Use of Grøbner Bases to Prove Nonrecoverability of Networks in Special Cases

Justin Sinz
Cornell University

August 10, 2000

Abstract

In this report we outline the basic theory behind the use of Grøbner bases in the solution of nonlinear systems of equations. We then describe a specific instance of their use; namely, in the case of a layered annular network.

1 Grøbner Bases

In 1972, Bruno Buchberger developed a method for finding a certain useful ideal of a set of polynomial equations in a certain finite number of variables. This ideal, called a Grøbner Basis in homage to his advisor, provides a somewhat practical method for finding the solutions to a system of nonlinear equations through a process similar to back substitution. The fact that the system of equations resulting from taking a Schur complement is nonlinear is the reason for the applicability of the Grøbner basis algorithm.

Suppose we have a set of polynomials $p_1, \ldots, p_n$ in the variables $a, b, \ldots$. Generally, it is very difficult to solve the system of nonlinear equations that arises if one sets each of the polynomials equal to zero. The Grøbner basis algorithm attempts to solve this nonlinear system in a way similar to Gaussian elimination and back substitution.

The main power of the Grøbner basis algorithm is that it computes from the initial n polynomials a system of new polynomials that has the same solutions as the original equations, with the difference being that the first polynomial involves the minimum number of variables of all the others. An example of this will be given at the end. Also needed for the calculation of a Grøbner basis is a term ordering; in our case this ordering will be lexicographic.

Let f be in a field of polynomials, and let P be a finite subset of this field. A representation of f as the sum of pairwise products of polynomials $p_i \in$ P and monomials m_i is called a standard representation of f with respect to P and a term order if the maximum lead term of all the pairwise products is less than or equal to the lead term of f. We state without proof the following: A finite subset G of a ring of polynomials is a Grøbner basis if and only if every f in the ideal of G has a standard representation. From this, the use of a lexicographic Grøbner basis is clear: if we can find a Grøbner basis for a system of polynomials, and if we can find roots of the Grøbner basis, then we will get the roots of the original system of equations.

2 The Main Problem

Consider an annular network with two circles intersected by three lines. Assume that the conductivity is constant on layers. Through symmetry arguments, it is seen that the Lambda matrix (or response matrix)for this special case will have only four distinct entries at most. This suggests strongly that the conductivities are not recoverable from the Lambda matrix; to rigorously demonstrate that this happens at least once, we analyze the response matrix symbolically.

3 The Entries

The Kirchoff matrix for the 3-2 network has the following form:

$$K = \begin{pmatrix} A & B \\ B & E \end{pmatrix}$$

with

$$A := \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & e \end{bmatrix}$$

$$B := \begin{bmatrix} -a & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -e & 0 & 0 \\ 0 & 0 & 0 & 0 & -e & 0 \\ 0 & 0 & 0 & 0 & 0 & -e \end{bmatrix}$$

$$E := \begin{bmatrix} 2b+c+a & -b & -b & -c & 0 & 0 \\ -b & 2b+c+a & -b & 0 & -c & 0 \\ -b & -b & 2b+c+a & 0 & 0 & -c \\ -c & 0 & 0 & 2d+c+e & -d & -d \\ 0 & -c & 0 & -d & 2d+c+e & -d \\ 0 & 0 & -c & -d & -d & 2d+c+e \end{bmatrix}$$

4 First Attempt at a Counterexample

We compute numerically the Lambda matrix for the 3-2 networks with conductivities = 1. We then set each rational function from the symbolic Lambda matrix equal to its corresponding numerical value, clear denominators, and attempt to find a Grobner basis for the four resulting polynomial equations in five unknowns.
The symbolic lambda matrix is

$$\Lambda := \frac{1}{common} \,\&*\, \begin{bmatrix} \eta & \alpha & \alpha & \iota & \delta & \delta \\ \alpha & \eta & \alpha & \delta & \iota & \delta \\ \alpha & \alpha & \eta & \delta & \delta & \iota \\ \iota & \delta & \delta & \theta & \varepsilon & \varepsilon \\ \delta & \iota & \delta & \varepsilon & \theta & \varepsilon \\ \delta & \delta & \iota & \varepsilon & \varepsilon & \theta \end{bmatrix}$$

where

$$\alpha := -a^2\,(b\,c^2 + 3\,b\,d\,c + 2\,c\,b\,e + 3\,d\,b\,e + b\,e^2 + d\,c^2)$$

$$\iota := -a\,(c\,a + a\,e + a\,d + c\,b + d\,c + e\,c + b\,e + 3\,b\,d)\,c\,e$$

