Directed Graphs

Jaime Lust

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Abstract

In [1] Curtis and Morrow have extensively studied the recoverability properties of undirected graphs and their conductances. Here directed graphs are defined and their properties are discussed. The uniqueness of the Dirichlet problem for directed graphs is proved, the recoverability of certain conductivities are outlined, and connection equivalences within graphs are listed.

1 Defining a Directed Graph

For an electrical network, represented by an undirected graph, $\Gamma = (G, \gamma)$, the conductivities are defined along the edges of the graph. The magnitude of the current read at the nodes at the opposite ends of an edge is the same, namely

$$|(u_p - u_q)\gamma_{pq}|$$

the potential drop times the conductivity of the edge as defined by Ohm's Law. However, in the case of directed graphs the conductivity is defined for an edge at each vertex and the resulting current is allowed to differ depending at which node the current is measured. For a network with current flow out of a node defined to be positive, the current at node p with neighbors q is

$$\sum_{q \sim p} \gamma_{pq}(u_p - u_q),$$

where γ_{pq} is the conductivity on edge pq as seen by node p. This can lead to counterintuitive results as, for example, the conductivity on edge pq can be defined to be zero according to p and some positive quantity, γ_{qp}, according to q. Then q will read a current of $\gamma_{qp}(u_q - u_p)$ with the current flowing towards q for $u_q < u_p$ and current flowing away from q for $u_q > u_p$ whereas p will see no current at all. For this reason, it may be easier to understand directed graphs as a set of defined relationships between nodes rather than as symbolic representations of electrical networks. As was the case with undirected graphs for a γ-harmonic potential function $u(p)$ on the interior of the graph $\sum_{q \sim p} \gamma_{pq}(u_p - u_q) = 0$ for each interior node. As will be necessary below it is important that not all $\gamma_{pq} = 0$ for an interior node p in order to prove the uniqueness of the Dirichlet problem.

The Kirchhoff matrix K for a graph with m nodes that satisfies the properties named above is the m x m matrix with entries defined as:

(1) $K_{i,j} = -\gamma_{i,j} \quad for\, i \neq j$

(2) $K_{i,i} = -\sum_{j:j\neq i} K_{i,j}$

This matrix has the properties that row sums are equal to zero and the off-diagonal entries are negative or zero, but it is no longer symmetric as was the case with undirected graphs. Writing K in block form

$$\mathbf{Ku} = \begin{bmatrix} K(B;B) & K(B;I) \\ K(I;B) & K(B;B) \end{bmatrix} \begin{bmatrix} \psi \\ x \end{bmatrix} = \begin{bmatrix} \phi \\ 0 \end{bmatrix} = \mathbf{i}$$

where u is the vector of potentials, i is the resulting vector of currents, B are boundary nodes, and I are interior nodes. Because u is γ-harmonic on the interior of G, $K(I;B)\psi + K(I;I)x = 0$ so $x = -K(I;I)^{-1}K(I;B)\psi$ and $\phi = K(B;B)\psi - K(B;I)K(I;I)^{-1}K(I;B)\psi$. Let

$$\Lambda = K(B;B) - K(B;I)K(I;I)^{-1}K(I;B)$$

be the response matrix, which when multiplied by ψ, the vector of boundary potentials, results in ϕ the vector of boundary current. In the above derivation it is important that $K(I;I)$ be invertible which is the case and is shown below.

For directed graphs the response matrix Λ of flow at the boundary nodes, resulting from potentials imposed at the boundary nodes, is a Kirchhoff matrix for a well-connected graph.

