Uniqueness of sources in certain 3-Dimensional Networks

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Abstract

This paper will investigate pairs of resistor arrays of the same shape and with the same boundary data, and will show that two such arrays have the same internal source configurations. For certain rectangular and cylindrical networks, source location is uniquely determined by boundary data.

1 Introduction

Consider the two resistor arrays referred to above; call them Γ and Σ. Γ and Σ are the same shape; each have unit resistors and no sinks. Let Δ be the "difference" array of Γ and Σ, that is $I(\Delta) = I(\Gamma) - I(\Sigma)$. Δ, by definition, has sources where Γ has sources, sinks where Σ has sources, and unit resistors. Thus, if both Γ and Σ have sources at a particular node, then Δ will have a net current zero in that position. Also, it has been proven that Δ has boundary voltages equal to zero. Note that all potentials in Δ are zero if and only if there are no sources or sinks present. If Δ has neither sources nor sinks, then Γ and Σ have the same internal source configuration. In many of the networks studied here, it is necessary to restrict the sources in Γ and in Σ to unit size. Thus Δ's sources and sinks are restricted to unit size. Both Γ and Σ have the same boundary currents and thus the same number of internal sources. In this paper, the idea of the difference network Δ will be used to determine uniqueness of sources in 3-dimensional rectangular and cylindrical networks.

2 Proof of Uniqueness in a 3-D Rectangular Network one unit in width

Before proving that two such arrays have the same internal source configuration, it is necessary, in this case, to first prove that the voltage at every node in the difference array must be an integer. Note that the nodes in the array are not necessarily Γ-harmonic.

Lemma 1 *If a 3-dimensional rectangular resistor array (one unit in width) with unit resistors has boundary currents and voltages zero, then the voltage at every node in the array must be an integer.*

Proof. The currents in the boundary spikes and between the first layer in are zero because the net currents exiting the boundary spikes are zero. Thus the nodes on the first layer in have zero potential. Consider a node on the first layer in. Please refer to figure one.

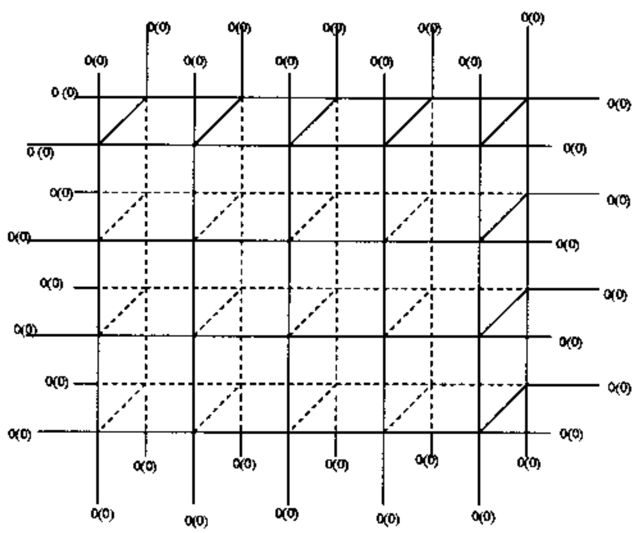

Figure 1: 3-dimensional rectangular resistor array one unit in width

Four currents flowing into this node are all zero. This node may be a unit source, unit sink, or harmonic node, so the current flowing from this node must be zero, 1, or -1. Therefore, all voltages on the second layer in are either zero, -1, or 1. Four of the five nodes surrounding a node on the second layer have integer-valued potential. Because the resistors have unit resistance, four of the five currents flowing into (or out of) this node are integer-valued. Thus, the fifth current is also of integer value. By induction, all nodes in the array have integer-valued potential.

Theorem 1 *Uniqueness Theorem*
If two 3-dimensional rectangular arrays (one unit in width) are the same size, have unit resistors, and have the same boundary data, then these two arrays have the same internal source configuration. Note that the sources in this problem are restricted to unit size. Remember that Γ and Σ have no sinks.

Method. Show that there are no sources or sinks in the difference array Δ. All potentials in Δ are zero if and only if there are no sources or sinks present.

Proof. Let μ be the maximum voltage in the difference array. Call v the node on diagonal ∂ with the highest potential μ. Since μ is the highest value of any potential, v must be a unit source or harmonic node. Please refer to figure two.

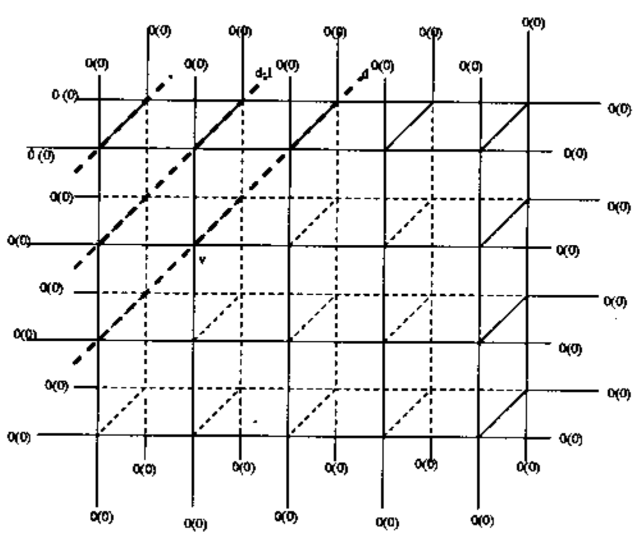

Figure 2: Previous resistor array cut by diagonal planes

If v is a harmonic node:

All neighbors of v must have potential μ. This is because the potential at a node is the average of the surrounding potentials, and since the highest is in the center, there can not be any surrounding nodes at higher potential.