$$\delta := -a\,(a\,d + d\,c + c\,b + b\,e + 3\,b\,d)\,c\,e$$

$$\varepsilon := -e^2\,(d\,a^2 + 2\,a\,d\,c + 3\,b\,d\,a + d\,c^2 + b\,c^2 + 3\,b\,d\,c)$$

and

$$\begin{aligned} common := {} & a^2\,c^2 + 3\,a^2\,d\,c + 2\,a^2\,e\,c + 3\,a^2\,d\,e + a^2\,e^2 + 3\,a\,b\,c^2 + 3\,a\,d\,c^2 + 2\,a\,c^2\,e \\ & + 2\,a\,c\,e^2 + 6\,a\,c\,d\,e + 9\,a\,b\,d\,c + 6\,a\,c\,b\,e + 9\,a\,d\,b\,e + 3\,a\,b\,e^2 + c^2\,e^2 + 3\,c^2\,d\,e \\ & + 3\,c^2\,b\,e + 9\,b\,c\,d\,e + 3\,b\,c\,e^2 \end{aligned}$$

The numeric lambda matrix is

$$M := \begin{bmatrix} \frac{23}{36} & \frac{-11}{72} & \frac{-11}{72} & \frac{-5}{36} & \frac{-7}{72} & \frac{-7}{72} \\ \frac{-11}{72} & \frac{23}{36} & \frac{-11}{72} & \frac{-7}{72} & \frac{-5}{36} & \frac{-7}{72} \\ \frac{-11}{72} & \frac{-11}{72} & \frac{23}{36} & \frac{-7}{72} & \frac{-7}{72} & \frac{-5}{36} \\ \frac{-5}{36} & \frac{-7}{72} & \frac{-7}{72} & \frac{23}{36} & \frac{-11}{72} & \frac{-11}{72} \\ \frac{-7}{72} & \frac{-5}{36} & \frac{-7}{72} & \frac{-11}{72} & \frac{23}{36} & \frac{-11}{72} \\ \frac{-7}{72} & \frac{-7}{72} & \frac{-5}{36} & \frac{-11}{72} & \frac{-11}{72} & \frac{23}{36} \end{bmatrix}$$

The corresponding polynomials are

$$\begin{aligned} p_1 := {} & -a^2\,(b\,c^2 + 3\,b\,d\,c + 2\,c\,b\,e + 3\,d\,b\,e + b\,e^2 + d\,c^2) + \frac{11}{72}\,a^2\,c^2 + \frac{11}{24}\,a^2\,d\,c + \frac{11}{36}\,a^2\,e\,c \\ & + \frac{11}{24}\,a^2\,d\,e + \frac{11}{72}\,a^2\,e^2 + \frac{11}{24}\,a\,b\,c^2 + \frac{11}{24}\,a\,d\,c^2 + \frac{11}{36}\,a\,c^2\,e + \frac{11}{36}\,a\,c\,e^2 + \frac{11}{12}\,a\,c\,d\,e \\ & + \frac{11}{8}\,a\,b\,d\,c + \frac{11}{12}\,a\,c\,b\,e + \frac{11}{8}\,a\,d\,b\,e + \frac{11}{24}\,a\,b\,e^2 + \frac{11}{72}\,c^2\,e^2 + \frac{11}{24}\,c^2\,d\,e + \frac{11}{24}\,c^2\,b\,e \\ & + \frac{11}{8}\,b\,c\,d\,e + \frac{11}{24}\,b\,c\,e^2 \end{aligned}$$

$$p_2 := -a(ad+dc+cb+be+3bd)ce + \frac{7}{72}a^2c^2 + \frac{7}{24}a^2dc + \frac{7}{36}a^2ec + \frac{7}{24}a^2de + \frac{7}{72}a^2e^2 + \frac{7}{24}abc^2 + \frac{7}{24}adc^2 + \frac{7}{36}ac^2e + \frac{7}{36}ace^2 + \frac{7}{12}acde + \frac{7}{8}abdc + \frac{7}{12}acbe + \frac{7}{8}adbe + \frac{7}{24}abe^2 + \frac{7}{72}c^2e^2 + \frac{7}{24}c^2de + \frac{7}{24}c^2be + \frac{7}{8}bcde + \frac{7}{24}bce^2$$

