An Introductory Look at the Discrete Scattering Problem

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Summer 2001

Abstract

I will discuss the discrete inverse scattering problem, taking as known the power expansion of the function $\Lambda(\lambda)$. I will introduce the reader to this version of the inverse problem and proceed to makes some easy observations about the data taken as given. Then I will present an attempt (only marginally successful) to characterize the given sequences based on the number of interior nodes in the networks from which the sequences arose. I will describe the difficulties in making such a characterization, and present some observations I made while attempting to resolve these difficulties. Finally, I will conclude with a brief discussion of the given data for layered networks.

1 Defining the Scattering Problem

1.1 Introduction to the Scattering Problem

The typical inverse problem on electrical networks takes as given a response matrix, Λ, and attempts to recover the Kirchhoff matrix, K, from this given data. The Kirchhoff matrix contains in the ijth entry, where $i \neq j$, the negative of the conductance of the edge between nodes i and j, or, where $i = j$, the sum of the conductances of the edges adjacent to node i. We think of K as being composed of four blocks:

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where A is the block with indices corresponding only to boundary nodes; C is the block with indices corresponding only to interior nodes; and B and B^T the blocks with column indices corresponding to interior nodes and row indices corresponding to boundary nodes, or vice versa, respectively:

$$K = \begin{array}{c} \\ \text{boundary} \\ \text{interior} \end{array} \begin{array}{c} \begin{array}{cc} \text{boundary} & \text{interior} \end{array} \\ \begin{pmatrix} \quad A \quad & \quad B \quad \\ \quad B^T \quad & \quad C \quad \end{pmatrix} \end{array}.$$

A different kind of inverse problem, called a *scattering problem*, assumes an input with a frequency, λ, and takes as known a response matrix $\Lambda(\lambda)$, from $K - \lambda I$, which can be represented by an infinite series of the form

$$\Lambda(\lambda) = -\lambda I + \Lambda_0 + \frac{\Lambda_1}{\lambda} + \frac{\Lambda_2}{\lambda^2} + \frac{\Lambda_3}{\lambda^3} + \cdots + \frac{\Lambda_n}{\lambda^n} + \cdots,$$

where I is the identity matrix, and again attempts to recover a Kirchhoff matrix, as defined above, from this data. In this infinite series representation of $\Lambda(\lambda)$, which is only valid for $\lambda > \|C\|$ (i.e. $\lambda >$ the absolute value of all eigenvalues of C), $\Lambda_0 = A$, and $\Lambda_n = BC^{n-1}B^T$, where A, B, B^T and C are submatrices of K as defined above. In looking at this inverse problem, we can take as given $\Lambda(\lambda)$, or the sequence Λ_n, from the series representation. I have chosen to take as known the sequence Λ_n.

1.2 Some Observations About the Data

Looking at the sequence of matrices taken as given in the scattering problem, we see that this data contains much more explicit information about K than the single matrix Λ taken as given in the usual formulation of the inverse problem. For example, we are given explicitly the portion of K containing only the boundary information, A, in the first term of the sequence. By examining the given data, we can make some observations about the terms of the sequence Λ_n:

Observation 1 *For all n, Λ_n will be symmetric.*

By definition, both K and the submatrices A and C must be symmetric. Thus, Λ_0 must be symmetric. By definitions of matrix multiplication and transposition, we see that Λ_n is symmetric for all n: $(\Lambda_n)^T = (BC^{n-1}B^T)^T = (B^T)^T(C^{n-1})^T(B)^T = BC^{n-1}B^T = \Lambda_n$.

Observation 2 *Λ_0 contains explicit information about the boundary structure of the network.*

Because $\Lambda_0 = A$, and the indices of A represent the boundary nodes, the dimension of Λ_0 will be $n \times n$, where $n =$ the number of boundary nodes in the given network. The off-diagonal entries in Λ_0 tell us if two boundary nodes are adjacent (if the entry $\neq 0$), and if so, (the negative of) the conductance of the edge between them. Further, since diagonal entries contain the sum of conductances into a given node, by summing the entries of the ith column of Λ_0, we can discover if the ith node is adjacent to any interior nodes (if the sum $\neq 0$), as well as the sum of the conductances on these boundary-to-interior edges. If two sequences have identical Λ_0's, the two networks will have identical boundary structures.

Observation 3 *Λ_1 only contains information about the boundary-to-interior connections of the network.*

Because $\Lambda_1 = BB^T$, the information contained in the matrix will be more indirect than the information contained in Λ_0. If it were possible to uniquely factor Λ_1 into BB^T, such factorization would make the information contained in the matrix explicit. In most cases, however, this is far easier said than done. Even before factoring, though, it is clear that if two sequences have the same number of interior nodes, as well as identical Λ_0's and Λ_1's, not only will the boundary structure of the two networks be identical, but the boundary-to-interior structure will also be identical, leaving possibility of differences only in the interior-to-interior edges of the network.

Observation 4 *Λ_2 contains information about the interior structure of the network.*

Similar difficulties arise in attempting to extract the information contained in Λ_n where $n > 1$. Because these terms are again multiples of submatrices of K, it is necessary to factor the terms into products of these submatrices before explicit information about the network can be obtained. However, we see that if two sequences have the same number of interior nodes, and the same Λ_0 through Λ_2, the sequences must describe the same network: they must have the same A, B, B^T, and C; thus, none of the subsequent terms will differ.

