RECOVERING NETWORKS WITH SIGNED CONDUCTIVITES

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ABSTRACT. It is known that a critical circular planar network can be recovered if conducitivies are restricted to positive real numbers. If the range of conductivities is extended to all nonzero real numbers, we are still able to recover the network if the network response exists.

CONTENTS

INTRODUCTION

Let $G = (V, V_b, E)$ be a graph for which V is the set of vertices, V_b is a subset of V denoting boundary vertices, and E is the set of edges. Let γ be the conductivity function on E, which assigns each edge e to a nonzero real number, not necessarily positive. Γ is the resistor network that is G combined with γ. The Kirchoff matrix K is the matrix defined as follows. $K_{i,j} = -\gamma(i, j)$, where $\gamma(i, j)$ is the conductivity of the edge joining nodes i and j, and $i \neq j$. $K_{i,j} = 0$ if no edge joins nodes i and j. $K_{i,i} = \sum_{j \neq i} \gamma(i, j)$.

∂ refers to the set of boundary nodes, and int refers to the interior nodes. K has the block structure:

$$K = \begin{bmatrix} A(\partial \times \partial) & B(\partial \times \text{int}) \\ B^T(\text{int} \times \partial) & C(\text{int} \times \text{int}) \end{bmatrix} \tag{1}$$

Date: August 13, 2003.

As before, Λ_γ denotes the response matrix, which is the linear map which sends the voltage to a current on the boundary nodes. We compute $\Lambda = K/C$, that is, we take the Schur complement of K with respect to the entries corresponding to the interior nodes. Λ is defined if and only if $\det(C) \neq 0$. This identity holds if $\gamma > 0$ for all edges. However, when γ is allowed to range over nonzero real numbers, we are not always able to define Λ. The simplest example is a network with two boundary nodes, labeled 1 and 2, and one interior node, labeled 3. Let $\gamma(1,3) = 1$, $\gamma(2,3) = -1$, and $\gamma(1,2) = 0$. Then Λ cannot be defined.

The cells of G will be bounded by the edges of G as well as the boundary circle. In this paper we will only consider critical circular planar networks, unless otherwise specified. We will prove that a critical circular planar network, with either positive or negative, but nonzero, conductivities is recoverable if $\det K(I;I) \neq 0$ in the Kirchoff matrix, so we can define a response matrix. We will see that the boundary edge and boundary spike formulas may be used for networks with signed conductivities, and they may also be used when the broken connection is not a circular pair. The only requirement for using a connection (P, Q) to recover a boundary edge or boundary spike is $\det \Lambda(P;Q) \neq 0$ before the deletion of the edge or contraction of the spike, and $\det \Lambda(P;Q) = 0$ after the deletion or contraction. The connection (P, Q) we use will have the feature that it can be formed through only one permutation τ of P_i to $Q_{\tau(i)}$, it can only be connected with one set of paths, and the paths use every interior node in a connected component of interior nodes. We will give an algorithm for building this connection and prove that it has the desired properties. Then we offer a procedure for continuing the recovery algorithm in the event that a recovered boundary spike cannot be contracted.

Suppose p and q are boundary nodes on a graph G. Then $\widehat{(p,q)}$ refers to the counter-clockwise arc from p to q on the boundary circle. This notation will be used throughout the paper. Two boundary nodes p and q will be called *consecutive* if q is immediately before or after p in clockwise circular order. If p is immediately after q in clockwise circular order, p is clockwise consecutive to q, and q is counter-clockwise consecutive to p. Also, always assume a circular planar graph G is given with a fixed embedding.