$$p_3 := -a(ca+ae+ad+cb+dc+ec+be+3bd)ce + \frac{5}{36}a^2c^2 + \frac{5}{12}a^2dc + \frac{5}{18}a^2ec + \frac{5}{12}a^2de + \frac{5}{36}a^2e^2 + \frac{5}{12}abc^2 + \frac{5}{12}adc^2 + \frac{5}{18}ac^2e + \frac{5}{18}ace^2 + \frac{5}{6}acde + \frac{5}{4}abdc + \frac{5}{6}acbe + \frac{5}{4}adbe + \frac{5}{12}abe^2 + \frac{5}{36}c^2e^2 + \frac{5}{12}c^2de + \frac{5}{12}c^2be + \frac{5}{4}bcde + \frac{5}{12}bce^2$$

$$p_4 := -e^2(da^2+2adc+3bda+dc^2+bc^2+3bdc) + \frac{11}{72}a^2c^2 + \frac{11}{24}a^2dc + \frac{11}{36}a^2ec + \frac{11}{24}a^2de + \frac{11}{72}a^2e^2 + \frac{11}{24}abc^2 + \frac{11}{24}adc^2 + \frac{11}{36}ac^2e + \frac{11}{36}ace^2 + \frac{11}{12}acde + \frac{11}{8}abdc + \frac{11}{12}acbe + \frac{11}{8}adbe + \frac{11}{24}abe^2 + \frac{11}{72}c^2e^2 + \frac{11}{24}c^2de + \frac{11}{24}c^2be + \frac{11}{8}bcde + \frac{11}{24}bce^2$$

5 The Death of Maple

When I attempted to use gbasis on the four above polynomials with respect to any lexicographic ordering, Maple refused.

6 The use of Eigenvalues and Eigenvectors

We obtained new polynomials from symbolically computing the eigenvalues of the eigenvectors of the symbolic Lambda matrix. Normally, this is impossible to do; however, I had the hunch that the eigenvector of a particular lambda matrix for the 3-2 network would be an eigenvector to all of them.

The eigenvectors and corresponding eigenvalues of the lambda matrix for the network with conductivities = 1 were

$$[\frac{2}{3}, 1, \{[-1, -1, -1, 1, 1, 1]\}], [0, 1, \{[1, 1, 1, 1, 1, 1]\}],$$
$$[\frac{3}{4}, 2, \{[1, -1, 0, 1, -1, 0], [0, -1, 1, 0, -1, 1]\}],$$
$$[\frac{5}{6}, 2, \{[1, 0, -1, -1, 0, 1], [1, -1, 0, -1, 1, 0]\}]$$

When the eigenvectors were multiplied by the symbolic Lambda matrix, the following symbolic eigenvalues were computed:

```
> evalm(Lambda&*vector([1, 1, 1, -1, -1, -1])):
```

$$\left[2\,\frac{c\,a\,e}{c\,e+c\,a+a\,e},\,2\,\frac{c\,a\,e}{c\,e+c\,a+a\,e},\,2\,\frac{c\,a\,e}{c\,e+c\,a+a\,e},\,-2\,\frac{c\,a\,e}{c\,e+c\,a+a\,e},\,-2\,\frac{c\,a\,e}{c\,e+c\,a+a\,e},\,-2\,\frac{c\,a\,e}{c\,e+c\,a+a\,e}\right]$$

```
> evalm(Lambda&*vector([1, 0, -1, 1, 0, -1])):
```

$$\left[3\,\frac{(c\,d+b\,c+e\,b+3\,b\,d)\,a}{\%1},\,0,\,-3\,\frac{(c\,d+b\,c+e\,b+3\,b\,d)\,a}{\%1},\,3\,\frac{(c\,d+b\,c+a\,d+3\,b\,d)\,e}{\%1},\,0,\,-3\,\frac{(c\,d+b\,c+a\,d+3\,b\,d)\,e}{\%1}\right]$$

$$\%1 := c\,e+3\,c\,d+3\,b\,c+3\,e\,b+9\,b\,d+c\,a+a\,e+3\,a\,d$$

```
> evalm(Lambda&*vector([-1, 0, 1, 1, 0, -1])):
```

$$\left[-\frac{(2\,c\,e+3\,c\,d+3\,b\,c+3\,e\,b+9\,b\,d)\,a}{\%1},\,0,\,\frac{(2\,c\,e+3\,c\,d+3\,b\,c+3\,e\,b+9\,b\,d)\,a}{\%1},\,\frac{(2\,c\,a+3\,c\,d+3\,b\,c+3\,a\,d+9\,b\,d)\,e}{\%1},\,0,\,-\frac{(2\,c\,a+3\,c\,d+3\,b\,c+3\,a\,d+9\,b\,d)\,e}{\%1}\right]$$

