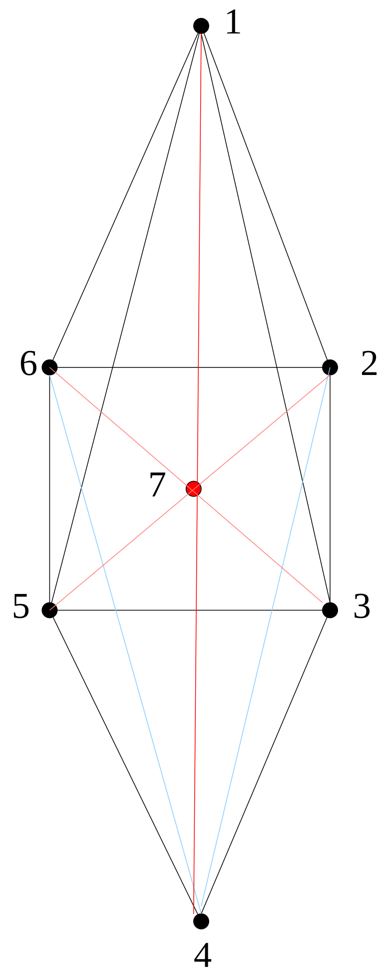

Figure 6: Eight Tetrahedron Network Without Spikes

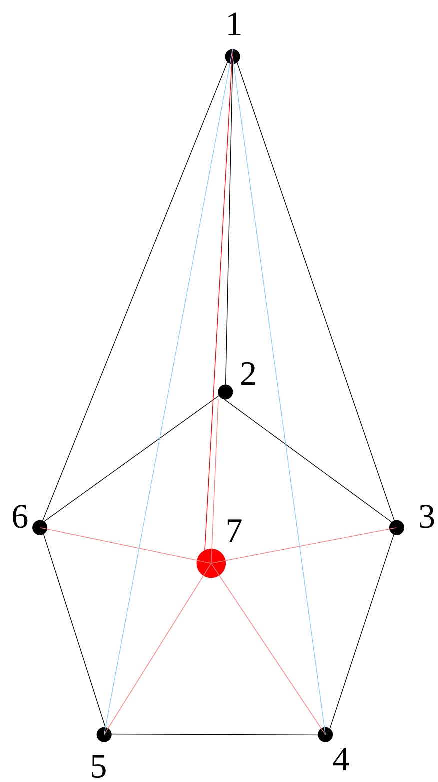

Figure 7: Five Tetrahedron Network

The next network we will attempt (unsuccessfully) to recover consists of five tetrahedrons glued together so that their base looks like a pentagon.

Here we make our attempt at recovery of this network. We want to compute the conductivity along a boundary edge. First we will try to get the conductivity along one of the edges coming down from node 1. Again, we assign node 1 a voltage of a. We want current to flow along only one edge, so we need node 2 to have a voltage of b, and the other nodes to have voltage of a. In order to ensure that node 7 has voltage a, we must assign a voltage of a and current of zero to node 4 or 5, as 2 already has voltage of b, and nodes 3 and 6 are directly connected to node 2. We choose node 4. This makes it so that current flows down from node 1 and to node 2 and nowhere else. However, we run into the same problem we did earlier: there is a voltage drop between nodes 2 and 7. Current flows from node 7 to node 2, but no current flows into node 7. This does not satisfy Kirchhoff's Law as it is required, thus there is no way to compute these edges directly. We will now try to compute one of the edges of the face of pentagon. We want current to flow from node 2 to node 3 and not to any other nodes. We set node 2 to voltage a, and node 3 to voltage b. We need all the nodes connected to node 2 to have voltage a. We set node 1 to have voltage a, and node 6. In order that node 7 have voltage a, we must have one of the boundary nodes adjacent to node 7 have zero current and voltage a. This cannot be nodes 1 or 4 because they are directly connected to node 3 and will have a voltage drop. That leaves either node 5 or node 6. It does not matter which we pick, because of the symmetry of the network. We choose node 5. This ensures that nodes 4, 6, 7, and 1 will have voltage a. However, we again are faced with the same problem as when we tried to recover the other edges – node 7 has current flowing OUT to node 3, because there is a voltage drop, but no current flows IN to node 7. As node 7 is an interior node, it must obey Kirchhoff's Law. Therefore, we cannot recover this network, and by the Lemma, we cannot recover it with spikes either.

We can attempt to recover the same network with an additional boundary node added at the base, so that the base is a hexagon instead of a pentagon. That network is still not recoverable. Similarly, if we add another node below the pentagon, so that we have two networks like above glued at the pentagon – that network is not recoverable either. Now on to some networks we can actually recover...

3 Spiky and Not-So-Spiky Cylinders

3.1 Cylinders with an Odd Number of Boundary Nodes

We consider a set of networks which are recoverable in the plane. These networks consist of n circles and either $4n+1$ rays emanating from the center or $4n+3$ spikes emanating from the center. These networks are recoverable for all n (see paper on Circular Planar Graphs and Resistor Networks by Curtis, Ingerman, and Morrow). We take these same networks and place one below the other, attaching corresponding boundary nodes, as well as the central node of each network to the central node of the other network. In the case where we have spikes, we can either attach the ends of the spikes to each other, or we can attach where the spike hits the outside circle. We

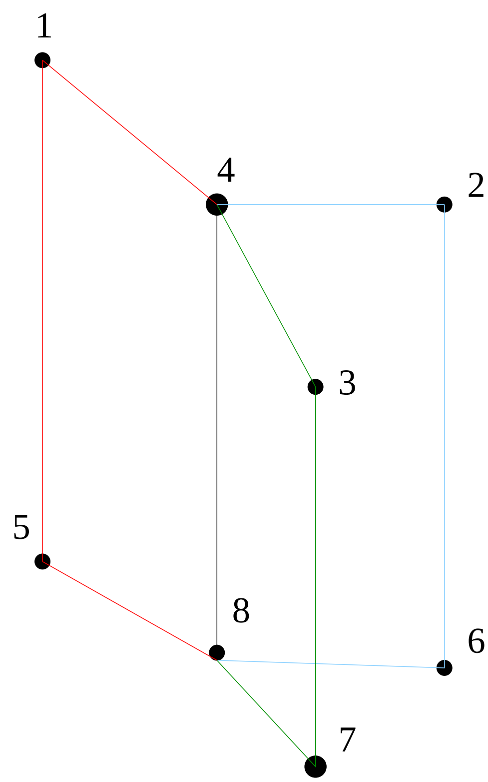

Figure 8: Paddle Wheel Network with $n = 0$

refer to the former as "paddle wheels" and the latter as "spikes." The first network, in which the rays end at the outermost circle, we refer to as "rays."