1. Important Formulas

To recover critical circular planar networks with signed conductivities, we first need the determinental identity from [1]. P and Q are disjoint sets of boundary nodes, each of size k. The first summation is over the permutation group S_k, and the second is over all paths α in a connection from P_i to $Q_{\tau(i)}$. E_α is the set of edges used in α. J_α is the set of interior nodes not used in α, while D_α is $\det K(J_\alpha, J_\alpha)$.

$$\det \Lambda(P;Q) \cdot \det K(I;I) = (-1)^k \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \left\{ \sum_{\substack{\alpha \\ \tau_\alpha = \tau}} \prod_{e \in E_\alpha} \gamma(e) \cdot D_\alpha \right\} \tag{2}$$

We also need the boundary edge and boundary spike formulas. In a network Γ, Let $P = (p_1, \ldots, p_k)$, $Q = (q_1, \ldots, q_k)$, $P' = (p, p_1, \ldots, p_k)$, $Q' = (q, q_1, \ldots, q_k)$, and pq be a boundary edge. Suppose $\det \Lambda_\Gamma(P;Q) \neq 0$. Let Γ' be the network with the edge pq deleted, and suppose $\det \Lambda_{\Gamma'}(P;Q) = 0$. $(P;Q)$ is not necessarily a circular pair. Then,

$$\gamma(pq) = -\Lambda(p;q) + \Lambda(p;Q) \cdot \Lambda(P;Q)^{-1} \cdot \Lambda(P;q) \tag{3}$$

Similarly, in a network Γ, let pr be a boundary spike between boundary node p and interior node r. Suppose there are disjoint sets of boundary nodes P and Q such that $\det \Lambda_\Gamma(P;Q) \neq 0$. Let Γ' be the network obtained after the contraction of pr, and suppose $\det \Lambda_{\Gamma'}(P;Q) = 0$. Again, (P,Q) is not necessarily a circular pair. Then,

$$\gamma(pr) = \Lambda(p;p) - \Lambda(p;Q) \cdot \Lambda(P;Q)^{-1} \cdot \Lambda(P;q) \tag{4}$$

We will see that, for a critical network, we can recover a boundary edge or a boundary spike if we merely know that the conductivities are nonzero. Unlike the case of all positive conductivities, the existence of a connection between a circular pair $(R;S)$ does not imply $\det \Lambda(R;S) \neq 0$. Using the determinental identity above, we have $\det \Lambda(R;S) \neq 0$ if the following three conditions are satisfied on the sets $(R;S)$.

- There is only one τ so that there is a set of paths α from R_i to $S_{\tau(i)}$.
- For the fixed τ, there is only one α joining (R,S).
- $D_\alpha \neq 0$.

One way to insure the third condition is $J_\alpha = \varnothing$. Alternately, suppose there are two subsets of interior nodes I_1 and I_2 such that there is no edge joining a node of I_1 to a node of I_2. Then $K(I;I)$ has the following form:

$$K(I;I) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{5}$$

$A = K(I_1;I_1)$ and $B = K(I_2;I_2)$. Then $\det K(I;I) = \det A \cdot \det B$. So, $\det B \neq 0$, which means it is sufficient that $J_\alpha = I_2$.

If we have a method of constructing the connection (P,Q), we can use a process similar to the process for recovering a circular planar network with positive conductivities. Recover the boundary edges and boundary spikes using the given formulas, then remove the edges and contract the spikes, performing the appropriate operations on the response matrix. Because the network is critical, the process will terminate with every edge recovered.

Suppose we adjoin a boundary edge of conductivity ξ between boundary nodes 1 and 2. If the old response matrix looks like this:

$$\Lambda = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} & a \\ \lambda_{2,1} & \lambda_{2,2} & b \\ d & e & C \end{bmatrix} \tag{6}$$

The new response matrix looks like this:

$$\Lambda' = \begin{bmatrix} \lambda_{1,1}+\xi & \lambda_{1,2}-\xi & a \\ \lambda_{2,1}-\xi & \lambda_{2,2}+\xi & b \\ d & e & C \end{bmatrix} \tag{7}$$

This operation can be performed for any value ξ. A boundary edge of known conductivity ξ can be deleted by adjoining a boundary edge of conductivity $-\xi$.