2 The Dirichlet Norm

In the undirected case there is a useful relation between the Kirchhoff matrix and a discrete analog of one of Green's identities. This is the Dirichlet Norm of K where

$$u^T K u = \sum_{q\sim p} \gamma_{pq}(u_p - u_q)^2.$$

This is minimized when $u(p)$ is a γ-harmonic function and defines the power loss of the system. For directed graphs K is no longer symmetric and a correction factor must be added to the Dirichlet Norm. Let

$$K = \frac{1}{2}(K + K^T) + \frac{1}{2}(K - K^T).$$

Then

$$u^T K u = u^T[\frac{1}{2}(K + K^T)]u$$

as

$$[\frac{1}{2}(K - K^T)]^T = -\frac{1}{2}(K - K^T)$$

and therefore

$$[u^T[\frac{1}{2}(K-K^T)]u]^T = -u^T[\frac{1}{2}(K-K^T)]u = 0.$$

$\frac{1}{2}(K+K^T)$ is symmetric but the rows no longer sum to zero so a diagonal correction matrix must be added with entries defined by:

(1) $D_{p,p} = \sum_{q\sim p} \frac{1}{2}(\gamma_{pq} - \gamma_{qp})$

(2) $D_{p,q} = 0$

Summing over the average of the two vertex conductivities with these symmetrizing corrections the corresponding Dirichlet Norm for directed networks is:

$$\sum_{q\sim p} \frac{1}{2}(\gamma_{qp} + \gamma_{pq})(u_p - u_q)^2 = u^T K u + u^T D u$$

A few observations. First, in the case of undirected graphs $\gamma_{qp} = \gamma_{pq} \Rightarrow D = 0$ and the new formula simplifies to the original formula. Second, this formula can no longer be used to prove the uniqueness of the Dirichlet problem as K is no longer positive definite and $u^T K u$ no longer has a definite sign.

3 Uniqueness of the Dirichlet Problem

3.1 The Maximum Principle

Because $u(p)$ is $\gamma-$harmonic on the interior of Γ, $\sum_{q\sim p} \gamma_{pq}(u_p - u_q) = 0$ for each interior node p. Rearranging (where $\sum_{q\sim p} \gamma_{pq} \neq 0$)

$$u_p = \frac{\sum_{q\sim p} \gamma_{pq} u_q}{\sum_{q\sim p} \gamma_{pq}}$$

The potential at p is the weighted average of the potential at its neighbors, therefore either $u_p = u_q$ for all $q \sim p$ or there exist q_1, q_2 such that $u_p > u_{q_1}$ and $u_p < u_{q_2}$. To prove uniqueness of the Dirichlet problem the additional condition that for each interior node there is a directed path to the boundary is placed on Γ. Then for an interior node p there exists a directed path $pq_1q_2\ldots q_\delta$ where q_δ is a boundary node. If $u(p)$ were a maximum at p then the value of $u(p)$ at all $q \sim p$ would equal that at p and this would propagate along the directed path to the boundary. Therefore, on each directed path of nodes either $u(p)$ is constant or the maximum (similarly minimum) value is attained at the boundary. This yields the result that if $u(p) = 0$ for all boundary nodes then $u(p) = 0$ for all interior nodes.

3.2 'Positive Definite' Submatrices

Although $C = K(I; I)$, the submatrix of K containing interior node data, is not symmetric and positive definite it can be shown that all principal submatrices of C have positive determinants (with the conditions that for every interior node p $\sum_{q \sim p} \gamma_{pq} \neq 0$ and there exists a directed path from p to the boundary). The process of Gaussian elimination produces a lower triangular matrix with diagonal entries that are ratios of sums of monomials in γ_{pq} with positive integer coefficients. In particular, when taking the Schur complement $K(I; I)$ is invertible and there exists a unique solution to the Dirichlet problem.