Two of the neighboring nodes to v are on the $(\partial - 1)$ diagonal. Thus there is a node on ∂ with the maximum voltage.

If v is a unit source:

Because v has the maximum voltage value μ and all voltages are integer-valued, four neighbors of v have potential μ and the fifth has potential $\mu - 1$. Therefore, because there are two neighboring nodes on $\partial - 1$, at least one node on $\partial - 1$ has potential μ.

The maximum is eventually found on the first diagonal. Note that the only node on the first diagonal has potential zero. Thus the highest potential in the array is zero.

Conversely, by making the necessary subtitutions ("minimum" for "maximum," "lowest" for "highest," etc.), this proof shows that the minimum is also found on the first diagonal, so the minimum voltage value is zero.

The minimum and maximum voltage values in the array are both zero. Therefore, all nodes in the array have potential values of zero. The array is unique.

This proof, when modified, can be applied to any 3-D rectangular network with no restriction on the number of rows or columns on any of the sides.

3 Uniqueness in all 3-Dimensional Rectangular Networks

Theorem 2 *If a 3-dimensional rectangular resistor array with unit resistors has boundary currents and voltages zero, then the voltage at every node in the array must be an integer.*

Proof. This proof works the same way as the one above. The currents in the boundary spikes and between the first layer in are zero because the net currents exiting the boundary spikes are zero. Thus the nodes on the first layer in have zero potential. Consider a node on the first layer in. Please refer to figure three.

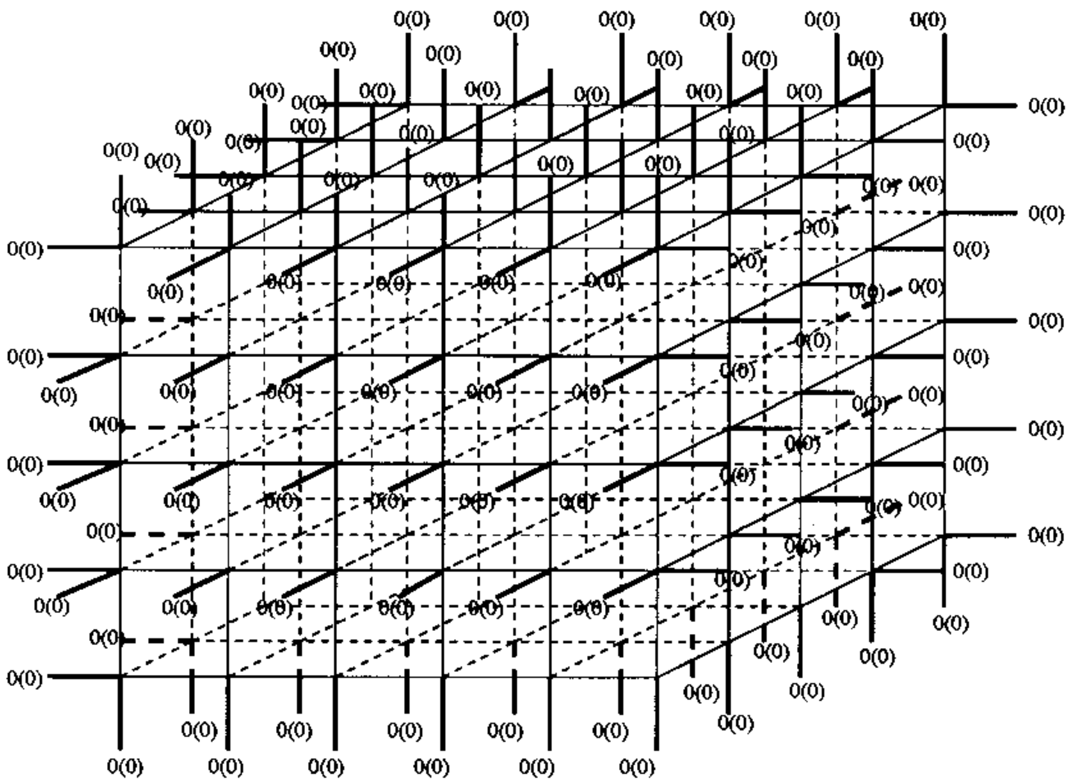

Figure 3: Rectangular resistor array with arbitrary dimensions

If the node is on one of the outside faces of the network, then the above proof applies. If the node is not on one of the outside faces, five out of six currents flowing into it are all zero. This node may be a unit source, unit sink, or harmonic node, so the current flowing from it must be zero, 1, or -1. Therefore, all voltages on the second layer in are either zero, -1, or 1. Five of the six nodes surrounding a node on the second layer have integer-valued potential. Because the resistors have unit resistance, five of the six currents flowing into (or out of) this node are integer-valued. Thus, the sixth current is also of integer value. By induction, all nodes in the array have integer-valued potential.