1.3 A Conclusion, and a Question

Having made the three above observations, we can easily conclude that if a network has no interior nodes, the sequence Λ_n will have only one meaningful term. The Kirchhoff matrix for a network with no interior nodes will have no B, B^T, or C submatrices, since these submatrices contain information about interior nodes. Or, put another way, the submatrices B, B^T, and C of K will all be equal to zero matrices. Thus, the only meaningful term of the sequence Λ_n will be Λ_0, which, in this case, is equal to K.

Thus, we have characterized all sequences which describe networks with no interior nodes. Now, too, if the enemy gives us a sequence, we can immediately tell if the given sequence describes such a network. For, if a sequence has more than one meaningful, non-zero term, it must have a Kirchhoff matrix with non-empty submatrices B, B^T, and C, and thus, it must have interior nodes.

Having come to this conclusion, I began to wonder if there were such characterizations of Λ_n sequences for networks that did contain interior nodes. In theory, it should be possible to determine the number of interior nodes in a given network from it's Λ_n sequence, for, as shown in [1], this information can be determined from the function $\Lambda(\lambda)$, itself. Since the infinite series contains all the information contained in the function, in theory the same conclusions can be drawn from either presentation of the data. However, such characterization for networks with greater than one interior node and arbitrary numbers of boundary nodes proved difficult. In the following section, I will present my characterizations for one-interior-node networks, along with my attempt to characterize two-interior node networks, the problems I encountered in this attempt, and some further information about nature of the Λ_n sequence which I uncovered while trying to overcome these problems.

2 Characterizing One-Interior-Node Networks

Before we can characterize the Λ_n sequence for single-interior-node networks, we must first examine a general Kirchhoff matrix for such a network, and the Λ_n sequence that this K will produce. If a network has exactly one interior node, and n boundary nodes, we see that the submatrix B of K will have dimension $(n-1) \times 1$, and B^T, dimension $1 \times (n-1)$. When multiplying BB^T, to obtain Λ_1, we see that the ith column of Λ_1 is obtained by multiplying the $(n-1)$-dimensional vector, B, by the ith entry in the same vector. This has several consequences:

- Each column of Λ_1 will be a scalar multiple of B. Thus, Λ_1 will be a rank-1 matrix.
- The iith entry of Λ_1 (that is, all diagonal entries) will be equal to the square of the ith entry in B. Thus, from Λ_1, we can exactly recover B.

Now, since the network has only one interior node, the submatrix C of K will have dimension 1×1. This means that when multiplying, C will behave like a scalar. Thus, $BC^kB^T = BB^Tc^k$, where c is the only entry in C. From this, we see that $\Lambda_k = BB^Tc^{k-1} = \Lambda_1 c^{k-1}$. The number, c, will be the nnth entry in K. Recall that from the definition of K, then, $c = -\sum_{i=1}^{n-1} k_{in}$, where k_{in} is the inth entry of K. In other words, $c =$ sum of the entries in $B(=$ sum of entries in $B^T)$.

Following directly from the above observations, we can outline the criteria for determining if a given Λ_n sequences describes a network with exactly one interior node:

- Is Λ_1 a rank 1 symmetric matrix? If yes,
- Is $\Lambda_k = \Lambda_1 c^{k-1}$, where c is the same constant for all Λ_k? If yes,
- Does $c = -\sum_{i=1}^{n-1} \sqrt{l_{ii}}$, where l_{ii} represents the diagonal entry in the ith row (and ith column) of Λ_1? If yes,
- Does $c = -\sum_{j=1}^{n-1}(\sum_{i=1}^{n-1} l_{ij})$, where l_{ij} is the ith row in the jth column of Λ_0? If yes, the sequence describes a one-interior-node network.

If we must answer no to even one of the above questions, the given sequence cannot describe a one-interior-node network. Further, we can say that if the sequence meets all the above criteria, *and* Λ_0 is a diagonal matrix with the sum of the diagonal entries equal to c, as defined above, then not only does the network have only one interior node, but is also star-shaped (that is, with one central interior node, possessing no boundary-to-boundary edges).

3 Characterizing Two-Interior-Node Networks: An Attempt

A network with more than one interior node will have a more complex interior, and thus, more information must be taken from the terms of the Λ_n sequence where $n > 1$, where the necessary information is more obscured. Thus, we have more possibilities and no easily distinguishable pattern as we had in the single-interior-node case. When I first learned about the scattering problem, I was told that the originators of the problem had yet to look at networks with more than one boundary node. At first, this seemed strange to me. However, one-boundary-node networks have the neat property, as do one-interior-node networks, that B will be composed of a single vector, and thus prevent some of the difficulties that arise due to the trickiness of matrix multiplication. As will be shown in this section, the one-boundary, two-interior-node networks are easily characterized, while other two-interior-node networks present problems.

3.1 One-Boundary, Two-Interior-Node Networks: A Very Un-general Case

- Is Λ_1 a 1×1 matrix?

In this case, we know that the network described by the Λ_n sequence has only one boundary node. We proceed to determine if the network also has exactly two interior nodes.

- Compute a prospective B.