Let E be any principal submatrix of C corresponding to a set of indices J where $q \in J$. Then with an appropriate reordering of the indices extracted from C

$$\mathbf{E} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \dots & \kappa_{1m} & \kappa_{1n} & d_1 \\ \kappa_{21} & \kappa_{22} & \dots & \kappa_{2m} & \kappa_{2n} & d_2 \\ \vdots & \vdots & \ddots & \dots & \dots & \vdots \\ \kappa_{m1} & \kappa_{m2} & \vdots & \kappa_{mm} & \kappa_{mn} & d_m \\ \kappa_{n1} & \kappa_{n2} & \vdots & \kappa_{nm} & \kappa_{nn} & d_n \end{bmatrix}$$

where $1, \dots, n$ are interior nodes and $(d_1, \dots, d_n)$ is the column of row sums of E. From the definition of the Kirchhoff matrix,

(1) all off-diagonal entries $\kappa_{ij} = \gamma_{pq} \leq 0$ and

(2) all diagonal entries $\kappa_{ii} = \gamma_{pp} = d_i - \sum_{j \neq i} \kappa_{ij}$ where $d_i = -\sum_{q \notin J} \gamma_{iq} \geq 0$.

(3) All $\kappa_{ii} \neq 0$ from the condition that $\sum_{q \sim p} \gamma_{pq} \neq 0$.

Also at least one $\sum_{j=1}^{n} \kappa_{ij} > 0$ from the condition that there exist a directed path from node i to the boundary (this prevents C from having a kernel). Then Gaussian elimination using the bottommost, righthand entry, κ_{nn}, to reduce the entries above it to zero produces a new matrix

$$\mathbf{E}' = \begin{bmatrix} \kappa'_{11} & \dots & \kappa'_{1m} & 0 & d'_1 \\ \vdots & \ddots & \dots & 0 & \vdots \\ \kappa'_{m1} & \vdots & \kappa'_{mm} & 0 & d'_m \\ \kappa_{n1} & \vdots & \kappa_{nm} & \kappa_{nn} & d_n \end{bmatrix}$$

where $(d'_1, \dots, d_n)$ is the column of row sums of E'. From Gaussian elimination

$$\kappa'_{ij} = \kappa_{ij} - \frac{\kappa_{in}\kappa_{nj}}{\kappa_{nn}} = -\frac{(-\kappa_{nn}\kappa_{ij}) + \kappa_{in}\kappa_{nj}}{\kappa_{nn}}.$$

From above

$$(-\kappa_{nn}\kappa_{ij}) \geq 0, \ \kappa_{in}\kappa_{nj} \geq 0, \ \text{and } \kappa_{nn} > 0$$

which implies that κ'_{ij} is a negative ratio of sums of monomials in γ_{pq} with positive integer coefficients. This satisfies (1). Also

$$d'_i = d_i - \frac{\kappa_{in} d_n}{\kappa_{nn}} = \frac{\kappa_{nn} d_i + (-\kappa_{in} d_n)}{\kappa_{nn}}$$

where

$$\kappa_{nn} d_i \geq 0, \ (-\kappa_{in} d_n) \geq 0, \text{ and } \kappa_{nn} > 0$$

implies that d'_i is a ratio of sums of monomials in γ_{pq} with positive integer coefficients. In addition

$$d'_i = \sum_{j=i}^{m} \kappa'_{ij} \ \Rightarrow \kappa'_{ii} = d'_i - \sum_{j=1(i \neq j)}^{m} \kappa'_{ij}$$

Therefore κ'_{ii} is also a ratio of sums of monomials in γ_{pq} with positive integer coefficients. This satisfies (2). From above $\kappa_{ii} \neq 0 \ \Rightarrow$ either some $\kappa_{ij} \neq 0$ or $d_i \neq 0$. For $j \neq n$ from the first expansion above it can be clearly seen that $\kappa_{ij} \neq 0 \ \Rightarrow \kappa'_{ij} \neq 0$. Similarly $d_i \neq 0 \ \Rightarrow d'_i \neq 0$. If $\kappa_{ii} = -\kappa_{in}$ then node i has a directed connection to node n and is directly connected to no other node. However, there must be a directed path from i to a boundary node. Therefore, there must be either be a directed edge from n to a boundary node so $d_n \neq 0$ or n is directly connected to another interior node that is eventually directly connected to a boundary node so some $\kappa_{nj} \neq 0$. $d_n \neq 0 \ \Rightarrow \ d'_i \neq 0$ from the second term in the above expansion for d'_i and similarly $\kappa_{nj} \neq 0 \ \Rightarrow \ \kappa'_{ij} \neq 0$. Thus,