$$\%1 := c\,e+3\,c\,d+3\,b\,c+3\,e\,b+9\,b\,d+c\,a+a\,e+3\,a\,d$$

By setting each of the symbolic eigenvalues equal to the corresponding numeric eigenvalue, the following three auxillary equations in a,b,c,d,e were obtained:

$$r_1 := \frac{2}{3}\,c\,e+\frac{2}{3}\,c\,a+\frac{2}{3}\,a\,e-2\,c\,a\,e$$

$$r_2 := \frac{5}{6}\,c\,e+\frac{5}{2}\,c\,d+\frac{5}{2}\,b\,c+\frac{5}{2}\,e\,b+\frac{15}{2}\,b\,d+\frac{5}{6}\,c\,a+\frac{5}{6}\,a\,e+\frac{5}{2}\,a\,d - (2\,c\,e+3\,c\,d+3\,b\,c+3\,e\,b+9\,b\,d)\,a$$

$$r_3 := \frac{3}{4}\,c\,e+\frac{9}{4}\,c\,d+\frac{9}{4}\,b\,c+\frac{9}{4}\,e\,b+\frac{27}{4}\,b\,d+\frac{3}{4}\,c\,a+\frac{3}{4}\,a\,e+\frac{9}{4}\,a\,d - 3\,(c\,d+b\,c+e\,b+3\,b\,d)\,a$$

These 3 new equations were augmented to one of the previous equations, and Maple had no trouble in calculating the Gröbner basis with respect to a particular lexicographic ordering.

```
> g4:=gbasis([r[1],r[2],r[3],p[4]],plex(e,b,c,d,a));
```

$$g4 := [-36\,a^6\,d^2-25\,a^4\,d^2+5\,a^4\,d^3-15\,a^5\,d^3+11\,a^6\,d^3+60\,d^2\,a^5,\; a^4\,d^2\,c+30\,a^4\,d^2-6\,a^4\,d^3+11\,a^5\,d^3-36\,d^2\,a^5,\; -6\,a^2\,c^2+7\,c^2\,a^3-3\,c\,a^3\,d+a^2\,d\,c+d\,a^3,\, 36\,c^2\,a^2\,d+11\,a^3\,c\,d^2-6\,c\,d^2\,a^2 - 1848\,a^5\,d^3+1057\,a^4\,d^3+6048\,d^2\,a^5-5292\,a^4\,d^2-6\,a^3\,d^2,\, 432\,a^2\,c^4$$

$$+11a^3d^3c - 2cd^3a^2 - 24a^3cd^2 - 162404a^5d^4 + 94201a^4d^4 + 314496a^5d^3$$
$$-349524a^4d^3 - 2a^3d^3 + 710208d^2a^5 - 607392a^4d^2,$$
$$24bda^3 - 19bda^2 + 7ca^3d - 6a^2dc - 6da^3, 361d^2a^2b + 11a^3cd^2$$
$$+114cd^2a^2 - 17160a^5d^3 + 9936a^4d^3 + 56160d^2a^5 - 49536a^4d^2 + 114a^3d^2,$$
$$11bc^2 + 22acd + 33bcd + 11c^2d - 72bda^2 + 11da^2 + 33abd - 49a^2c^2,$$
$$-6ca^2 - 6a^2e + 7ca^3 + 7a^3e - 3da^3 + da^2, 42768a^2de - 3888a^2e$$
$$-13068a^2c^3 - 7128a^2c^2 - 1331a^3cd^2 + 363cd^2a^2 + 43956a^2dc$$
$$-3888ca^2 - 1880703a^5d^3 + 1088516a^4d^3 + 6155028d^2a^5 - 5427576a^4d^2$$
$$+13431a^3d^2 - 7128a^2d^2 + 648da^2,$$
$$11ca + 11ae + 11ce - 21ca^2 - 21a^2e + 9da^2,$$
$$11be - 49a^2e + 11bc + 33bd + 11cd - 49ca^2 + 21da^2 + 11ad]$$

Note that the first equation involves only a and d the third involves these previous two and the variable c, and so on. It was found that the outer conductivity a could be varied arbitrarily above a fixed number and the others could always be computed from the initial choice by back substitution.

7 Conclusions

I have no idea why the use of the eigenvalues helped maple to calculate the Grøbner basis; however, these eigenvalues were rather pretty. In retrospect, it seems obvious that four equations can never determine five unknowns when the beginning equations come from a Lambda matrix.