If the sequence does indeed describe a two-interior node network, the Kirchhoff matrix for the network will be given by

$$K = \begin{pmatrix} a & -b_1 & -b_2 \\ -b_1 & c+b_1 & -c \\ -b_2 & -c & c+b_2 \end{pmatrix},$$

where $\Lambda_0 = a = b_1 + b_2$, and $\Lambda_1 = b_1^2 + b_2^2$. By this second equality, we can think of b_1 and b_2 as two edges of a right triangle with hypotenuse of length $\sqrt{\Lambda_1}$ (remember that all Λ_n's in this case are 1×1 matrices, and thus can be treated as scalars). Thus, we can say that, for some θ,

$$b_i = \sqrt{\Lambda_1}(\cos\theta), \quad \text{and} \quad b_2 = \sqrt{\Lambda_1}(\sin\theta).$$

We can further see that

$$\Lambda_0 = b_1 + b_2 = \sqrt{\Lambda_1}(\cos\theta + \sin\theta)$$

and thus

$$\frac{\Lambda_0}{\sqrt{\Lambda_1}} = \cos\theta + \sin\theta.$$

Call this number t. Then there must exist some point (x, y) on the unit circle, such that

$$x = \cos\theta, \quad y = \sin\theta, \quad x + y = t, \text{ and } x^2 + y^2 = 1.$$

¿From these equalities, we see that $0 \le t^2 \le 2$. Solving for x and y algebraically, we find that either

$$x = \frac{t}{2} + \sqrt{2 - t^2} \text{ and } y = \frac{t}{2} - \sqrt{2 - t^2},$$

or

$$x = \frac{t}{2} - \sqrt{2 - t^2} \text{ and } y = \frac{t}{2} + \sqrt{2 - t^2}.$$

Without loss of generality, we can assume that $x = \frac{t}{2} + \sqrt{2 - t^2}$ and $y = \frac{t}{2} - \sqrt{2 - t^2}$, and calculate b_1 and b_2.[1] If both b_1 and b_2 are positive, we have the possibility of a two-interior node network, and can proceed.

- Compute a prospective C.

Since $\Lambda_2 = BB^T$, we can compute c in the Kirchhoff matrix at the beginning of this section algebraically. Thus, we have that

$$c = \frac{\Lambda_2 - b_1^3 - b_2^3}{(b_1 - b_2)^2}.$$

Note that b_1 and b_2 are both positive, and thus $b_1 - b_2$ will never equal zero. With b_1, b_2 and c known, we can compute our prospective C.

- Check to see if C fits with the rest of our computed Kirchhoff matrix.

If the matrix $K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$, using the blocks computed above, has row and column sums equal to zero, then we have found the Kirchhoff matrix of a two-interior-node network described by the given Λ_n sequence. If not, we know that no such network exists, since the uniqueness of our calculations proves that there are no other possible two-interior-node networks that could fit this data.

3.2 The More General Cases

Obviously, the situation where a network has only one boundary nodes does not represent the general case for two-interior-node networks. The general case occurs when there are n boundary nodes, all Λ_n are $n \times n$ matrices, and, if the network has two interior nodes, B is an $n \times 2$ dimensional matrix, with two linearly independent column vectors, making all Λ_n matrices rank two. A variation on this general case occurs when the column vectors of B are (coincidentally) linearly dependant. What follows is my attempted characterization of the general two-interior-node network case, and a discussion of the problems with this characterization.

- Is Λ_1 rank 2?

- Compute a prospective B.

Let O be an orthogonal $(n-2) \times (n-2)$ matrix such that, if Λ_1 has eigenvalues a and b, and corresponding unit eigenvectors $\mathbf{u_a}$ and $\mathbf{u_b}$,

$$O = \begin{pmatrix} \mathbf{u_a} & \mathbf{u_b} & & E & \end{pmatrix}, \quad \text{and} \quad O^T \Lambda_1 O = \begin{pmatrix} a & 0 & 0 & \cdots \\ 0 & b & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

[1]Switching x and y here only changes the order of the (in this case, one-dimensional) column vectors in our subsequent B. A switch of the column vectors of B does not change the network represented by the Kirchhoff matrix. Interior nodes, boundary nodes, connections and conductances remain the same. We can even show algebraically that the same Λ_n sequence is produced by any arrangement of the column-vectors of B, and, if we are attempting to reconstruct a network's graph from the given sequence, the graph we draw will be the same, irrespective of the order of B's columns. We can think of the switch as a renaming. The nature of the node does not change if I choose to call it b_1 instead of b_2.

where E is a submatrix of O containing the appropriate number of column vectors, all orthogonal to the span of $\mathbf{u_a}$ and $\mathbf{u_b}$. We know that such an O exists because Λ_1 is symmetric. Further, we can write $O^T\Lambda_1 O$ as the square of a uniquely determined matrix, R:

$$R = \begin{pmatrix} \sqrt{a} & 0 & 0 & \cdots \\ 0 & \sqrt{b} & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus, we have that

$$O^T\Lambda_1 O = \begin{pmatrix} a & 0 & 0 & \cdots \\ 0 & b & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \sqrt{a} & 0 & 0 & \cdots \\ 0 & \sqrt{b} & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \sqrt{a} & 0 & 0 & \cdots \\ 0 & \sqrt{b} & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = R^2.$$