$$\kappa_{ii} \neq 0 \ \Rightarrow \ \text{some } \kappa'_{ij} \neq 0 \text{ or } d'_i \neq 0 \ \Rightarrow \kappa'_{ii} \neq 0.$$

This satisfies (3).

The upper, lefthand m x m submatrix of E' is of the same form as C. Proceeding by induction a lower triangular matrix with diagonal entries that are ratios of sums of monomials in γ_{pq} with positive integer coefficients, and off-diagonal entries that are either zero or negative ratios of sums of monomials in γ_{pq} is formed. In particular all the diagonal entries of $K(I;I)$ are positive so C has a positive determinant and is invertible. When taking the Schur complement of K with respect to C the Dirichlet problem has a unique solution.

The process of taking the Schur complement is equivalent to the process of Gaussian elimination of interiorizing one node at a time, so using the above described method on the entire K matrix, leaves the response matrix Λ, the upper lefthand square matrix that remains after eliminating the interior nodes, with almost the same form as E'. Λ has rows that sum to zero with diagonal entries that are either zero or ratios of sums of monomials in γ_{pq} with positive integer coefficients and off-diagonal entries that are either zero or negative ratios of sums of monomials in γ_{pq} with positive integer coefficients.

4 Recoverability of Conductances

In looking at many directed graphs the only ones found to be totally recoverable were those with no interior nodes in which case the Kirchhoff matrix is the response matrix. It is conjectured that these are the only fully recoverable directed graphs. There are, however, pieces of every graph that can be recovered.

4.1 One-way Boundary Spikes

A boundary spike is a boundary node connected only to one interior node and nothing else. Let a graph contain a boundary spike consisting of a boundary node p connected to an interior node q where $\gamma_{pq} \neq 0$ and $\gamma_{qp} = 0$. Then the Kirchhoff matrix has the form

$$\mathbf{K} = \begin{bmatrix} \gamma_{pq} & -\gamma_{pq} & 0 & 0 \\ 0 & \ddots & \dots & \dots \\ 0 & \vdots & \ddots & \dots \\ 0 & \vdots & \vdots & \ddots \end{bmatrix}$$

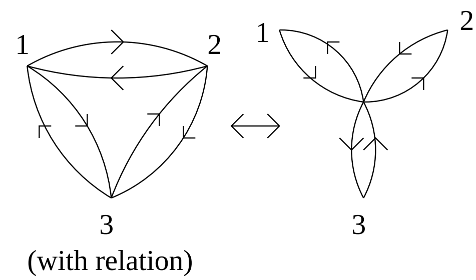
Here $\kappa_{pp} = \gamma_{pq}$ and $\kappa_{qp} = 0$ for all $q \neq p$. The process of taking the Schur complement does not affect this column as the only non-zero entry is due to a boundary conductivity. Thus, the response matrix has the form

$$\mathbf{\Lambda} = \begin{bmatrix} \gamma_{pq} & \ddots & \dots \\ 0 & \vdots & \ddots \\ 0 & \vdots & \vdots \end{bmatrix}$$

The conductivity γ_{pq} can be directly read from the response matrix.