The first network we consider is not really a cylinder. It has three boundary nodes on each face. It does not have a circle enclosing them, so it could be considered the paddle wheel case with $n = 0$.

Now we give the recovery procedure for this network. There are two steps.

1. First, we assign a voltage of zero and current of zero to nodes 2 and 6. This forces nodes 4 and 8 to have voltage zero. Next, we assign a voltage of 1 to nodes 1 and 5. This forces a current to flow from nodes 1 and 5 into nodes 4 and 8, respectively. By Kirchhoff's Law, an equal amount of current must flow out of nodes 4 and 8 to nodes 3 and 7, respectively. These nodes must have voltages a and b, respectively. We have the same amount of unknowns as conditions, and there is only

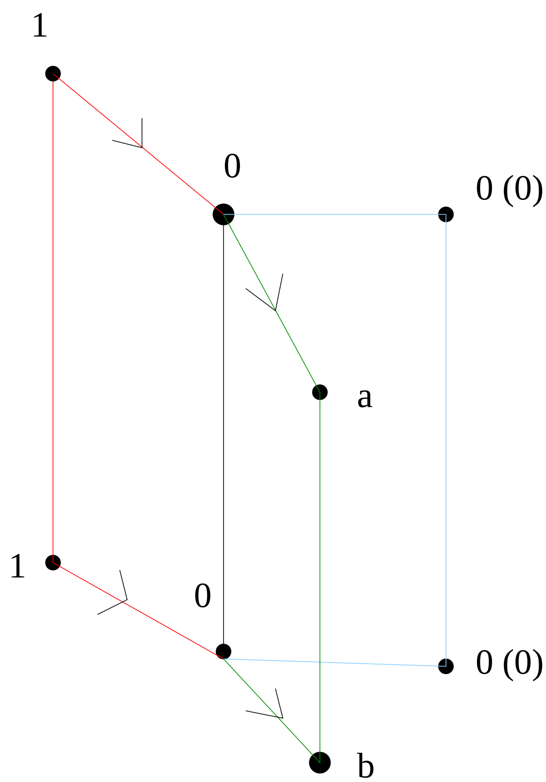

Figure 9: First Voltage Pattern for Recovery of $n = 0$ Paddle Wheel Network

one way to connect nodes 2 and 6 to nodes 3 and 7 without crossing. Thus that determinant in the Λ-matrix is not zero and we can solve for the unknowns. We can measure the current flow into the network at nodes 1 and 5, and we know the only path that current takes as well as the voltage drop along that path. Thus we can solve for the conductivity along those edges. We can now use the same pattern of voltages and currents on each face, just rotated one node clockwise on each face. Our unknowns may be different, but because the network is symmetric we can still solve for them. We can continue to rotate until we have found all the horizontal conductivities.

2. Now we will recover the vertical conductivities. We assign a voltage of 1 to nodes 1 and 5. We assign a voltage of 1 and current of zero to node 2, and a voltage of zero and current of zero to node 6. This forces nodes 3 and 7 to have voltages of a and b, and nodes 4 and 8 to have voltages of

v and w, respectively. We have the same number of unknown boundary voltages as conditions and only one way to connect those nodes without crossing. This determinant in the Λ-matrix cannot be zero, so we can always solve for the unknowns. We can get the current flow out of node 1 from the Λ-matrix, and because we know the conductivity along that horizontal edge we can solve for v. We can also solve for w by the same reason. Now we can calculate the current flow into node 2 from node 4 and because the net current is equal to zero, we know that that must be the same as the current flow from node 2 to node 6. We know all the voltage drops, thus we can solve for that vertical conductivity. We can also find the conductivity along the central vertical edge, for we can calculate the net current flow into the network as well as the net current flow out, and we know the voltage drop, so we can solve for the conductivity. We rotate this pattern as before to get the last of the vertical conductivities. Now we have recovered the whole network.

The next network we consider is the ray network with $n = 1$. We have five boundary nodes on each face in this network.

Now we shall recover this network. This requires three steps.

1. First we recover the faces. We assign nodes 5 and 11 a current of zero and voltage of zero. This forces nodes 1, 4, 6, 7, 10, and 12 to have voltage zero. Next we assign a voltage of 1 to nodes 2 and 8. This causes current to flow into the central nodes, so current must also flow out of the cenral nodes. This forces the voltages at nodes 3 and 9. They are respectively a and b. We have the same number of unknowns as conditions and a unique way to connect them, so that determinant is not zero and we can solve for the unknowns. We know the current flow out of node 1, and all that current came from node 2. We can solve for that boundary edge. We can rotate as before to get all the edges around the circle. Now we know the net amount of current that goes into node 2, as well as how much travels around the circle. We can thus calculate the conductivity from node 2 to the central node (6). By rotating we can get the conductivity along all the rays. We have now recovered the faces.