Because R is symmetric, we can say that $O^T\Lambda_1 O = RR^T$. By the orthogonal nature of O, we can go further, to say that $\Lambda_1 = OO^T\Lambda_1 O^T O = ORR^T O^T = (OR)(OR)^T$. Now,

$$OR = \begin{pmatrix} \mathbf{u_a} & \mathbf{u_b} & E \end{pmatrix} \times \begin{pmatrix} \sqrt{a} & 0 & 0 & \cdots \\ 0 & \sqrt{b} & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} & & 0 & \cdots \\ & & 0 & \cdots \\ \sqrt{a}\mathbf{u_a} & \sqrt{b}\mathbf{u_b} & 0 & \cdots \\ & & 0 & \cdots \\ & & \vdots & \ddots \end{pmatrix},$$

and does not depend on the vectors we pick for E. Further, we see that only the first two columns of OR contain non-zero entries, and thus, only these columns are necessary to obtain the product Λ_1. So it is shown that $\Lambda_1 = BB^T$, where

$$B = \begin{pmatrix} \sqrt{a}\mathbf{u_a} & \sqrt{b}\mathbf{u_b} \end{pmatrix},$$

the first two columns of OR, and B^T is the first two rows of $(OR)^T$. Thus, we have factored Λ_1 into BB^T.

- Does this B fit the given A $(= \Lambda_0)$?

By definition, all row and column sums of K must equal zero. Thus, the computed B is only accurate for the given sequence if the matrix composed of the two submatrices, $\begin{pmatrix} A & B \end{pmatrix}$, has row sums equal to zero. If it does,

- Compute a possible C.

By definition of the matrix C, we see that

$$C = \begin{pmatrix} \sum_1 + c & -c \\ -c & \sum_2 + c \end{pmatrix}.$$

where $\sum_1$ = the negative of the sum of the entries in the first column vector of B, and $\sum_2$ = the negative of the sum of the entries in the second column vector of B. To find c, we see that the ijth entry in Λ_2, $(\Lambda_2)_{ij} = (x_i - y_i)(x_j - y_j)c + x_i x_j \sum_1 + y_i y_j \sum_2$, where x_i is the ith entry in the first column vector of B, and y_i is the ith entry in second column vector of B. Thus,

$$c = \frac{(\Lambda_2)_{ij} - x_i x_j \sum_1 - y_i y_j \sum_2}{(x_i - y_i)(x_j - y_j)}$$

for all k where $x_k \neq y_k$. We know that at least one such k exists, because the two column vectors are linearly independent. If the values of c obtained from different k's are inconsistent, we know that no appropriate c exists, and thus that the sequence does not describe a two-interior-node network. If the values of c are consistent, however, we can compute the second possible C matrix by substituting the computed value of c into the formula for C above.

- Does this C fit the sequence?

Using the same test as was used above, determine if this new C produces the appropriate values for all Λ_n. If it does, we have now found the B and C of the two-interior node network represented by the given sequence. If not, the sequence does not describe a general case two-interior-node network.

3.3 The Problem

The above process will only prove effective if the factorization of B is unique. If there is more than one possible factorization of Λ_1 into BB^T, we cannot eliminate the possibility of a sequence describing a two-interior-node network even if the above process fails to produce an appropriate B and C. I believed that the factorization was unique, until I was presented with a counterexample:

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 8 & 3 \\ 8 & 13 & 5 \\ 3 & 5 & 2 \end{pmatrix} = \begin{pmatrix} \sqrt{5} & 0 \\ \frac{8}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{3}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \times \begin{pmatrix} \sqrt{5} & \frac{8}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}.$$

This shows that there may be many possible factorizations of Λ_1. Also, $(OR)(OR)^T = OROO^TRO^T = (ORO)(ORO)^T$, so when B is a square matrix, we see that both OR and ORO as defined above are potential candidates for B. I have yet to find a way of resolving the problem. I cannot find a way to produce all possible factorizations of Λ_1, and thus, I cannot determine if there exists one such factorization which satisfies the given $\Lambda_0 = A$ and produces an appropriate C. This lead me to attempt to discover some kind of pattern in the terms of the Λ_n sequence, a pattern that would identify the sequence as definitely coming from a two-interior-node network, a pattern like the one that existed in the single-interior-node case. I was able to see no pattern easily distinguishable from the terms of the sequence themselves, so I attempted to find such a pattern in the subdeterminants of the terms of the sequence. Though I have not yet found a useful pattern, I have found some interesting characteristics of the data sequence, which will be the topic of the next section.

3.4 Nifty Sub-Determinant Tricks

- In general, determinants of $k \times k$ submatrices of Λ_1 are not equal to zero when $k <$ the number of linearly independent column vectors of B. There are coincidental situations where these determinants equal zero, but for the most part, this statement is true.
- If the determinant of $\Lambda_1 \neq 0$, B has at least as many columns as Λ_1. For Λ_1 to have full rank, B must also have the same rank, which can only happen when B has as many linearly independent columns as Λ_1.
- If B has k linearly independent columns, the determinants of the symmetric $k \times k$ submatrices on the diagonal of Λ_n will be equal to x times the determinants of the corresponding submatrices of Λ_{n-1} for some constant x. Further, if all columns of B are linearly independent , x will be equal to the determinant of C.
- When B has k linearly independent columns, $\det(\Lambda_1(I;J)) = \sum_{\forall K} \det B(I;K) \times \det B(J;K)$, where I and J have cardinality k, and K is the set of all subsets of the columns of B with cardinality k. When B has exactly k columns, which are linearly independent, $\det \Lambda_1(I;J) = \det B(I;1,\ldots,k) \times \det B(J;1,\ldots,k)$.