4.2 Boundary Edges

The method of recoverability of conductivities as outlined here is adapted from Curtis and Morrow in [1] with only minor changes from undirected to directed graphs. A boundary to boundary directed edge can be recovered if deleting that edge breaks a directed connection in the original graph. Let $P = (p_1, \dots, p_k)$ and $Q = (q_1, \dots, q_k)$ be sequences of boundary nodes in a graph G such that there is a disjoint directed connection from each node p_n to node q_n. Then from Blunk and Coskey in [2] it is known that $\det\Lambda(P;Q) \neq 0$ in the original graph G. The response matrix from the new graph G' obtained by deleting the directed edge p_1q_1 has $\det\Lambda'(P';Q') = 0$. The new Kirchhoff matrix K' differs from the original K by $\kappa'_{p_1q_1} = \kappa_{p_1q_1} + \gamma_{p_1q_1} = 0$ and $\kappa'_{p_1p_1} = \kappa_{p_1p_1} - \gamma_{p_1q_1}$, all other entries staying the same. The process of taking the Schur complement only adds and multiplies with data from nodes being interiorized so the response matrix Λ' only differs from the original Λ by $\lambda'_{p_1q_1} = \lambda_{p_1q_1} + \gamma_{p_1q_1}$ and $\lambda'_{p_1p_1} = \lambda_{p_1p_1} - \gamma_{p_1q_1}$,



all other entries being the same. This being known $\gamma_{p_1q_1}$ is found by solving

$$\begin{vmatrix} \lambda_{p_1q_1}+\gamma_{p_1q_1} & \lambda_{p_1q_2} & \cdots & \lambda_{p_1q_k} \\ \lambda_{p_2q_1} & \lambda_{p_2q_2} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots \\ \lambda_{p_kq_1} & \vdots & \vdots & \lambda_{p_kq_k} \end{vmatrix} = 0.$$

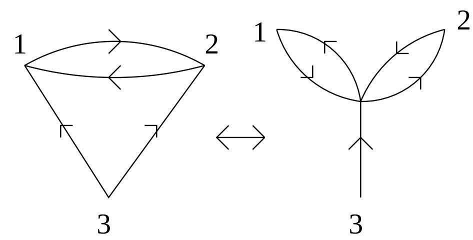
All directed boundary edges recovered so far have been shown to break a connection. It is conjectured that only directed boundary edges that break a connection can be recovered.

5 Recoverability of Graphs

5.1 Connection Equivalences

As motivation, given a square matrix with row sums equal to zero is it possible to find a graph or "Y – $\triangle$" equivalent set of graphs which would have the given matrix as its response matrix? Before tackling this question perhaps it would be better to start by seeing if there is a corresponding "Y – $\triangle$" transformation for directed graphs, i.e. a set of circular planar directed graphs, linked by a specific group of substitutions, that have the same connections for all circular pairs of boundary nodes. This is the case but it is more complicated for directed graphs as there are many more substitutions. These substitutions are found from looking at the response matrix for a three boundary node graph. Because in the directed case the Kirchhoff matrix is no longer symmetric and the graphs can by double edged there are many more possibilities for equivalences. Unlike the undirected case, there are also three boundary node directed graphs for which there are no equivalent substitutions.