2. To recover the outside vertical edges, we use the same pattern, but instead of aligning the top and bottom faces, we make them so that

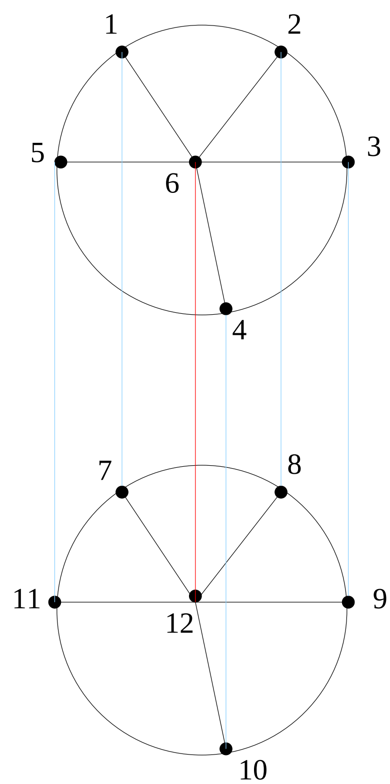

Figure 10: Ray Network with $n = 1$

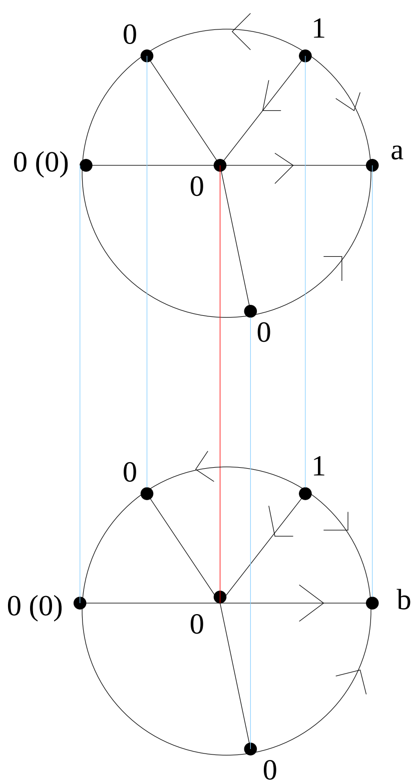

Figure 11: Voltage Pattern for Recovering $n = 1$ Ray Network

top face stays the same as before, but the bottom face has voltage c at node 8 and 1 at node 1. By applying Kirchhoff's Law to the central node on the bottom face (12), we know that $c < 0$. So current must flow from node 2 to node 8. We can still calculate the unknowns, so we can still use Λ to calculate the net current out of each boundary node. We know the net current out of node 2, as well as how much flows along the top face. We know the voltage drop between node 2 and node 8, so we can solve for the conductivity along that edge. By rotating we can get the conductivity along all those vertical edges.

3. Now we wish to recover the vertical edge running down the center. To do this, we assign a voltage of 1 to all the boundary nodes on the top face and a voltage of zero to all the boundary nodes on the bottom face. This causes a voltage of v at node 6 and w at node 7. We cannot have a voltage of 1 at node 6 and a voltage of 0 at node 7, for then they would not obey Kirchhoff's Law. Similarly, we cannot have $v = w$, and both not equal to 1 or 0, because then they would not obey Kirchhoff's Law. So we know that $v \neq w$ and that current flows from the top boundary nodes down to the bottom nodes and also in to node 6, and current flows from node 6 to node 12, and out at all the bottom boundary nodes. We know the conductivities on the faces and the vertical exterior edges. We can measure the current flow into node 1 using Λ. We know the amount of current that travels down to node 7, and the conductivity along the edge connecting node 1 to node 6, and the net amount of current leaving node 1, so we can solve for v. Similarly, we can solve for w. Now we simply apply Kirchhoff's Law to node 6 or node 12 to calculate the amount of current flowing between the two nodes, and since we know the voltage drop, we can calculate that conductivity. We have now recovered the whole network.

The next network we consider is the spike network with $n = 1$ (7 spikes). The recovery of this network is also in three steps.

1. Again, we recover first part of the faces of the cylinder. We assign voltage and current zero to nodes 4, 5, 6, 19, 20, 21. This forces nodes 10, 11, 12, 13, 14, 15, 25, 26, 27, 28, 29, and 30 to have voltage zero. We assign a voltage of one to nodes 7 and 22. This forces the voltages in the rest of the network to be determined. Current will flow in at nodes

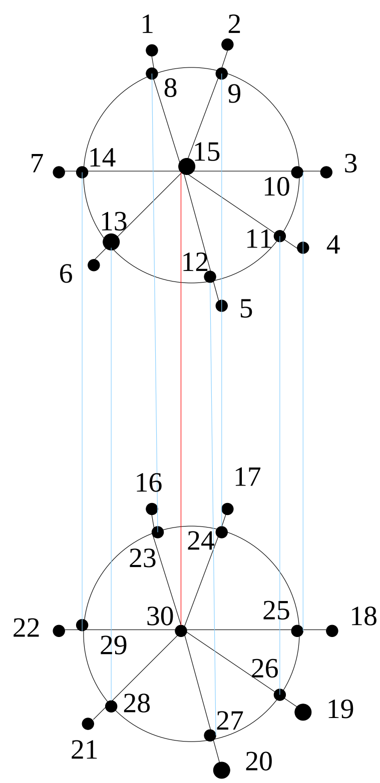

Figure 12: Spike Network with $n = 1$

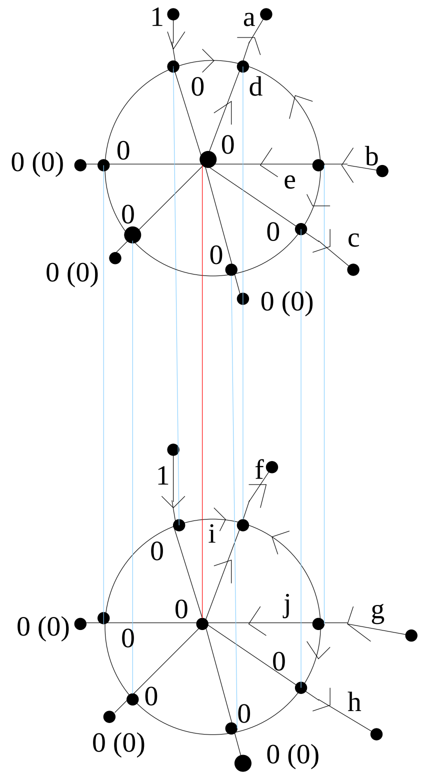

Figure 13: Voltage Pattern For Recovering $n = 1$ Spike Network

7 and 22, around from node 14 to 8, out at node 1, in at node 2, out at node 9 to nodes 8, 10, and 15, out at node 3, and the same pattern will be on the bottom face. We now have three unknown boundary voltages on each face, and three conditions on each face. Again we have a unique connection so we can solve for the unknowns. Now we can readily get the conductivity along the spike at node 7, for we know the current in and the voltage drop. We rotate and solve for the conductivity along all the spikes. Next, we can solve for the interior unknown voltages at nodes 8, 9, 23, and 24. We can now get the conductivity from node 14 to node 8. We rotate again, and recover the conductivities along the circle.