These last two items follow directly from the Cauchy Product Theorem, which relates subdeterminants of non-square matrices to sub determinants of their products, as shown in [3].

4 The Scattering Problem for Layered Networks

Attempting to narrow my field of inquiry, I now turn to an examination of the scattering problem for layered networks. The physical characteristics of layered networks are described in detail in [2].

4.1 Characteristics of Layered Networks

Graphs of these networks are composed of a discrete number of circles, with a discrete number of radial lines, at the outermost ends of which we find the boundary nodes, and which may or may not extend beyond the inner-most circle to meet at a central interior node, and which also may or may not extend beyond the outer-most circle to in boundary spikes. I have chosen to look at the case where both of these occur, that is, where the network has boundary spikes and a central interior node, in which case the network will always have an odd number of layers. To be considered a layered network, such a graph must have the property that corresponding edges have the same conductances. This means that each of the boundary spikes will have conductance e_0, and subsequent edges on each of the radial lines will have conductances e_1, e_2, etc., while each of the edges on the outer-most circle will have conductance d_1, each of the edges on the next outer-most circle will have conductances d_2, and so on. This produces a Kirchhoff matrix with a very unique pattern. For such a network, with, for example, three radial lines and two circles,

$$K = \left(\begin{array}{ccc|ccccccc}
e_0 & & & -e_0 & & & & & & \\
 & e_0 & & & -e_0 & & & & & \\
 & & e_0 & & & -e_0 & & & & \\
\hline
-e_0 & & & e_0+2d_1+e_1 & -d_1 & -d_1 & -e_1 & & & \\
 & -e_0 & & -d_1 & e_0+2d_1+e_1 & -d_1 & & -e_1 & & \\
 & & -e_0 & -d_1 & -d_1 & e_0+2d_1+e_1 & & & -e_1 & \\
 & & & -e_1 & & & e_1+2d_2+e_2 & -d_2 & -d_2 & -e_2 \\
 & & & & -e_1 & & -d_2 & e_1+2d_2+e_2 & -d_2 & -e_2 \\
 & & & & & -e_1 & -d_2 & -d_2 & e_1+2d_2+e_2 & -e_2 \\
 & & & & & & -e_2 & -e_2 & -e_2 & 3e_2
\end{array}\right)$$

(note that blank spaces represent zeros). The blocks of K are given as follows:

$$A = \begin{pmatrix} e_0 & & \\ & e_0 & \\ & & e_0 \end{pmatrix}, \qquad B = \begin{pmatrix} -e_0 & & & & & & \\ & -e_0 & & & & & \\ & & -e_0 & & & & \end{pmatrix}, \qquad \text{and}$$

$$C = \begin{pmatrix}
e_0+2d_1+e_1 & -d_1 & -d_1 & -e_1 & & & \\
-d_1 & e_0+2d_1+e_1 & -d_1 & & -e_1 & & \\
-d_1 & -d_1 & e_0+2d_1+e_1 & & & -e_1 & \\
-e_1 & & & e_1+2d_2+e_2 & -d_2 & -d_2 & -e_2 \\
 & -e_1 & & -d_2 & e_1+2d_2+e_2 & -d_2 & -e_2 \\
 & & -e_1 & -d_2 & -d_2 & e_1+2d_2+e_2 & -e_2 \\
 & & & -e_2 & -e_2 & -e_2 & 3e_2
\end{pmatrix}.$$

Each circle added to the graph will produce a new block in the C block of K, because it will add two new layers to the interior. Adding a new circle is like taking an existing circle and extending it to two circles with new segments of radial line between them. Thus, a layer will be added with the edges of the new circle itself, and a second layer will be added with the new radial line segments. The following is the Kirchhoff matrix of a layered network with three radial lines and one circle, which can be compared to the Kirchhoff matrix given above for a network with three radial lines and two circles, to illustrate this phenomenon:

$$K = \left(\begin{array}{ccc|cccc} e_0 & & & -e_0 & & & \\ & e_0 & & & -e_0 & & \\ & & e_0 & & & -e_0 & \\ \hline -e_0 & & & e_0+2d_1+e_1 & -d_1 & -d_1 & -e_1 \\ & -e_0 & & -d_1 & e_0+2d_1+e_1 & -d_1 & -e_1 \\ & & -e_0 & -d_1 & -d_1 & e_0+2d_1+e_1 & -e_1 \\ & & & -e_1 & -e_1 & -e_1 & 3e_2 \end{array}\right).$$

4.2 Λ_n sequences for Layered Networks

As we have seen, it is the structure of the K matrix of a network that determines the structure and properties of the terms of the network's Λ_n sequence. Thus, the Λ_n sequences for layered networks, being as they are made up of products of blocks of these special K matrices, must have some special properties also. First, we see that

$$\Lambda_0 = \begin{pmatrix} e_0 & & \\ & e_0 & \\ & & e_0 \end{pmatrix},$$

and that, because of the structure of B,

$$\Lambda_1 = \begin{pmatrix} e_0^2 & & \\ & e_0^2 & \\ & & e_0^2 \end{pmatrix} = \Lambda_0^2.$$