Theorem 3 *Uniqueness Theorem*
If two 3-dimensional rectangular arrays with unit resistors are the same size (no restrictions on the dimension size) and have the same boundary data, then these two arrays have the same internal source configuration. Note that the sources in this problem are restricted to unit size.

Method. Show that there are no sources or sinks in the difference array Δ. All potentials in Δ are zero if and only if there are no sources or sinks present. This method is the same as the one above witht he exception that the diagonal dividers are 2-dimensional sheets as opposed to being one-dimensional lines.

Proof. Let μ be the maximum voltage in the difference array. Call v the node on diagonal ∂ with the highest potential μ. Since μ is the highest value of any potential, v must be a unit source or harmonic node. Please refer to figure four.

If v is a harmonic node:

All neighbors of v must have potential μ. This is because the potential at a node is the average of the surrounding potentials, and since the highest is in the center, there can not be any surrounding nodes at higher potential.

Two of the neighboring nodes to v are on the $(\partial - 1)$ diagonal. Thus there is a node on ∂ with the maximum voltage.

If v is a unit source:

Because v has the maximum voltage value μ and all voltages are integer-valued, five neighbors of v have potential μ and the sixth has potential $\mu - 1$. Therefore, because there are two neighboring nodes on $\partial - 1$, at least one node on $\partial - 1$ has potential μ.

The maximum is eventually found on the first diagonal. Note that the only node on the first diagonal has potential zero. Thus the highest potential in the array is zero.

Conversely, by making the necessary subtitutions ("minimum" for "maximum," "lowest" for "highest," etc.), this proof shows that the

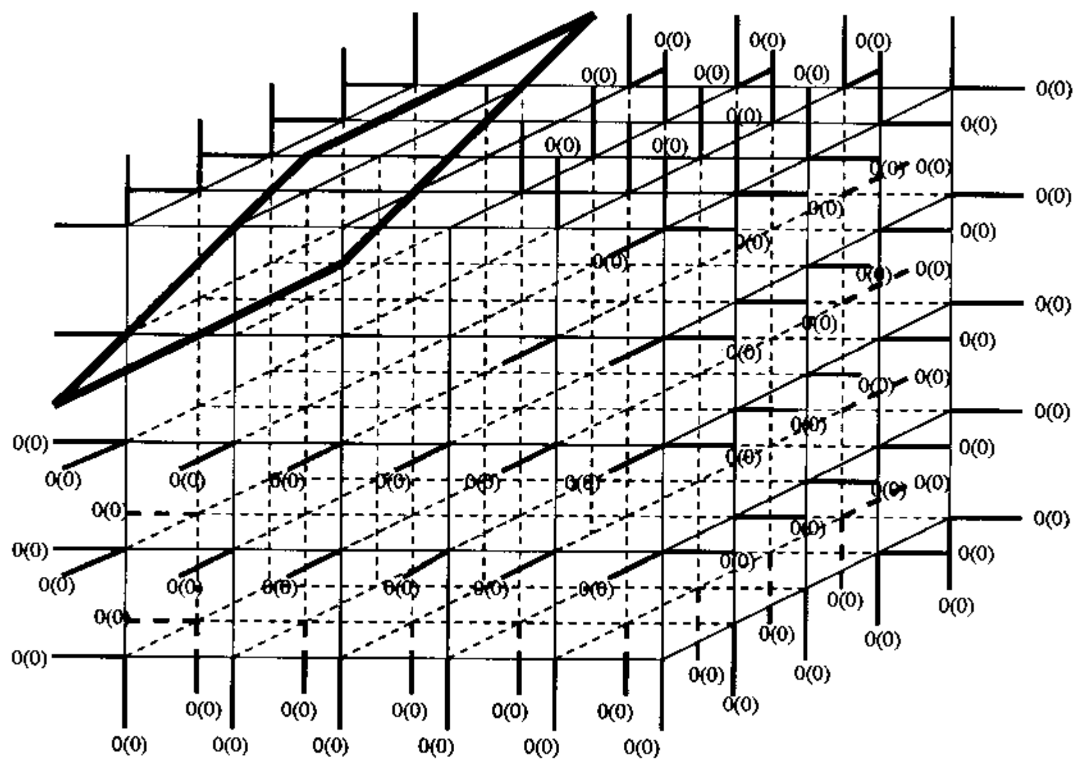

Figure 4: Previous resistor array cut by diagonal planes

minimum is also found on the first diagonal, so the minimum voltage value is zero.

The minimum and maximum voltage values in the array are both zero. Therefore, all nodes in the array have potential values of zero. The array is unique.

The method this proof is based upon can be applied to different networks of unrelated shapes, as will be demonstrated later in this paper.

Important note:

It was not necessary to carry out the proof to this length. If it is proven that every node in this network has an integer-valued voltage, and if the number of sources in this particular difference network is limited to strictly less than six, then the network is unique. This is because μ is the maximum voltage but μ always has some (not all) neighboring nodes with this same value μ. In the rectangular network being examined, when looking at an interior node with six neighboring nodes, it is known that at least five of the six neighbors have a voltage with value μ, as does the node in question itself. This makes a total of at least six nodes with the maximum voltage μ. However, not all of these nodes can be source nodes because the number of sources is restricted to be strictly less than six. So, one of these nodes with the maximum potential μ must be an interior node, neither a source nor a boundary node. This cannot happen.