2. Now we will leave all the conditions we have imposed the same, except we will change the voltage at node 22 so that the resulting voltage at node 23 is the same as the voltage at node 7. We can easily solve

for these voltages as we can measure the current flow out of all the boundary spikes using Λ and we know the conductivity along all the spikes. We can thus ensure that no current will flow between node 23 and node 8. We apply Kirchhoff's Law to node 8, and we can compute the conductivity between nodes 8 and 15. Again, we rotate this pattern and compute the conductivities of all the edges leading into the center of the circle on each face. Now we adjust the voltage at node 22 so that the resulting voltage at node 23 is different from the voltage at node 7. This causes current to flow between these two nodes. We again apply Kirchhoff's Law to node 8, and we can now compute the vertical conductivity along that edge. We rotate and can compute all the vertical conductivities. The only edge left is the center vertical edge...

3. We assign zero current flow to nodes 1, 6, 7, 16, 21, 22. We assign nodes 1, 6, 7 a voltage of one and nodes 16, 21, 22 a voltage of zero. We assign voltage of one to node 2 and node 17. The rest of the voltages in the network are forced. We have only one way of connecting the nodes with known current to the nodes with known voltage so we can solve for the unknown boundary voltages. We know the current flow in/out of each boundary node by the Λ-matrix and the conductivity along each spike so we can solve for the unknown interior voltages. The voltage at nodes 8, 13, and 14 must be one, so that no current flows to nodes 1, 6, and 7. Similarly the voltage at nodes 23, 28, 29 must be zero so that no current flows to nodes 16, 21, 22. This forces current to flow vertically downward. Nodes 8, 13, 14, 23, 28, 29 are all interior nodes and must obey Kirchhoff's Law, so node 15 must be positive and node 30 must be negative. We have current flowing along this interior vertical edge. We can calculate the voltage at nodes 15 and 30 by applying Kirchhoff's Law to nodes 8 and 23 and solving for the unknown interior voltages. Now we again apply Kirchhoff's Law, to node 15, and we know the voltage drop and the current flowing in so we can calculate the current flowing out and the conductivity along the edge from node 15 to node 30. We have thus calculated all the conductivities in the network.

The next network we consider is the paddle wheel network with $n = 1$. This network is easily recovered.

Now we will show how to recover this paddle wheel network. It requires

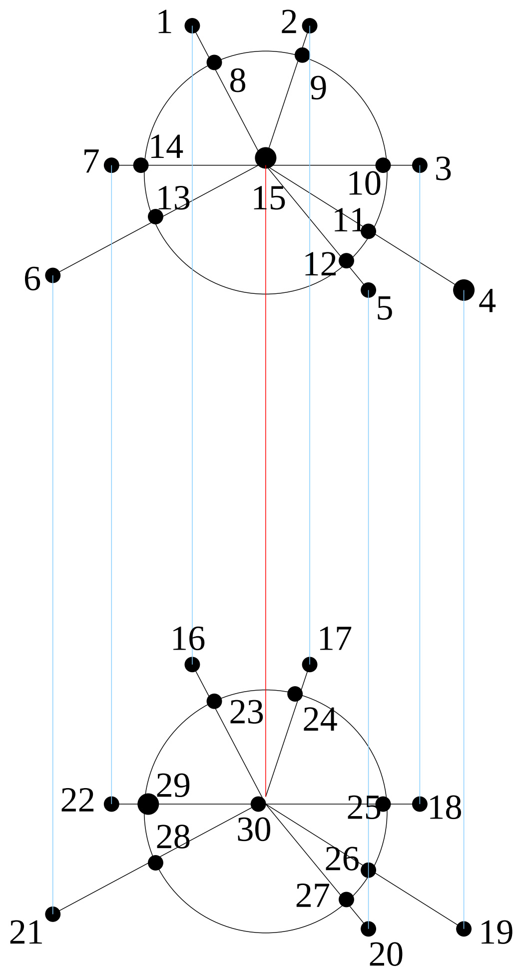

Figure 14: Paddle Wheel Network with $n = 1$

3 steps.

1. We will first recover the conductivity from the circle to the points of the paddles. We assign a voltage of zero and current of zero to nodes 1, 2, 3, and 16, 17, 18. This forces nodes 8, 9, 10, 11, 14, 15, and 23, 24, 25, 26, 29 to have zero voltage. We assign nodes 7 and 22 a voltage of one. The rest of the network is forced by these conditions. We have six unknown voltages at nodes 4, 5, 6, 19, 20, and 21. These are alternately negative and positive. We also have four unknown interior voltages at nodes 12, 13, 27, 28. There is only one way to connect the set of boundary nodes with unknown voltage to the set of nodes with current zero, so we can solve for the unknowns. We can now calculate the conductivity from node 7 to node 14, for we can get the net current into node 7 from the Λ-matrix, after solving for the unknowns, and we know the voltage drop. We can rotate this pattern and find all those conductivities from the outer corner of the wheel to where it touches the circle.

2. Now we will find the vertical conductivities. We use the same pattern of voltages, except we assign voltage -1 to node 22. This causes a vertical current flow from node 7 to node 22. We can solve for the six unknowns, as before, and we can calculate how much current flows into node 7 from Λ. We know that the amount of current going into node 7 is equal to the sum of the current that flows to node 14 and to node 22. We know how much goes to node 14, and we know the voltage drop to node 22. We can now calculate the conductivity from node 7 to node 22. We can rotate our pattern and get the remaining vertical conductivities of the paddles. We now can solve for the four unknown interior voltages. We use node 13. We know how much current flows from node 21 to node 6. We know the amount of current flowing out of node 6, and we know the conductivity from node 6 to node 13. We can solve for the voltage at node 13. Using the same method we can solve for the voltages at nodes 12, 27, and 28. Now we can apply Kirchhoff's Law to node 14 to find the conductivity from node 13 to node 14. We can rotate the patter to get all the conductivities along the circle. We can again apply Kirchhoff's Law to node 13 to get the conductivity from node 15 to node 13. By rotating again we can get the conductivity from the center node to each of the nodes on the circle. We have now found the conductivity of the faces and the vertical exterior edges.