Suppose we adjoin a boundary pendant of conductivity ξ at node 1. The new boundary node is labeled 1 and all other boundary nodes are incremented. If the original response matrix has the following form,

$$\Lambda = \begin{bmatrix} \lambda_{1,1} & a \\ b & C \end{bmatrix} \tag{8}$$

Then, after the pendant is adjoined, the new response matrix has the following form,

$$\Lambda' = \begin{bmatrix} \xi & -\xi & 0 \\ -\xi & \lambda_{1,1} + \xi & a \\ 0 & b & C \end{bmatrix} \tag{9}$$

Suppose we adjoin a boundary spike of conductivity ξ to boundary node 1. If the old response matrix looks like this:

$$\Lambda = \begin{bmatrix} \lambda_{1,1} & a \\ b & C \end{bmatrix} \tag{10}$$

The new response matrix looks like this, with $\delta = \lambda_{1,1} + \xi$:

$$\Lambda' = \begin{bmatrix} \xi - \frac{\xi^2}{\delta} & \frac{a\xi}{\delta} \\ \frac{b\xi}{\delta} & C - \frac{ab}{\delta} \end{bmatrix} \tag{11}$$

This operation can be performed for any value ξ, except $\xi = -\lambda_{1,1}$. A boundary spike of known conductivity ξ can be contracted by adjoining a boundary spike of conductivity $-\xi$. We will deal separately with the case that we want to contract a boundary spike of conductivity ξ at node p and $\xi = \lambda_{p,p}$.

2. Removal Number

Suppose G is a circular planar graph. A vertex u is considered *near* a vertex v if u and v are on the boundary of the same cell in G. We use the notation $u \bowtie v$.

We will define the *removal number* of u with respect to v, or $\text{rem}(u, v)$, recursively. For any vertex u, $\text{rem}(u, u) = 0$. If $u \neq v$, we use the following formula:

$$\text{rem}(u, v) = 1 + \min_{w \bowtie u} \text{rem}(w, v) \tag{12}$$

Suppose a path $\alpha = (p_0, p_1, \ldots, p_n)$ joins boundary nodes p_0 and p_n through the interior. Suppse u is any node in G. Then we define $\text{rem}(u, \alpha)$ as follows.

$$\text{rem}(u, \alpha) = \min_{0 \leq i \leq n} \text{rem}(u, p_i) \tag{13}$$

A cell c is considered *adjacent* to a path α if some vertex on the boundary of c is on α. The region $\mathcal{Z}(\alpha)$ is the union of all cells adjacent to α. Note that if u is a vertex of a cell $c \in \mathcal{Z}(\alpha)$, then $\text{rem}(u, \alpha) = 0$ or 1.

We also define the removal set of α, $\mathcal{R}(\alpha)$, as follows. A vertex $u \in \mathcal{R}(\alpha)$ if and only if $\text{rem}(u, \alpha) = 1$. In words, $\mathcal{R}(\alpha)$ is the set of all nodes u that lie on the boundaries of cells adjacent to α, not including α itself.

3. Boundary Antennas

Definition 3.1. An *boundary antenna*, or just antenna, is a pair of boundary spikes that share a common vertex.

A boundary antenna is adjoined in the following manner. Suppose p is a boundary node. First adjoin a boundary pendant at p with a fixed conductivity ξ. Then adjoin a boundary spike of fixed conductivity ω at p. The resulting graph will have two boundary spikes at node p, and p becomes an interior node.

Fact 3.2. Adjoining an antenna to a critical graph results in a critical graph.

This can be observed by studying the effects of adjoining an antenna on the medial graph.

4. Minimal Paths

Suppose a graph G with boundary has boundary nodes p and q that can be joined by possibly many paths through the interior. Consider a directed path α from p to q to divide G into two components.

Definition 4.1. The *minimal oriented path* from p to q, denoted $\min(pq)$ is the path α joining p and q through the interior that minimizes the number of cells to the right of α, oriented from p.

Theorem 4.2. *The minimal oriented path* $\min(pq)$ *is unique.*

Proof. Suppose there are two distinct paths $\min(pq)$, α_1 and α_2, that have the same number of cells to their right. Construct a new path P as follows. Start at p and follow whichever path is farther to the right, assumed without loss of generality to be α_1. Whenever α_1 and α_2 intersect, continue P by following whichever path is farther to the right after the intersection point. Continue until P reaches q. P must follow α_2 at some point, or α_1 would have fewer cells to its right than α_2. The number of cells to the right of P will be fewer than the number of cells to the right of either α_1 or α_2, and α_1 and α_2 are not minimal oriented paths. □

Definition 4.3. The *minimal vertex path* through interior node b with respect to a path α or a boundary node a is the path between two boundary nodes through the interior that passes through b, does not intersect α or end in a, and minimizes the number of cells of the component of the graph containing α or a. It will be denoted $\min(b, \alpha)$ or $\min(b, a)$.