There are other networks where uniqueness can be proven if all the nodes in a particular network all have integer-valued voltages. Oftentimes, the number of sources in a given network must be limited to be able to use this argument.

4 Proof of Uniqueness in Cylindrical Networks

Definition: The word "table" indicates a circular edge that also separates rays and spikes.

The first cylindrical networks to be considered are those that have spikes. This network has two tables with one spike each. Please refer to figure five.

As done previously, the difference network will be used to prove uniqueness. In the difference network, it is known that the boundary voltages and boundary currents are equal to zero, so the voltages at nodes two and four are zero. There is thus no current flowing on the circular edges, and there is none flowing between nodes two and four. However, nodes two and/or four could be unit sinks, unit sources, or harmonic nodes. There is no way to tell whether or not there is current flowing from either of those nodes to nodes one and three, their respective neighbors. Thus, this network is not unique.

The next network to be examined has two tables with two spikes each. Please refer to figure six.

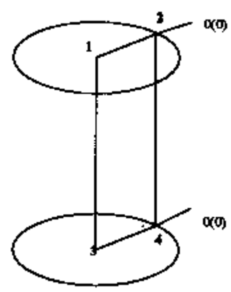

Figure 5: A very simple cylindrical network

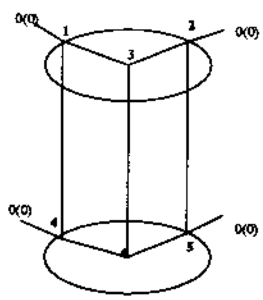

Figure 6: Cylindrical network with two tables that have two rays each

Proof that the nodes have integer-valued potential:
The boundary voltages and currents are zero, so the voltages at nodes one and two in this network are zero. Nodes one and two may each be a unit source, unit sink, or harmonic node. By definition, the resistors have unit resistance, so the current flowing from each of these nodes may be 1, zero, or -1, which are integer values. Thus the current flowing into node three must be integer-valued, so the voltage at node three is integer-valued as well. The same argument can be used to show that the voltage at node six also has an integer value. Thus all nodes in this array have integer-valued potentials.

Proof of Uniqueness:
The following equations may be obtained by looking at a sketch of

the network. $I(1)$ represents the net current at node one while $v(1)$ represents the voltage at node one, and so forth.

$$I(1) = I(2) = -v(3)$$
$$I(4) = I(5) = -v(6)$$
$$I(3) = 3v(3) - v(6)$$
$$I(6) = 3v(6) - v(3)$$

The net current at any of the nodes in this network must be 1, zero, or -1, because the sources in Γ and in Σ are restricted to unit size. By equations one and two, $|v(3)|$ and $|v(6)|$ must be less than or equal to one. $I(3)$ in equation three and $I(6)$ in equation four must be 1, 0, or -1. In order for equations three and four to have validity with these restrictions, $v(3)$ and $v(6)$ each must be zero. Thus, the net current at each and every node in the network is zero. There are no sources or sinks in Δ, so this network is unique.

Equations one through four were obtained directly by looking at the network. As long as there are two or more rays in each of the faces, equations such as these can be used to prove uniqueness.

Theorem 4 *Uniqueness Theorem depending on number of rays*
In a cylindrical network with two tables, there must be at least two spikes for the network to be unique. Please note that the spikes are connected to rays, which are, in turn, connected to one center node in the middle of each table.

Proof. It has already been shown that if there is only one ray and spike in each table, then the network is not unique. It has also been shown that if there are two rays on each table, then the network is unique. In a similar manner, it will be shown that if there are two or more rays on each table, then the network is indeed unique.

The difference network, once again, is to determine uniqueness. It is known that both the currents and the voltages at the ends of the spikes are zero. Thus, the voltages on the first and only layer of the network are also zero. The nodes on the first layer are either unit sources, unit sinks, or harmonic nodes. Thus the current flowing into each center node is of integer valued, which determines that the voltages at each of the center nodes are of integer-valued as well. All nodes in the network are determined to have integer-valued voltage, so the equation method can be used to solve this problem, as it was used to solve the above problem. Please refer to figure six. Note that $I(3)$ equals the number of rays coming into node three, multiplied by the voltage at node three, minus the voltage at node six. Note that $I(6)$ equals the number of rays coming into node six, multiplied by the voltage at node six, minus the voltage at node three. The same "rules" as in the above paragraph apply to this network.

In order for the equations for $I(3)$ and $I(6)$ to have validity with these restrictions, $v(3)$ and $v(6)$ each must be zero at all times (unless there are less than two rays). Thus, the net current at each and every node in the network is zero. There are no sources or sinks in Δ, so this network is unique.

The next network to be examined has two tables (tables A and C, which are also referred to here as rims) with a horizontal connecting line in between (line B). The two tables on the ends have rays that extend into spikes. Each of the two tables has the same number of rays as the other; let that number be six. Please refer to figure seven.