We then see that subsequent terms of the sequence, $\Lambda_n = BC^{n-1}B^T$ will equal

$$e_0^2(C^{n-1}(1,2,3;1,2,3)),$$

where the notation $M(I;J)$ denotes the submatrix of M made up by taking the matrix whose rows and columns correspond to the intersection of the row and column indices that appear in the sets I and J, respectively.[2] This is because the portion of the B matrix which is composed of zeros will serve to eliminate from Λ_n all information contained in C beyond the intersection of the first three rows of C with the first three columns (where the only non-zero entries appear in B). Next, we can observe that

$$BC^{n-1}B^T = \begin{pmatrix} -e_0 & & \\ & -e_0 & \\ & & -e_0 \end{pmatrix} \mathrm{ulc}(C^{n-1}) \begin{pmatrix} -e_0 & & \\ & -e_0 & \\ & & -e_0 \end{pmatrix}.$$

Matrices of the form $\begin{pmatrix} x & & \\ & x & \\ & & x \end{pmatrix}$, where x is some constant, are scalar multiples of the identity. Since scalar multiplication of matrices is commutative, and since multiplication by the identity is commutative, we see that multiplication of matrices of this form must also be commutative, implying that

$$\begin{pmatrix} -e_0 & & \\ & -e_0 & \\ & & -e_0 \end{pmatrix} \mathrm{ulc}(C^{n-1}) \begin{pmatrix} -e_0 & & \\ & -e_0 & \\ & & -e_0 \end{pmatrix} = e_0^2(\mathrm{ulc}(C^{n-1})).$$

[2] From this point on I will refer to the matrix $C^n(1,2,3;1,2,3)$ as $\mathrm{ulc}(C^n)$ for "upper left corner" of C^n.

Thus, all new information presented to us in subsequent terms of the Λ_n sequence will come from powers of C. In other words, knowing that a network is layered allows us to factor out an e_0^2 from each Λ_n and look directly at the upper left $n \times n$ block of C^{n-1}, where n is the number of boundary nodes, in this case three. I have also observed, though I am at this moment unable to provide a proof, that, in the three-boundary-node case, each Λ_n will have the same structure as the upper right 3×3 block of C, that is $\Lambda_n = \begin{pmatrix} \alpha_n & \beta_n & \beta_n \\ \beta_n & \alpha_n & \beta_n \\ \beta_n & \beta_n & \alpha_n \end{pmatrix}$ for all n.

4.3 C in Block Form

I soon found that the matrix multiplication necessary to compute increasing powers of C produces pages upon pages of Mathematica output, which quickly becomes unreadable. Because of this, it seemed more reasonable to look at C in block form. We can consider C to be composed of many $n \times n$ blocks, where n is the number of boundary nodes of the network. The structure of C seems to suggest such block form, as can be seen above in the C matrices I have constructed. In these examples, the blocks are 3×3, though I believe that none of the conclusions which will follow depend upon the size of the blocks, and thus they can be generalized to layered networks with any number of radial lines. There will be two types of blocks: scalar multiples of the $n \times n$ identity, as in

$$\begin{pmatrix} -e_1 & & \\ & -e_1 & \\ & & -e_1 \end{pmatrix};$$

and symmetric blocks with all diagonal entries equal and all off-diagonal entries equal to either zero or a scalar (the same scalar for each off-diagonals in a given block), as in

$$\begin{pmatrix} e_0 + 2d_1 + e_1 & -d_1 & -d_1 \\ -d_1 & e_0 + 2d_1 + e_1 & -d_1 \\ -d_1 & -d_1 & e_0 + 2d_1 + e_1 \end{pmatrix}.$$

We will refer to these blocks by the capital of the diagonal entry in the first case, and by the capital of the off diagonal entry in the second. Thus, we have

$$C = \begin{pmatrix} D_1 & E_1 & & & \\ E_1 & D_2 & E_2 & & \\ & E_2 & D_3 & E_3 & \\ & & E_3 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}.$$

We can treat these blocks as scalars because they behave that way: their symmetry is such that they have multiplicative commutativity.

4.4 The "Moving Up" Property of Powers of C

In computing powers of C in block form, I was able to see an important property of the Λ_n sequence of layered networks: *Each term in the Λ_n sequence will contain information about a block of C previously unknown.* In other words, $\mathrm{ulc}(C^n)$ will show the influence of one more block of the matrix C than did the matrix $\mathrm{ulc}(C^{n-1})$. There is, in effect, a "moving up" of the blocks of C in each subsequent power, which allows us to reconstruct C from the Λ_n sequence. This moving up happens in the following manner: We know that the various entries in the block form of any power of C will be polynomials in D_1, D_2, etc., and E_1, E_2, etc. If the "moving up" conjecture is correct, then each power of C will have a new block, a new D_k or E_k, appearing in the polynomial corresponding to $\text{ulc}(C^n)$, that is, the first entry in C^n in block form. To show that this is in fact the case, I will proceed to show simplified versions of the powers of C. These $(C^k)'$will have as their entries only the blocks of C that appear in the polynomial of a given entry, and that have not appeared in or above that entry before. By the term “above” in this case, I mean to suggest that the block has not appeared in any of the polynomial entries above the right to left diagonal in which the given entry is located, in the previous powers of C. For example,

$$C = \begin{pmatrix} D_1 & E_1 & & & \\ E_1 & D_2 & E_2 & & \\ & E_2 & D_3 & E_3 & \\ & & E_3 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix} \qquad C^2 = \begin{pmatrix} D_1^2 + E_1^2 & E_1(D_1 + D_2) & E_1 E_2 & \\ E_1(D_1 + D_2) & E_1^2 + D_2^2 + E_2^2 & \ddots & \ddots \\ E_1 E_2 & \ddots & \ddots & \\ & \ddots & & \end{pmatrix}$$