Note that $\min(b, \alpha)$ exists if and only if there is a path between two boundary nodes through the interior that intersects b but does not intersect α.

Theorem 4.4. *Suppose α is a path between two boundary nodes through the interior, and b is an interior node not on α. If the minimal vertex path* $\min(b, \alpha)$ *exists, it is unique.*

Proof. Suppose there are two distinct paths $\min(b, \alpha)$, β_1 and β_2. Suppose the endpoints of α are p and q. Label the endpoints of β_1, p_1 and q_1 so that p, p_1, q_1, q are in circular order. Similarly label the endpoints of β_2, p_2 and q_2. Without loss of generality, assume p, p_1, p_2 are in clockwise order if p_1 and p_2 are not the same point.

Construct a new path β beginning at p_1 and follow β_1. Whenever β_1 and β_2 intersect, follow whichever path is farther to the right after the intersection point. The two paths have at least one intersection point, namely b. β must follow β_2 at some point, or β_1 have fewer cells to its right than β_2. Then β has fewer cells to its right then either β_1 or β_2. □

Theorem 4.5. *Suppose a is a boundary node. If the minimal vertex path $min(b, a)$ exists, it is unique.*

Proof. The proof is very similar to the case of $\min(b, \alpha)$. □

5. Maximal Connection

Suppose we are given a critical circular planar graph G. We will build a maximal connection $\mathcal{M}$ with respect to consecutive boundary nodes p and q, or a boundary spike sr, with s as the boundary node. We will use a restricted portion of G as follows. Suppose there is a boundary edge between two non-consecutive boundary nodes p' and q'. The edge $p'q'$ divides G into two regions. One region, G', contains the boundary nodes p and q, or the boundary spike sr. G' will include p', q', and the edge $p'q'$ as well. Taken as a graph, with the boundary nodes of G' exactly those which were boundary nodes in G, is circular planar and therefore critical by [2]. The interior nodes in G' will be denoted I_1, while the interior nodes in G but not in G' will be denoted I_2. It is sufficent that $\mathcal{M}$ use all of I_1 and none of I_2 because no there is no edge between a member of I_1 and I_2. This restriction will be relevant in the next section.

Definition 5.1. The *maximal connection* of a graph, with respect to consecutive boundary nodes p and q or with respect to boundary spike sr with s as the boundary node, is the set of paths α constructed by the following process. It is denoted $\mathcal{M}(pq)$ or $\mathcal{M}(s)$. It is assumed in this section that p is clockwise consecutive to q.

We will build $\mathcal{M}$ inductively, creating a set of minimal vertex paths $\mathcal{L}_i$ at step i. Suppose we are building $\mathcal{M}(pq)$ with respect to consecutive boundary nodes p and q. The first set of minimal paths, $\mathcal{L}_1$, is the singleton $\{\min(pq)\}$. Put $p \in P$ and $q \in Q$. If we are building $\mathcal{M}(s)$ with respect to boundary spike sr, $\mathcal{L}_1$ is the singleton $\{\min(r, s)\}$. The endpoints of $\min(r, s)$ are labeled p and q so s, p, q are in clockwise circular order. Put $p \in P$ and $q \in Q$. We will also consider $\mathcal{L}_0 = \{s\}$, even though s by itself is a single boundary node instead of a path. For the remainder of this paper, the endpoints of the single path of $\mathcal{L}_1$ will be denoted p_0 and q_0 so that $p_0 \in P$ and $q_0 \in Q$.