As stated above making any of the above substitutions in a directed graph does not change the set of connections between the boundary nodes. This is similar to the undirected case as studied extensively by Curtis and Morrow in [1]. Let there be a graph G' obtained by making one of the above substitutions to a graph G. Referring to the graph fragments above, suppose that in G the original form was similar to a $\triangle$ with a directed path from 3 to 2 and that in G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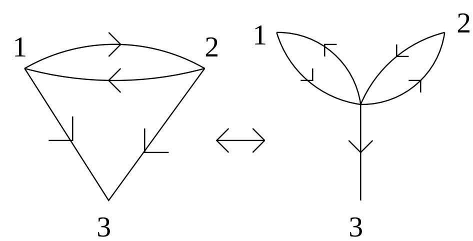
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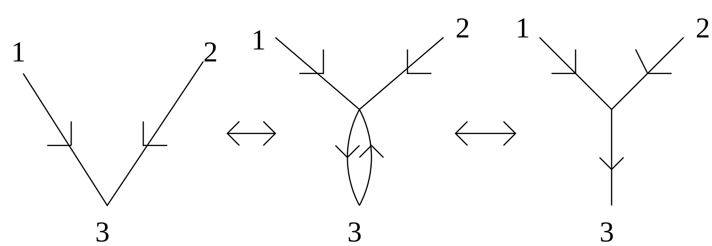
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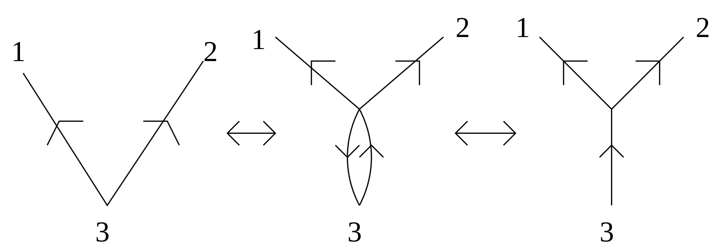
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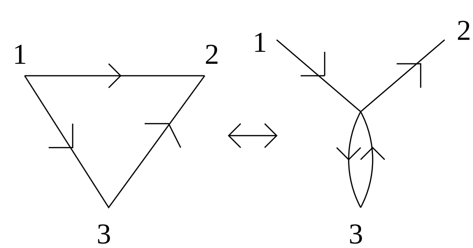
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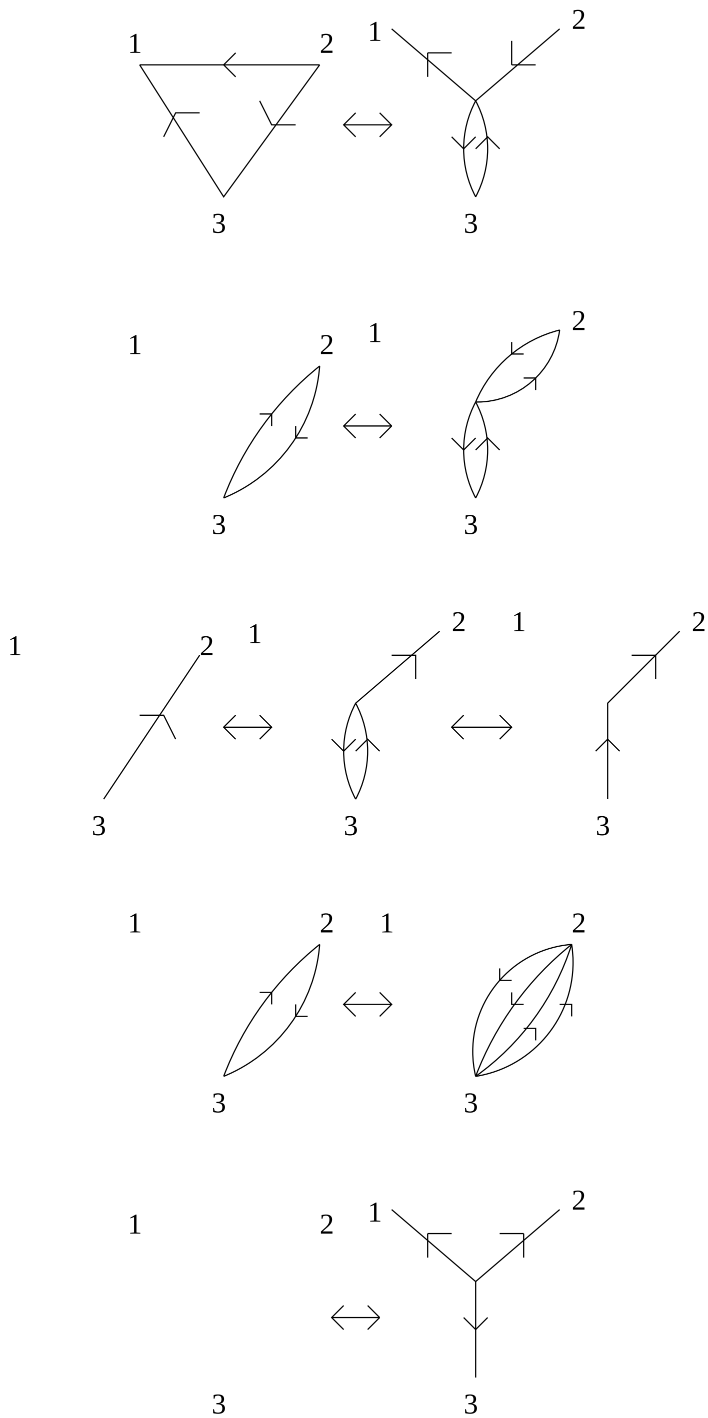
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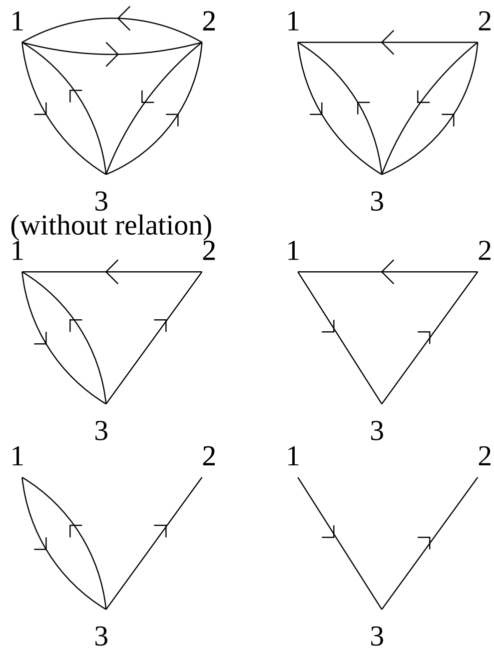