3. The last conductivity we find is from the top center node to the bottom center node. We fix the voltage at nodes 2, 3, 4, 5 to one. We fix the voltage at nodes 17, 18, 19, 20 to zero. We now make the current flow at nodes 2, 4, 17, 19 be zero. We make the current flow at node 3 very very negative, and the current flow at node 18 very very positive. We can easily compute the voltage drop along the vertical edges from nodes 2, 3, 4 to nodes 17, 18, 19. By applying Kirchhoff's Law to nodes 2, 3, 4 and nodes 17, 18, 19 we can easily find the value of the unknown interior voltages at those nodes. We now want to make sure that the current at node 3 is so negative that the voltage at node 10 is more than the voltage at nodes 9 and 11, and we want the current at node 18 to be so positive that the voltage at node 25 is much lower than the voltage at nodes 24 and 26. This will make it so that the voltage at node 15 will be very positive and the voltage at node 30 very negative, and current will flow down. We can solve for the 6 unknown boundary voltages as before, as well as all the interior voltages. We can then apply Kirchhoff's Law to node 15 and calculate the conductivity from node 15 to node 30. We have now recovered the whole network.

The next network we consider is the ray network with $n = 2$ (nine rays). This network is also recoverable.

The recovery of this network requires three steps.

1. First we will recover the faces of the cylinder as before. We assign a voltage of zero and current of zero to nodes 5, 6, 7, and 24, 25, 26. This causes nodes 4, 8, 12, 13, 14, 15, 16, 17, 18, 19, and 23, 27, 30, 31, 32, 33, 34, 35, 36 to have voltage zero. We assign node 3 and 22 a voltage of one. Now the rest of the voltages in the network are forced. We have 6 conditions and 6 unknown boundary voltages and a unique way to connect them, so we can solve for the unknowns. Now we can compute the current out of node 4 and get the conductivity from node 3 to node 4. We can rotate and get the conductivity for all the edges along the outer circle. Now we know the current into node 3 by Λ, and we know how much flows along the edge of the circle, so we can calculate how much flows in and get the conductivity from node 3 to node 12. We rotate again and get all these conductivities.

2. Now we will solve for the conductivities along the vertical edges. We use the same pattern as above, except we assign a voltage of -1 to node

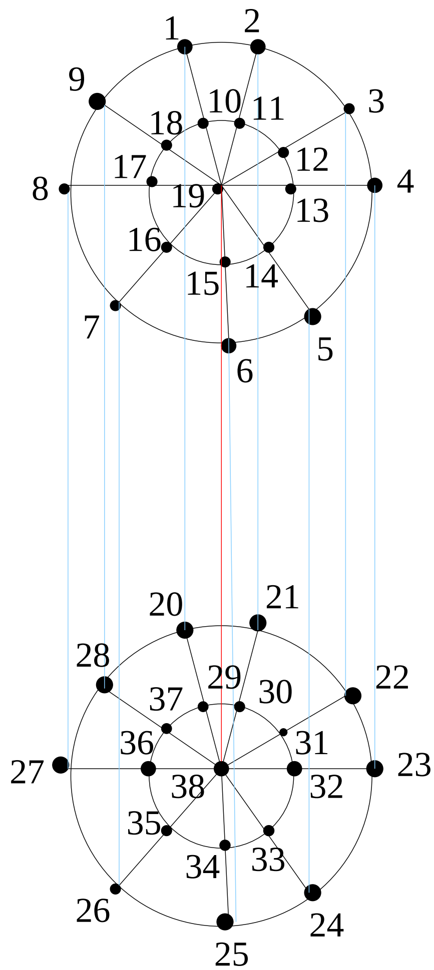

Figure 15: Ray Network with $n = 2$

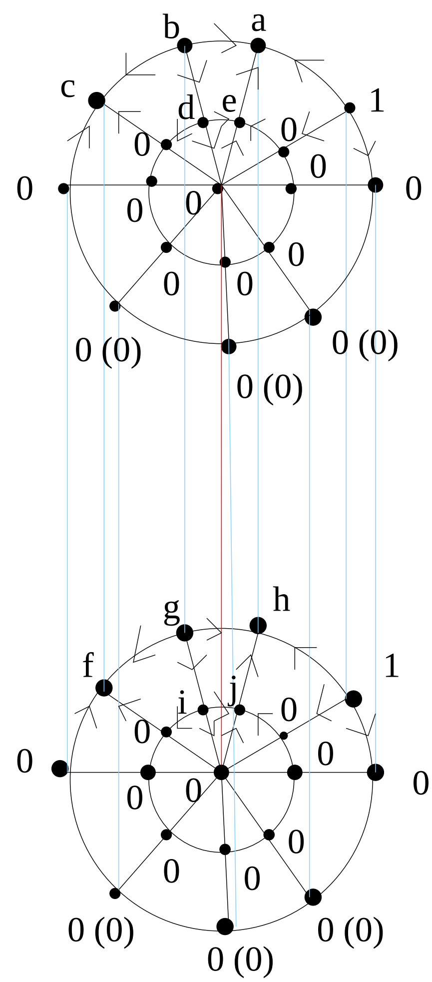

Figure 16: Voltage Pattern for Recovering $n = 2$ Ray Network

22. We can solve for the unknown boundary voltages as before, and thus we can compute the current into node 3. We know how much flows into the inner circle and how much flows along the edges, and we know the voltage drop from node 3 to node 22, so we can compute the conductivity along that edge. We rotate and get all the vertical conductivities. We are now in a position to solve for the unknown interior voltages. We know how much current flows out of node 2, and how much flows along each edge connected to that node, so we can calculate the unknown voltage at node 11. We can calculate all the interior voltages in this fasion. Now we can apply Kirchhoff's Law to node 12, and compute the conductivity from node 11 to node 12. By rotating we can compute the conductivity along the whole inner circle. Again we apply Kirchhoff's Law to node 11, and we can calculate the conductivity from the center of the top face to the inner circle. We rotate this pattern of voltages and compute the rest of them. Now we have both faces and all the vertical edges.