This proof of uniqueness is much like the one shown earlier for three-dimensional rectangular networks. This sketch (please see figure seven) is of the difference network of a network of this shape: Note that the boundary currents and voltages are zero, so the voltages on the first and only layers on the end tables are zero. The boundary nodes are those at the ends of the spikes on the end tables. It will be shown that wherever the maximum voltage is (and likewise, the minimum), its value is zero. Note that it is known that the potentials on the outside end rims are zero.

If the maximum is at the center node on either end:

- If the maximum voltage μ is a unit source, all neighbors except for one have voltage μ also. All of the neighbors except for one are on the rim of table A or of table C (depending on which end is in question), where all potentials are zero.
- If the maximum μ is a harmonic node, then all of its neighbors have the same potential, which is zero. Thus, the maximum μ has potential zero.

If the maximum is on the center axis, not on either end:

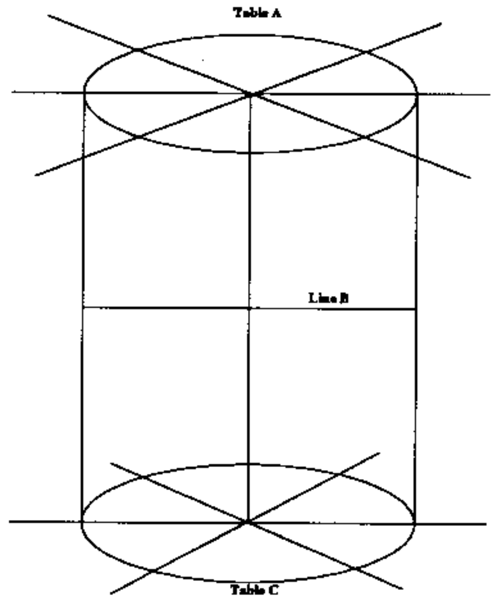

Figure 7: Another cylindrical network with spikes

• If the maximum is a unit source, all neighbors have potential μ except for one that has potential $\mu - 1$. Therefore at least one node on rim B has potential μ (unit source; also the maximum). All neighbors of this μ have potential μ except for one that has potential $\mu - 1$. There are two neighboring nodes; one of each is on rim A or rim C. One of these neighbors has potential μ (unit source; also the maximum). But all the potentials on the rim A and rim C are zero: Thus, the maximum potential is zero.
•If the maximum is harmonic, all of its neighbors have potential μ. At least one of those neighbors lies on line B. All neighboring potentials to that potential also have a value of μ; neighboring potentials on rim A or rim C have values of zero. Thus, the maximum potential in the network, μ, is zero.

If the maximum is on line B:
• If the maximum μ is a unit source, at least one node with the same voltage μ will be on rim A or rim C, where all the voltages have been determined, already, to be zero.

• If the maximum μ is harmonic, all of its neighbors have potential μ as well; two neighboring nodes are on rims A and C, where all the potentials are zero.

With any of these locations, the maximum μ is found to be on the edge of rim A and/or the edge of rim C, where all of the potentials are zero. Thus the highest potential in the array is zero.

By making the necessary subtitutions ("minimum" for "maximum," "lowest" for "highest," etc.), this proof shows that the minimum is also found to be on rim A and/or on rim C, so the minimum voltage value in the array is zero.

The minimum and maximum voltage values in the array are both zero. Therefore, all nodes in the array have potential values of zero. The array is unique.

5 Proof of Uniqueness in Cylindrical Networks

The next cylindrical networks to be considered are those with connected spikes; please refer to figure eight during the next proof.

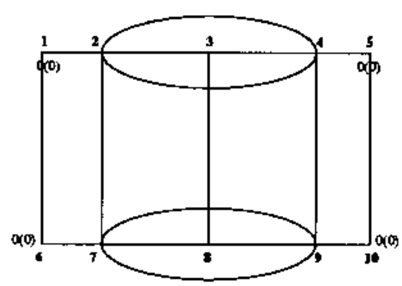

Figure 8: Simple cylindrical network with connected spikes

Nodes one, six, five, and ten have voltages and currents of zero. Because there is neither a voltage drop between nodes one and six and between nodes five and ten, no current flows along the respective boundary edges connecting them. Thus, that edge can be ignored and this case reduces to one already proven earlier in this paper (please see figure six and the accompanying proof).

6 Proving the uniqueness of a Cylindrical Network: a different approach

In the following specific networks, it is not simple to prove that the nodes must have integer-valued potentials. Thus, a slightly different approach to the problem may be required.
Quick note For the purposes of simplification, the use of the word "sources" refers to both sources and sinks when referring to the difference network Δ (neither Γ nor Σ have sinks).

Method of Approach.
If both Γ and Σ are restricted to one source each, then Δ would have one source and one sink if the network was not unique. If the network *was* unique, then the sources in Γ and Σ would overlap (that is, their locations would overlap), and Δ would have zero potential and zero net current at each and every node.

The method is to show that one source, one sink cannot exist anywhere on Δ. This will show that a network of this shape is uniquely determined in the single-source case. The next case to examine, naturally, is the double-source case, and so on.