Here we see that in $(C^2)_{12}$ both D_1 and D_2 appear for the first time. However, D_1 has already appeared in C_{11}, which is on the diagonal above $(C^2)_{12}$. So, the matrix $(C^2)'$ is as follows:

$$(C^2)' = \begin{pmatrix} E_1 & D_2 & E_2 & \\ D_2 & E_2 & \ddots & \ddots \\ E_2 & \ddots & \ddots & \\ & \ddots & & \end{pmatrix}$$

To proceed:

$$(C^3)' = \begin{pmatrix} E_1 & D_2 & E_2 & \\ D_2 & E_2 & \ddots & \ddots \\ E_2 & \ddots & \ddots & \\ & \ddots & & \end{pmatrix} \times \begin{pmatrix} D_1 & E_1 & & & \\ E_1 & D_2 & E_2 & & \\ & E_2 & D_3 & E_3 & \\ & & E_3 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix} = \begin{pmatrix} D_2 & E_2 & D_3 & E_3 & \\ E_2 & D_3 & E_3 & \ddots & \ddots \\ D_3 & E_3 & \ddots & \ddots & \\ E_3 & \ddots & \ddots & & \\ & \ddots & & & \end{pmatrix}$$

$$(C^4)' = \begin{pmatrix} D_2 & E_2 & D_3 & E_3 & \\ E_2 & D_3 & E_3 & \ddots & \ddots \\ D_3 & E_3 & \ddots & \ddots & \\ E_3 & \ddots & \ddots & & \\ & \ddots & & & \end{pmatrix} \times \begin{pmatrix} D_1 & E_1 & & & \\ E_1 & D_2 & E_2 & & \\ & E_2 & D_3 & E_3 & \\ & & E_3 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix} = \begin{pmatrix} E_2 & D_3 & E_3 & D_4 & E_4 & \\ D_3 & E_3 & D_4 & E_4 & \ddots & \ddots \\ E_3 & D_4 & E_4 & \ddots & \ddots & \\ D_4 & E_4 & \ddots & \ddots & & \\ E_4 & \ddots & \ddots & & & \\ & \ddots & & & & \end{pmatrix}$$

At this point, the pattern seems evident. Although this is not a thorough proof, I am convinced that this pattern will continue until every block of C eventually appears in $\text{ulc}(C^k)$. In fact, I believe it is possible to predict at which k all blocks will appear in $\text{ulc}(C^k)$ for the first time. If a given network has n layers, there will be n distinct blocks of C, where the $n-1^{st}$ block, $E_{\frac{n-1}{2}}$, will be a row vector when it appears in the final row of C, and its transpose, a column vector, when it appears in the final column of C; and the n^{th} block, $D_{\frac{n+1}{2}}$, will be a 1×1 matrix. (See the K matrices for networks with three and five layers, respectively, to illustrate this point.) Based on the pattern we observe above, we can predict that $D_{\frac{n+1}{2}}$ will appear the upper right corner of C^k for the first time when $k = 2(\frac{n+1}{2}) - 1 = n$.

4.5 Recoverability and the Moving Up Property

In the previous section, we saw that all blocks of C eventually appear in the upper left block of some power of C, and we have seen previously that from Λ_k we know $\mathrm{ulc}(C^{k-1})$. In theory, this gives us the ability to recover all blocks of the C matrix from the sequence Λ_n. After having spent quite a bit of time examining the Λ_n sequence, I believe that an even stronger statement can be made as follows:

- When k is even, $\Lambda_k = \mathrm{ulc}(C^{k-1}) = P_k + E_1 \times \cdots \times E_{\frac{k}{2}-1} \times D_{\frac{k}{2}}$, where P_k is a polynomial in $D_1, \ldots, D_{\frac{k}{2}-1}$, and $E_1, \ldots, E_{\frac{k}{2}-1}$.
- When k is odd, $\Lambda_k = \mathrm{ulc}(C^{k-1}) = P_k + E_1 \times \cdots \times E_{\frac{k-1}{2}}$, where P_k is a polynomial in $D_1, \ldots, D_{\frac{k-1}{2}}$, and $E_1, \ldots, E_{\frac{k-1}{2}-1}$.

In many ways, this is simply a restatement of the moving up property; it says that each subsequent odd power of C will have a new D block in its upper left corner, and each subsequent even power of C will have a new E block in its upper left corner. However, notice that the polynomials P_k in each case are made up of blocks which will already be known from the upper left corners of previous powers of C, which we learned from previous terms of the Λ_n sequence. Thus, we know not only that we can recover these blocks, but we know how to proceed in their recovery: if we calculate the block-structure form of $\mathrm{ulc}(C^k)$ for each k, and proceed to use the blocks computed from previous powers of C, the invertibility of all the E_k's allows us to find each new block as we are given each subsequent term of the Λ_n sequence.