3. The last step (as usual) is computing the conductivity along the inside vertical edge. We assign a voltage of one to nodes 3, 4, 5, 6, 7, 8, 22 and a voltage of zero to nodes 23, 24, 25, 26, 27, and 2. We set the current equal to zero at nodes 5, 7, 24, and 26. We know the current that flows along the vertical edges from nodes 5, 6, 7 to nodes 24, 25, 26. We can apply Kirchhoff's Law to nodes 5, 6, 7 and 24, 25, 26 to find the voltage at nodes 14, 15, 16 and 33, 34, 35. We make the current at node 6 so negative that the voltage at 15 is much much higher than the voltage at 14 and 16. We make the current at node 25 so positive that the voltage at 34 is much much lower than the voltage at 33 and 35. This forces node 19 to have a very positive voltage and node 38 to have a very negative voltage, causing current to flow along that edge. We have the same number of unknowns as known current flow, and a unique connection, so we can solve for the unknown boundary voltages. We can also solve for the unknown interior voltages by using Kirchhoff's Law as we did before when we found the conductivity along the vertical edges and on the faces. We apply Kirchhoff's Law to node 19, and as we know the voltage drop from 19 to 38, we can solve for the conductivity along that edge. We have now recovered the entire network.

Now we are ready to state a theorem about the recovery of the ray, paddle wheel and spike networks in general.

Theorem: All networks made with two identically shaped (not necessarily identical conductivities) circular bases with n concentric circles and no less than $4n+1$ ($n \geq 1$) rays or $4n+3$ "paddle wheels" or "spikes" are recoverable.

Proof: For the proof we show how to recover these networks. There are three cases, the case where we have rays terminating at the outermost circle, the case where we have paddle wheels, and the case where we have spikes. We label these Case 1. Case 2. and Case 3.

Case 1: Assume we have a network as described above. We have shown how to recover this network for $n = 1$ and $n = 2$. We will now show the recovery process for an arbitrary n ($n \geq 1$). Let $n = m$. We will first recover part of the faces of the cylinder. Let $2m-1$ boundary nodes have zero voltage and zero current flow, for each face. Align these. Then let a boundary node one node away from these zeros have voltage one. Align these. This will force the voltages to be determined in the rest of the network: $2m+1$ nodes will have voltage zero on the outermost circle, and $2m+3$ nodes will have voltage zero on the next circle inside, $2m+5$ on the next, and so on, continuing until the innermost circle has $4n-1$ nodes with zero voltage. The central node will be forced to have voltage zero (these are for each face). $2m-1$ of the boundary nodes will have unknown voltages (on each face). There are the same number of unknowns as known current flow, and by Curtis, Ingerman and Morrow there is a unique way to connect these so that determinant cannot be zero and we can solve for the unknowns. Now we know the current flow out of all the boundary nodes. We can calculate the conductivity along the outermost circle by using the current flow out of one of the nodes with voltage zero adjacent to the nodes with current zero. The only place the current leaving that node came from is along the circle, from a node with known voltage, so we can calculate that conductivity. We can rotate our pattern of voltages and get the whole outer circle. Now we can use that information to calculate the current flowing in from the next circle inside at one of the nodes with voltage of one. We know the net current entering the network at that node, as well as how much flows out along the outer circle, so we can solve for the conductivity along the edge connecting that node to the next circle inside. We again rotate the pattern and calculate all these conductivities.

Now we change our pattern slightly. At the node on the bottom face whose voltage was set to one, we change that voltage to -1. Now we can calculate the unknown boundary voltages as we did before, and we know the current flowing out of the node with voltage 1 into the inside circle and along the edges, and we also know the total current going into that node, so we can compute the current flowing down to the node with voltage -1. We know the voltage drop along this edge, so we can calculate the conductivity there. We rotate our pattern and find all these conductivities. We now apply Kirchhoff's Law to one of the boundary nodes with an interior neighbor node with unknown voltage, and we can calculate that voltage, for all the other information is known. We can proceed in calculating all the interior voltages, using Kirchhoff's Law, and we can work our way inside the faces and get all the conductivities on the faces.

Now we must just find the conductivity along the middle vertical edge. For this we assume $m > 1$ (we have already shown the case $n = m = 1$). We let $2m + 2$ boundary nodes have voltage zero on the top face, and $2m + 2$ nodes on the bottom face have voltage one. We assign a current flow of zero to all these nodes except the first node, the $2m + 2$, $2m + 1$, and the $m + 1$ nodes on the top face, and the corresponding nodes on the bottom face. As before, we can now easily compute the current flowing from the top to the bottom as we know the voltage drop and the conductivity. We can apply Kirchhoff's Law to the nodes with zero current flow in order to solve for the voltages at those nodes' interior neighbor nodes. We now let the current at the $m+1$ node get so positive that the voltage at its interior neighbor is much less than all the other voltages on that same circle. We let the current at the corresponding node on the bottom face get so negative that the voltage at its interior neighbor is much more than all the other voltages on that same circle. We are then certain that current will flow from the bottom central node to the top central node. We know set the voltage to zero at one of the nodes on the bottom face adjacent to the group of nodes with voltage of one, and the corresponding node on the top face to voltage one. The voltages in the rest of the network will be forced, and we will again be able to solve for all the unknown boundary voltages as well as all the unknown interior voltages. We apply Kirchhoff's Law to the top central node, and as we know the current flowing out, the voltage drop, we can calculate the current flowing in and the conductivity along the interior vertical edge. We have calculated the conductivities for the whole network.