there were disjoint directed paths α that used node 1 and β that used nodes 2 and 3. Then if G' contains the equivalent Y form there exist disjoint paths α' that uses node 1 and β' that uses nodes 2 and 3. Again, if

$$\alpha = a_1 \ldots 1 \ldots a_2$$

$$\beta = b_1 \ldots 32 \ldots b_2$$

then

$$\alpha' = a_1 \ldots 1 \ldots a_2$$

$$\beta' = b_1 \ldots 3c2 \ldots b_2$$

where c is the center node between nodes 1, 2, and 3.

5.2 Ratio Relations

In undirected graphs if two graphs were connection equivalent they were also electrically equivalent and had similar conductance recoverability properties. For directed graphs this is not the case. For example, a given 3 x 3 matrix with negative (all non-zero) off-diagonal entries and row sums equal to zero could always have come from a $\triangle$ graph but not necessarily from a Y graph. For a $\triangle$ graph $K = \Lambda$ and is therefore fully recoverable but Y graphs are not recoverable. Boundary conductances $\gamma_{\delta I}$ are recoverable where

$$\gamma_{1I} = \lambda_{11} - \frac{\lambda_{21}\lambda_{13}}{\lambda_{23}}.$$

Interior conductances are not recoverable but are related by the ratios

$$\frac{\gamma_{I1}}{\gamma_{I2}} = \frac{\lambda_{31}}{\lambda_{32}}, \frac{\gamma_{I2}}{\gamma_{I3}} = \frac{\lambda_{12}}{\lambda_{13}}, \frac{\gamma_{I3}}{\gamma_{I1}} = \frac{\lambda_{23}}{\lambda_{21}}.$$