Suppose $\mathcal{L}_i$ is constructed and contains the paths $\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}$. Do the following for every path $\alpha_{ij} \in \mathcal{L}_i$. Consider $\alpha_i j$ to divide G into two regions, and consider the set of interior nodes α_{ij}^+ on the region not including p_0q_0. For every interior node $b \in \mathcal{R}(\alpha_{ij}) \cap \alpha_{ij}^+$, construct $\alpha_{i+1,k} = \min(b, \alpha_{ij})$, where $k-1$ is the number of paths already constructed in $\mathcal{L}_{i+1}$. Let $p_{i+1,k}$ and $q_{i+1,k}$ be the endpoints of $\alpha_{i+1,k}$, and assume $p, p_{i+1,k}, q_{i+1,k}, q$ are in circular order. Put $p_{i+1,k} \in P$, $q_{i+1,k} \in Q$, and $\alpha_{i+1,k} \in \mathcal{L}_{i+1}$. The process terminates when no more paths can be constructed in this manner.

Theorem 5.2. *Suppose α_{ij_1} and $\alpha_{ij_2} \in \mathcal{L}_i$ intersect at an interior node. Then $\alpha_{ij_1} = \alpha_{ij_2}$.*

Proof. Assume the maximal connection $\mathcal{M}$ has been constructed. Suppose $\alpha_{ij_1} = \min(u, \alpha_{i-1,k})$ and $\alpha_{ij_2} = \min(v, \alpha_{i-1,k})$ for interior nodes u, v and $\alpha_{i-1,k} \in \mathcal{L}_{i-1}$. The endpoints of α_{ij_1} and α_{ij_2} will respectively be called p_1, q_1, p_2, and q_2, with $p_1, p_2 \in P$ and $q_1, q_2 \in Q$. Without loss of generality, assume p, p_1, p_2 are in clockwise circular order if $p_1 \neq p_2$. Construct a new path α as follows. Follow α_{ij_1} from p_1 until the first intersection point of α_{ij_1} and α_{ij_2}. At every intersection point, follow whichever path is farther to the right after the intersection point. Because $u, v \in \mathcal{R}(\alpha_{i-1,k})$, u and v both lie on α. Every cell to the right of α, oriented from p_1, is also to the right of α_{ij_1} and α_{ij_2}. Therefore, $\alpha_{ij_1} = \alpha_{ij_2} = \alpha$. ☐

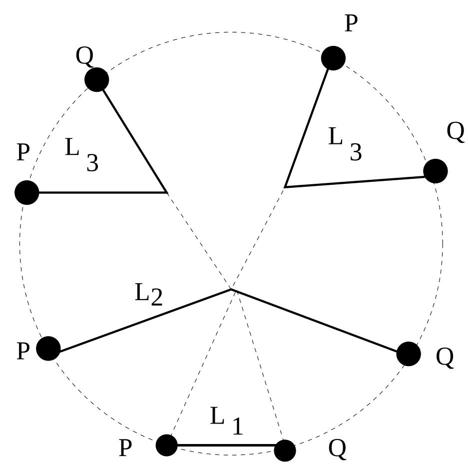

FIGURE 1. Example of branching paths

It is possible that α_{ij_1} and $\alpha_{ij_2} \in \mathcal{L}_i$ share a boundary endpoint r, which will lead to problems in constructing $(P; Q)$. In that case, adjoin a boundary antenna with known, fixed conductivities to r, which causes r to become an interior node. Then modify the given response matrix Λ to obtain Λ' as in equations (9) and (11). If Λ has dimension $n \times n$, then Λ' has dimensions $(n+1) \times (n+1)$. Then combine α_{ij_1} and α_{ij_2} into one path. See Figure 2 for an example.

Lemma 5.3. *Suppose sr is a boundary spike, with s as the boundary node. Then no boundary antenna is adjoined at s.*

Proof. Suppose two paths α_{ij_1} and $\alpha_{ij_2} \in \mathcal{L}_n$ have an endpoint at s. Then both paths use r. By Theorem 5.2, $\alpha_{ij_1} = \alpha_{ij_2}$, so only one path has an endpoint at s. So, no boundary antenna will be adjoined at s. ☐

Definition 5.4. The nth *level* in the maximal connection $\mathcal{M}$ is the set of paths created on the nth step of the construction of $\mathcal{M}$. We taken the nth level to be $\varnothing$ if the construction terminated in fewer than n steps. The nth level is denoted $\mathcal{L}_n$, and is identical to the $\mathcal{L}_n$ used above.