In the double-source case, Γ and Σ are restricted to two sources each. Then, to prove uniqueness for this case, it must be shown that two sources, two sinks cannot exist anywhere on Δ.

The next network to be considered is depicted in figure nine. The single-source case will be considered for this network.

Single-source case
Γ and Σ are restricted to one source each, no sinks, and unit resistors. To prove uniqueness, it is necessary to show that one sink and one source cannot exist at any possible location in the difference network Δ. This will show that single unit sources are uniquely determined in a network with the same shape as Γ and/or Σ.

This network can be reduced. Please refer to figure ten.
Proof of Uniqueness.

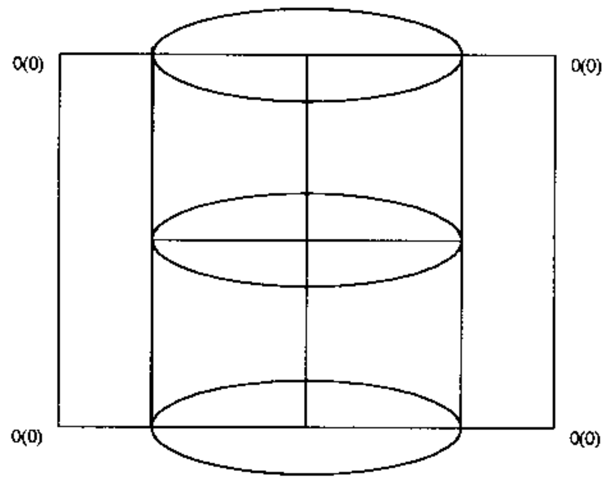

Figure 9: More complicated cylindrical network with connected spikes

There can be no sources at the corner nodes where the voltages are zero. If one of the sources were at a corner node, the positive one for example, then zero would be the maximum voltage in the network. However, zero would also be assumed at another corner, which are considered interior nodes if no sources are there. If both of the sources were at corner nodes, then zero would be the maximum and minimum voltage in the network. The network would thus have zero voltages at all nodes and would thus be unique.

Symmetry argument

This is a quick proof that if there are no sources at any of the four corners, then the voltages at nodes two and eight and at nodes four and six are respectively equivalent. In addition, the voltages at nodes two and eight are equal and opposite to those at nodes four and six.

Proof. Let v represent the voltage at node eight. Since there are no sources at nodes seven and nine, then the voltage at nodes four and six each must be equal to $-v$. Let w represent the voltage at node five. If there is a source at node four, a source of the same sign and magnitude must be at node six because both nodes four and six share the same voltage drops in all directions. However, there are only two sources in this difference network and they are of opposite signs.

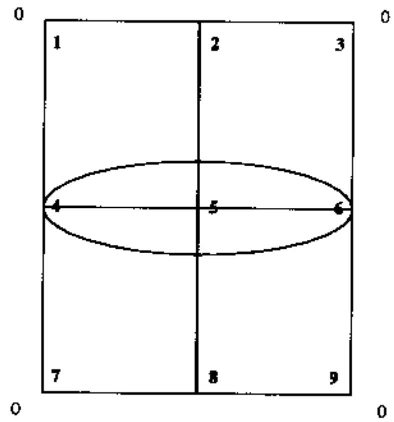

Figure 10: More complicated cylindrical network: reduced

Therefore, there are no sources at nodes four and six, so the v must represent the voltage at node two.

It has been proven that there cannot be sources at any of the corners and there cannot be sources at nodes four and/or six. Now it will be proven that there cannot be a source at either or both of nodes two and eight.

If there is a source at node eight (note that the following can be reversed for the case where there is a source at node two):

If the other source is at node two, then voltage v is the maximum and minimum voltage in the network. All voltage values of interior nodes fall in between the maximum and minimum voltage values in the network, including the value of zero which is on the interior. Therefore, all voltage values in the network are identically zero.

If the other source is somewhere other than node two, the voltage v is an extreme value in the network (i.e., it is the maximum or the minimum). However, it is assumed on the interior (at node two). This is invalid.

If there is a source at node five, the other source must fall at another point in the network. All other points have been ruled out. Therefore, single unit sources are uniquely determined for this network.

GETTING STARTED ON A DEMONSTRATION OF THE DOUBLE-SOURCE CASE

The next network to be examined is slightly more complicated: it has four tables with two rays upon each; on the end tables, the rays extend to spikes and are connected with the spikes on the opposite end table. Please refer to figure eleven.