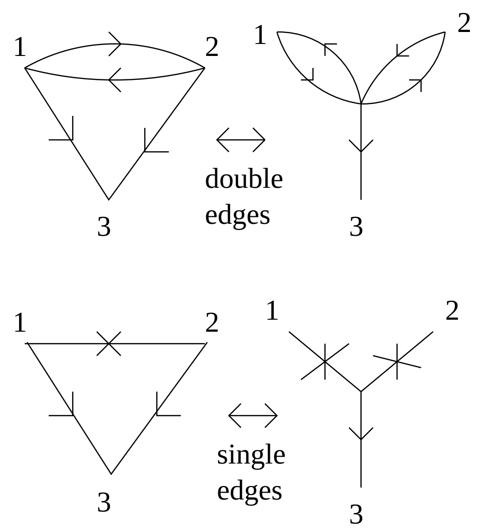
Rewritten this is the relation $\lambda_{12}\lambda_{23}\lambda_{31} = \lambda_{13}\lambda_{32}\lambda_{21}$ which is always present in response matrices from Y graphs. There are many of these nontrivial multiplicative relations in response matrices from directed graphs that are not a factor in their undirected counterparts with symmetric response matrices. These relations are a consequence of determinantal relationships from the connections present in the graph. These relations reduce the number of independent entries in the response matrix and may be a clue to why directed conductivities are so difficult to recover.

6 Directed Medial Graphs

Once again relying heavily upon the example of the undirected case, now that the class of connection equivalent graphs are known, is it possible to find a directed medial graph in the effort towards recovering the directed graph from information in the response matrix? This is still an open question. (The definition and usefulness of medial graphs is described in detail by Curtis and Morrow in chapters 8 and 9 of [1].) First, it is unknown what is the most helpful way to depict and consequently think about directed graphs. Between two nodes two edges can be drawn, one going each way, an edge being deleted if the conductivity on that edge is zero. Or, between two nodes one edge can be drawn labeled with a double headed arrow for a non-zero conductance both ways or labeled with a single headed arrow if the conductance in one way is zero.

In the case where doubled edged graphs are drawn there is an algorithm to draw medial graphs with consistently directed geodesics. For a geodesic that crosses two directed edges the relationship between the directions of the edges uniquely determines the direction of the geodesic. Consequently when drawing the graph from the medial graph, for an edge that is crossed by two geodesics the relationship of the directions of the geodesics uniquely determines the direction of the edge. The main problem found with this process is that due to the double edges in the original graphs there are many lenses in the medial graphs. Also, it is difficult if not impossible to understand how to manipulate the geodesics in these directed medial graphs to describe connection equivalent substitutions in the original graphs.

Looking at graphs drawn with single edges eliminates the problem of an excessive amount of lenses in the medial graph but leads to problems when trying to label the geodesics. No good scheme for drawing medial graphs with possibly one-way or two-way directed geodesics to correspond to graphs with one-way and two-way directed edges has been found.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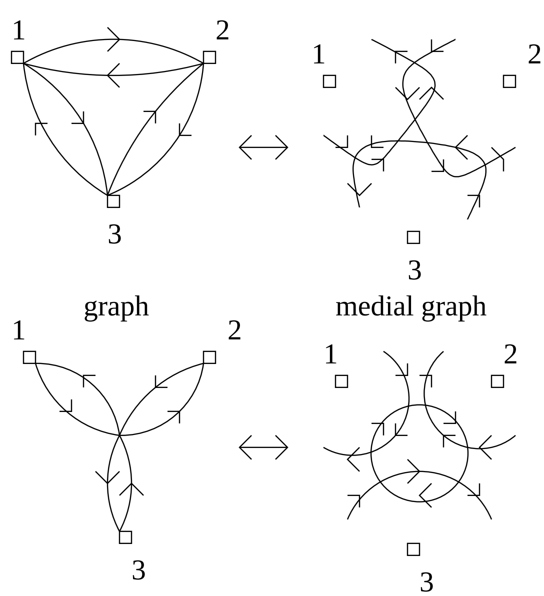
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[2] Blunk, Mark, and Sam Coskey. *Vertex Conductivity Networks,* University of Washington, 2001.