Case 2: In this case we have m concentric circles ($m \geq 1$) and $4m + 3$ paddle wheels. We start by solving for some of the conductivities on the faces. We assign $2m + 1$ nodes per face a voltage of zero and conductivity of zero. We align these. Now we assign a voltage of one to one of the nodes adjacent to the group of zeros, and the corresponding node on the bottom face. This forces $2m + 3$ nodes on the outermost circle (per face) to have voltage zero, $2m + 5$ nodes on the next circle inside, and so on until the innermost circle which has $4m + 1$ nodes with voltage of zero. The center node has voltage zero also. We have $2m + 1$ unknowns per face, the same number as known current flow, and by Curtis, Ingerman and Morrow, there exists a unique connection ensuring that the corresponding determinant is never zero so we can always solve for the unknown boundary voltages. Now we can compute the conductivity from the node at the outermost circle to the node with voltage one (on the top face). We rotate our pattern and get all these conductivities.

We now change the voltage of the node on the bottom face that was 1 to -1. We can now compute the conductivity along that vertical edge, for we know the net current out of the top node, and the current flow to its interior neighbor, and the voltage drop to the node below. We rotate this pattern and solve for all these voltages. We can now start applying Kirchoff's Law to the interior neighbors of the boundary nodes to solve for the unknown voltages at some of those nodes (m of them per face, to be exact). We can then work our way inside the faces and compute all the unknown voltages and apply Kirchhoff's Law to get all the unknown conductivities. Now we must get the conductivity along the inside vertical edge. We use a procedure quite similar to that in Case 1. We assign $2m + 2$ nodes per face a voltage of zero on the top face, and the corresponding nodes on the bottom face a voltage of one. We make all the nodes except the first node and the $m + 1$ node have zero current flow, and we do the same to the corresponding nodes on the bottom face. We now manipulate the voltages exactly as we did in Case 1 to enable us to have current flow from bottom to top and then we are able to compute it as we did before. We have $2m + 1$ nodes per face with known current and $2m + 1$ nodes per face with unknown current, and a unique way to connect these (as above), so we can solve for the unknowns. Now we can solve for the rest of the voltages in the network and then apply Kirchoff's Law to the top center node and find the current flow between the center nodes, and we will be able to solve for it as we will have calcaulted the voltage drop. We have shown how to recover the whole network.

Case 3: We have a spike network as described previously. We have shown how to recover the network for $n = 1$. We will now show how to recover the network for any $n \geq 1$. First we set $2n + 1$ nodes with zero current flow and zero voltage, per face. We align these. These zero voltage zero current boundary nodes force there to be $2n+3$ nodes with zero voltage the outermost circle, and $2m + 5$ nodes with zero voltage on the next circle, $2m + 7$ on the next and so on until the innermost circle has $4m + 1$ nodes with zero voltage and the center node, which also has zero voltage (this is on each face). We then set the node adjacent to zeros with voltage of one, and again align top and bottom. Now we can solve for the unknown boundary voltages, for there will be $2n + 1$ per face, and we know there is a unique way of connecting the nodes with unknown voltage to the nodes with unknown current by Curtis, Ingerman, and Morrow. That ensures that the determinant of that subsection of the Λ-matrix is non-zero so we can solve for the unknowns. We can now measure the current flow out of each boundary node. We measure the current flow into the node with votlage one. We know the voltage at this node's interior neighbor must be zero, so we can find the conductivity along the spike. We rotate this pattern of voltages and we can find the conductivity along all the spikes. We can then go back to the first pattern (after rotating) and find all the unknown voltages at the nodes on the first circle, for we know the current flow in and the conductivity on the spike and the voltage at the boundary node. We now apply Kirchhoff's Law to the interior neighbor of the node assigned voltage one (do same for each face). We can calculate the conductivity from that node to its neighbor farther around the outer circle. We rotate this pattern and we can get the conductivity on the whole outer circle.

Now we change the pattern slightly. We change the voltage of the node on the bottom face that lines up with the node on the top face with voltage one. We adjust the voltage of the bottom node so that the voltage at the node one place over around the circle (not towards the zeros, the other direction) has the same value as the voltage of the node directly above it. We can easily measure these voltages, as we know current flow into all the boundary nodes and conductivity along the spikes. This condition forces there to be no vertical current flow between those two nodes. Now we can calculate the conductivity from the node on the outer circle to its neighbor on the next circle inside, by applying Kirchhoff's Law to that node. We continue working our way inside the circle until we have recovered the whole face. Now we wish to get the vertical conductivities. We take the same node that we manipulated previously on the bottom face, and this time we make the voltage at the node on the circle on place over around the circle has a different voltage than the node directly above. This forces current to flow along that edge. We apply Kirchhoff's Law to one of these nodes, and then we can calculate the conductivity along that edge. We rotate this pattern and get all the vertical conductivities. The procedure for recovering the central vertical conductivity is identical to the procedure for the paddle wheel network. We have shown how to recover the whole network.

3.2 Cylinder Networks with an Even Number of Boundary Nodes

We now consider a network similar to the odd cylindrical networks discussed above. This network has six boundary nodes. It is made by taking a circle, adding three square petals, and connecting the corners of the petals down to a geometrically identical circular planar network below. More networks similar to this one can be constructed, but there must be $4n + 2$ boundary nodes for n circles. The petals start at the edge of the innermost circle. The six-node-cylinder (per face) is the simplest of these.

To recover this network we need two steps.