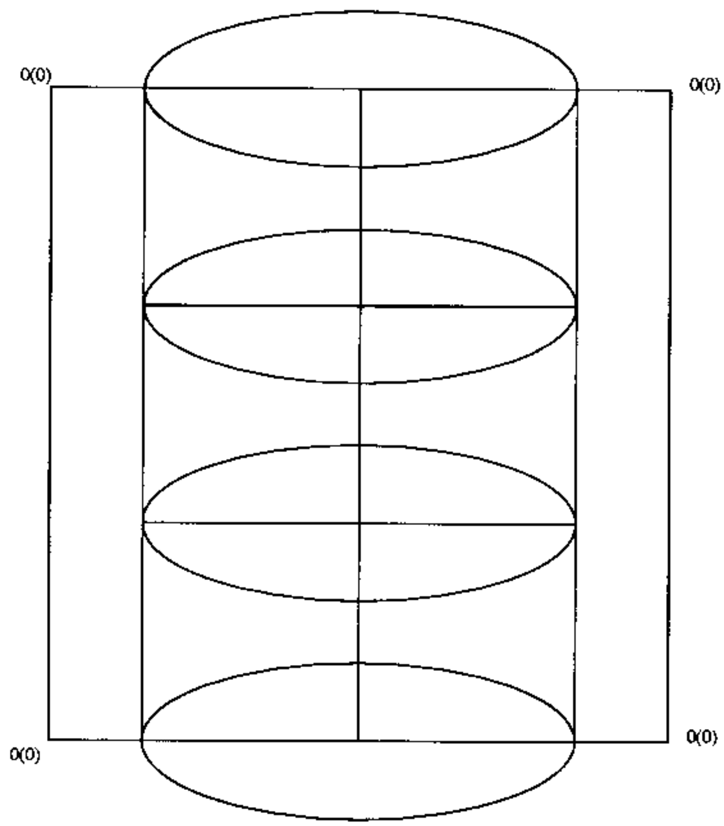

Figure 11: Cylindrical network with four tables and connected spikes

This network can be somewhat simplified to a network that looks like this, with the four corners as boundary nodes where it is known that the voltages are zero. Please refer to figure twelve.

For this network, the double-source case will be examined.

Double-source case

Γ and Σ are restricted to two sources each, no sinks, and unit resistors. Δ will have two sources and two sinks if the networks Γ and

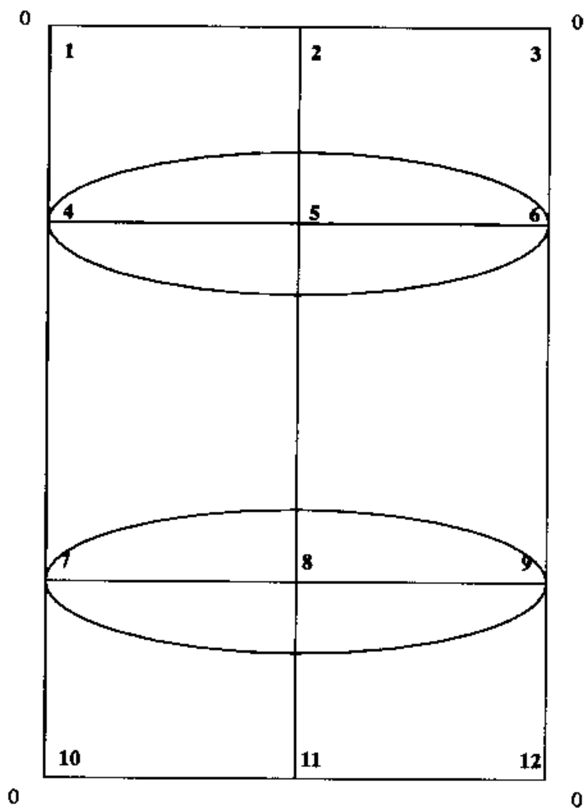

Figure 12: Previous network reduced to this network

Σ are not unique. Having one source and one sink present in Δ is also a possibility, if a source in Γ and a source in Σ overlap, but the other two sources in each one do not. If this does occur, the single-source case applies (which has not yet been proven).

So to prove uniqueness, it is necessary to show that two sinks and two sources cannot exist at any possible location in the difference network Δ. This will show that two unit sources are uniquely determined in a network with the same shape as Γ and/or Σ.

Recap. To prove uniqueness, it must be shown that each case of two sources and two sinks in the difference network cannot exist. The difference network that will now be looked at will have two sources and two sinks; each are of unit size.

Case one: Sources at corner nodes only
Proof. The maximum and/or minimum voltage must be located at a source or at a boundary node. In this network, the sources are at boundary nodes, where it is known that the voltages are zero. There-

fore, the maximum and minimum voltages in the network are both zero, so every voltage in the network is zero.

Case two: Sources at nodes one and three, and the other two located anywhere along the center line (the center line has end nodes two and eleven)

Proof. This argument is based upon the symmetry of the network down the center line. Let s, t, u, and v represent the voltages at nodes two, five, eight, and eleven, respectively. The voltages at nodes 7 and 9 are $-v$ because of the voltage of v at node eleven. It is apparent that the current between nodes seven and four and between nodes nine and six is equivalent, because the same currents flow into node seven from nodes eight and ten as those that flow into node nine from nodes eight and twelve. Therefore, the voltage at nodes four and six is the same; let w represent this voltage. Please refer to figure thirteen.