A brief note about recoverability: When we compute the D_1 block of C from Λ_2 (for example), we will find d_1 directly, because it appears alone in entries of D_1. However, knowing e_0 and d_1, we will also be able to compute e_1 algebraically, since the diagonal entries of D_1 will equal $e_0 + 2d_1 + e_1$. Thus, in a sense, we do not get any new information from the odd Λ_n's, which give us the E_k blocks of C. Also, for this same reason, we will be finished recovering the network at Λ_{m-1} where m is the number of layers in the network. We see that $m-1$ will be even, since m is odd for all layered networks of the type I have chosen to look at. Thus, from Λ_{m-1} we can compute a D block of C, and also an e conductance. As discussed previously, the next "block" in the C matrix will actually be a vector with all entries equal to the e conductance just computed, and its transpose, and the final block will simply be a 1×1 matrix, equal to the sum of the entries in the vector. Thus, all the conductances will already have been computed from Λ_{m-1}. From this observation, it follows that two layered networks with the same conductances on the same layers will have equal Λ_n for all $n < m$, where m is the number of layers of the smaller network.

4.6 C as a Sum of Matrices

In attempting to understand the behavior of C in block form, it has proved useful to look at C as a sum of two Matrices,

$$D = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & D_3 & \\ & & & \ddots \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} & E_1 & & & \\ E_1 & & E_2 & & \\ & E_2 & & E_3 & \\ & & E_3 & & \ddots \\ & & & \ddots & \end{pmatrix}.$$

So we have

$$
\begin{aligned}
C &= D + E \\
C^2 &= D^2 + ED + DE + E^2 \\
C^3 &= D^3 + ED^2 + DED + E^2D + D^2E + EDE + DE^2 + E^3 \\
C^4 &= D^4 + ED^3 + DED^2 + E^2D^2 + D^2ED + EDED + DE^2D + E^3D \\
&\quad + D^3E + ED^2E + DEDE + E^2DE + D^2E^2 + EDE^2 + DE^3 + E^4
\end{aligned}
$$

etc.

This formulation is helpful in understanding C because it allows us to see what parts of the C matrix cause it to behave as it does. For example, only terms containing an even number of E's will have a non-zero first entry. This can be proved by induction:

- Multiplication of E by D does not change E's structure. E has zero and non-zero entries in the same places as before. The only difference is the numeric values of these entries. This is true because D is a diagonal matrix.
- Base case: $E = \begin{pmatrix} & E_1 & \\ E_1 & & \ddots \\ & \ddots & \end{pmatrix}$, $E^2 = \begin{pmatrix} E_1^2 & & E_1E_2 & \\ & E_1^2+E_2^2 & & \ddots \\ E_1E_2 & & \ddots & \\ & \ddots & & \end{pmatrix}$.
- k implies $k+1$: If $E^k = \begin{pmatrix} X & & \cdots \\ & Y & \\ \vdots & & \ddots \end{pmatrix}$,

$$
E^{k+1} = \begin{pmatrix} X & & \cdots \\ & Y & \\ \vdots & & \ddots \end{pmatrix} \times \begin{pmatrix} & E_1 & \\ E_1 & & \ddots \\ & \ddots & \end{pmatrix} = \begin{pmatrix} & XE_1 & \cdots \\ YE_1 & \ddots & \\ \vdots & & \end{pmatrix},
$$

so E^{k+1} has no non-zero first entry. If $E^k = \begin{pmatrix} & X & \cdots \\ Y & \ddots & \\ & & \\ \vdots & & \end{pmatrix}$,

$$
E^{k+1} = \begin{pmatrix} & X & \cdots \\ Y & & \\ \vdots & \ddots & \end{pmatrix} \times \begin{pmatrix} & E_1 & \\ E_1 & & \ddots \\ & \ddots & \end{pmatrix} = \begin{pmatrix} E_1X & \cdots \\ \vdots & \ddots \end{pmatrix},
$$

so E^{k+1} has a non-zero first entry.

Thus, only some of the terms (those which are a product of an even number of E's and any number of D's) added to produce C^{k-1} will contribute to information that appears in Λ_k. Due to time constraints, I have been unable to examine the properties of D and E much further than this.

5 Conclusion

Obviously, there are many questions which remain unanswered. In fact, perhaps more questions are raised by this paper than are answered. As a first step into a new field, however, I believe my research will be useful, in that it illuminates areas that were before completely in the dark. In conclusion, I leave the reader with a summary of questions raised by this paper that have not been answered.

- Is it possible to determine the number of interior nodes in a network from the power series expansion of the function $\Lambda(\lambda)$?
- How many possible factorizations of a given Λ_1 matrix into a prospective B and B^T exist? Is this number finite? Can the process described in this paper be improved to provide a true algorithm for determining from the Λ_n sequence if a network has two interior nodes?
- If such an algorithm can be found for sequences with rank-two Λ_1's, can it be found for sequences with only rank-one Λ_1's, a variation on the general two-interior-node case mentioned only briefly in this paper.
- Can we develop a test to determine if a given sequence describes a layered network?
- What happens in the terms of the Λ_n sequence of a layered network for $n >$the number of layers?
- Can rigorous proofs be shown for the moving up property on C for layered networks and its effect on the recoverability of these networks?
- How is the structure of C^k influenced by the matrices D and E.

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