1. We assign zero voltage and zero current to nodes 1, 2, 13, and 14. This forces nodes 7, 8, 9, 12, 19, 20, 21, 24 to have voltage zero. We then assign voltage one to nodes 5, 6, 17, and 18. This forces the voltages to be determined in the rest of the network. There is a unique way to connect the nodes with known current to the nodes with unknown voltage, so that determinant in the Λ-matrix is always non-zero. We can solve for those voltages. We know the current into node 6, and we know the voltage drop from node 6 to node 12, so we can solve for the conductivity along that edge. We can rotate our pattern and get all those conductivities. We can then use that information to solve for the unknown voltages at nodes 10, 11, 22, and 23. We can apply Kirchoff's Law to node 12 and get the conductivity from node 11 to node 12. We can also apply Kirchhoff's Law to node 11 and get the conductivity from node 10 to node 11. We rotate our pattern and we can get the conductivity along the whole circle. We know the current flow into node 3 and the current flow from node 3 to node 9, so we can compute the current flow from node 3 to node 4. Again we rotate to

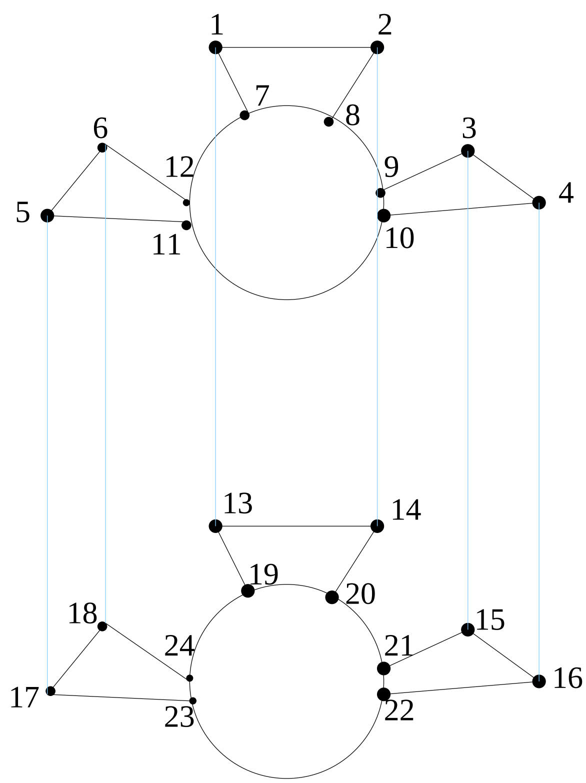

Figure 17: Cylindrical Network with 6 Boundary Nodes

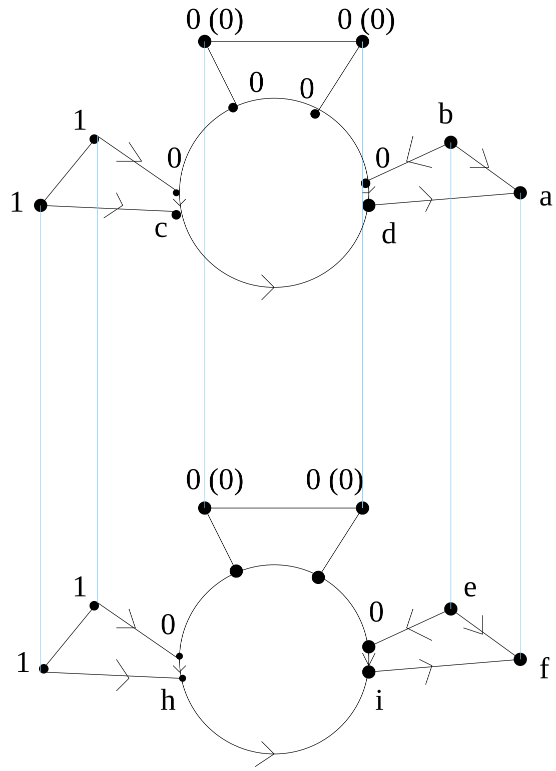

Figure 18: Voltage Pattern for Recovering $n = 1$ Petal Network

get the current flow along all these edges. We have now recovered the whole face.

2. We change our picture very simply now. We set the current flow at nodes 17 and 18 to -1. Now there is a vertical current flow. Again we can solve for the unknown boundary voltages. We know the current flow into node 6, and the current flow from node 6 to node 12, so we can solve for the current flow from node 6 to node 18. We know the voltage drop along that edge, so we can solve for the conductivity there. We rotate the voltage pattern and solve for all those conductivities. We have recovered the complete network.

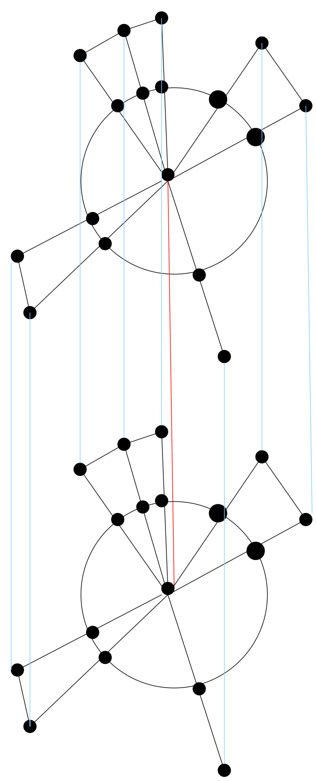

Figure 19: Eight Node Per Face Cylindrical Network

3.3 Conjectures

There is an eight-node-pattern similar to the six-node-pattern discussed above. We have not yet recovered it (lack of time), however the network is definately recoverable. The pattern for the 6, 10, 14,... node (per face) networks is that there are n concentric circles and $4n + 2$ boundary nodes, which begin at the innermost circle and terminate past the outermost circle. These boundary nodes are connected together in separate pairs. The pattern for the 8, 12, 16,... node (per face) networks is that there are n concentric circles and $4n + 4$ boundary nodes drawn straight from the center node of the circles. The boundary nodes stick out past the edge of the outermost circle. Three of the boundary nodes are connected to each other, and there is one spike, and the rest of the boundary nodes are connected to one other boundary node, in separate pairs.

We suspect that we could state and prove a theorem for recovery of cylindrical networks with an even number of boundary nodes similar to the odd-node theorem given previously in this paper.

The cylindrical networks discussed above could be slightly changed and would still remain recoverable (we are fairly certain): instead of having boundary nodes connected vertically, and the central node, connect every node on the top face to the node directly below it on the bottom face (assuming the faces are aligned). These networks are very similar to those discussed here, and most definately can be recovered similarly. The odd-node cylinder theorem and the even-node cylinder theorem (yet to be written) could very easily apply to those networks as well, though that has yet to be proven...

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