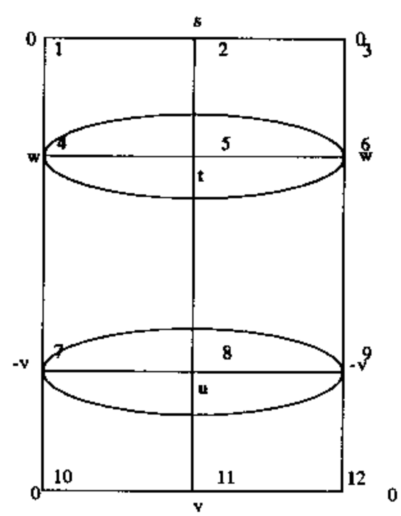

Figure 13: Case two

The voltage drop between nodes four and one and then nodes one and two is the same as that between nodes six and three and then nodes three and two. Therefore, the currents flowing along the edges from nodes four to one to five is the same that flows along the edges from nodes six to three to two. Therefore, the sources located at nodes one and three must be of not only the same sign, but of the same magnitude. The contradiction of this proof lies in the fact that if the current sources at nodes one and three are of the same sign, say positive, then the other two current sources in the network are of negative sign. This makes zero the maximum voltage in the network; however, zero is assumed on the interior too. This cannot happen.

Case three: Sources at nodes two, three, eight, and twelve. Please refer to figure fourteen.

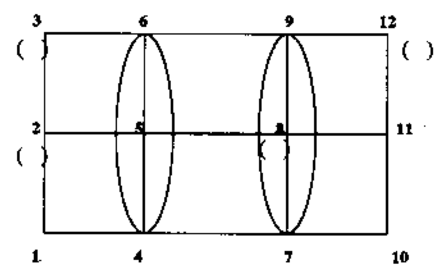

Figure 14: Case three

The two sources located at nodes with zero voltage have opposite signs. This is because if they both had the same sign, positive for example, then the other two sources would be negative. This would make zero the maximum potential in the network, but zero is also assumed at the interior. This cannot happen. So if one of the sources of node three and node twelve has a positive sign and the other has a negative sign, one of nodes two and eight have a positive and a negative sign. The source with the positive sign will be arbitrarily assigned to node two, so the source at node eight will have a negative sign. Thus, node two will be the location of the network's maximum voltage and node eight will be the location of the minimum voltage. Node four has a negative voltage because of current flowing out of node two and through node one. Node eleven has a negative voltage because its voltage is the average of those of its neighbors. Node seven thus has a positive voltage. Current flows in the directions indicated on this sketch (please refer to figure fifteen).

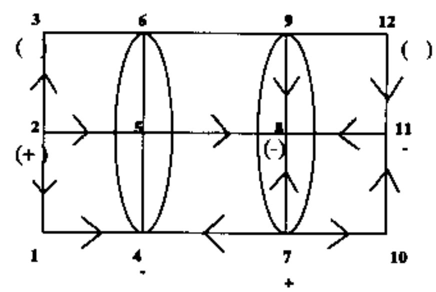

Figure 15: Case three

The two cases that must be considered are when the source at node three has a positive sign and that at node twelve has a negative sign, and vice versa.

Case a

The source at node three has a positive sign and that at node twelve has a negative sign. Current flows in the directions indicated on this sketch (please refer to figure sixteen).

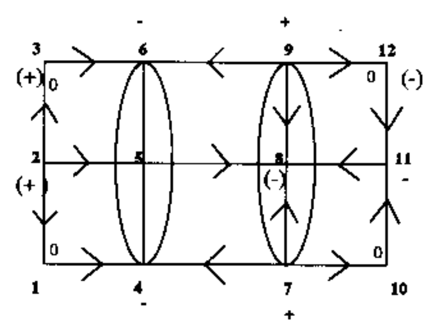

Figure 16: Case three "a"

Observe that no current is flowing into node nine. Current must flow out of node seven and into node nine, but this leads to a contradiction at node seven: Current is flowing out of node seven but none can flow in.

Case b

The source at node three has a negative sign and that at node twelve has a positive sign. Observe line l and the edges it crosses. Please refer to figure seventeen.

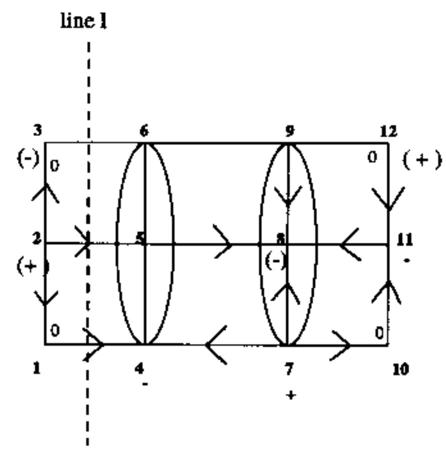

Figure 17: Case three "b"

On the left side on line l, there is a source and a sink. Two of the edges that line l cuts have currents flowing to the right. The third edge does not have a designated current direction yet, but current must flow to the left because an equal amount of current must flow into the section with the source and sink than flows out of the section. Therefore, node two must have a positive sign. Current must flow from node five into node two because current is flowing out of node two in every other direction. Node five therefore has a positive voltage. Observe that no current is flowing into node nine. Current must flow out of node seven and into node nine, but once again, this leads to a contradiction at node seven: Current is flowing out of node seven but none can flow in.

These three cases are only examples of the types pf cases that must be discussued in order to determine that the netowrk is unique for the double-source case.

References

[1] Margaret Chaffee and A.R.K. Whitley. *Recovering Sources in Rectangular Networks with Known Resistors*, submitted.

[2] Gilbert Strang. *Introduction to Linear Algebra*, Wellesley, MA: Wellesley-Cambridge Press, 1993.