A path $\alpha \in \mathcal{L}_n$ joining boundary nodes $p' \in P$ and $q' \in Q$ through the interior divides a graph G into two regions. The region to the right of α, oriented from p', is considered *below* α, while the region to the left of α, oriented from p', is considered

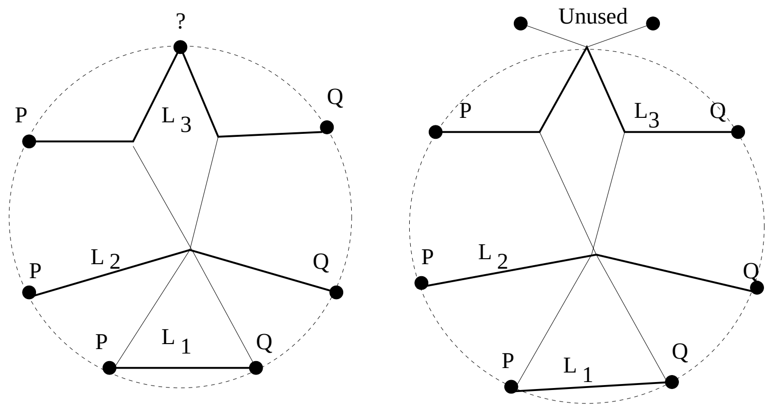

FIGURE 2. Example of adjoining a boundary spike

above α. If level $\mathcal{L}_i$ consists of many paths, the paths will divide G into many regions. The region that is below every path in $\mathcal{L}_i$ is considered below $\mathcal{L}_i$, while a region that is above any path in $\mathcal{L}_i$ is considered above $\mathcal{L}_i$.

Lemma 5.5. *Suppose $p_k \in P$ is the endpoint of a path $\alpha \in \mathcal{L}_n$ and p^* is clockwise consecutive to p_k. Then no boundary antenna is adjoined at p^* in the construction of $\mathcal{M}$.*

Proof. Suppose α_1 and $\alpha_2 \in \mathcal{L}_{n+1}$ have p^* as a common endpoint. Call their other respective endpoints q' and q''. Without loss of generality, suppose p, q', q'' are in clockwise circular order. Because q' and q'' are both above α, p_k, q', q'' are also in clockwise circular order, as are p^*, q', q''. Then either the region above α_1 is entirely above α_2, or α_1 and α_2 intersect in the interior. The latter possibility contradicts Theorem 5.2, and the former implies $\alpha_1 \in \mathcal{L}_j$ and $\alpha_2 \in \mathcal{L}_k$, with $j > k$. Then α_1 and α_2 could not have a common endpoint. So, p^* is not the endpoint of two paths in $\mathcal{M}$, and no boundary antenna is adjoined at p^*. $\square$

Suppose $q_k \in Q$ is the endpoint of a path $\alpha \in \mathcal{L}_n$, and q^* is counter-clockwise consecutive to q_k. By the same argument as the previous lemma, no boundary antenna is adjoined at q^* in the construction of $\mathcal{M}$.

Theorem 5.6. *Suppose α_1 and $\alpha_2 \in \mathcal{L}_{n+1}$ are distinct paths. Let their respective endpoints be p_1, q_1, p_2, q_2 with $p_1, p_2 \in P$ and $q_1, q_2 \in Q$. Also assume p_0, p_1, q_1, p_2, q_2 are in clockwise circular order. Then no path joins p_1 and q_2 without intersecting $\mathcal{L}_n$.*

Proof. Suppose there is a path α joining p_1 and q_2 that does not intersect $\mathcal{L}_n$. If α is entirely to the right of α_1, travelling from p_1, then $\alpha_1 \notin \mathcal{L}_{n+1}$, so α is not entirely to the right of α_1. However, $q_2 \in \alpha$ is to the right of α_1, so α and α_1 intersect in the interior. Construct a new path β as follows. Start at p_1, and follow whichever path between α and α_1 is farther to the right. At every intersection point, follow whichever path between α and α_1 is farther to the right after the intersection point. Eventually β will follow α to q_2. β does not intersect $\mathcal{L}_n$ and lies entirely on or to the right of α_1, so α_1 is not a minimal vertex path with respect to $\mathcal{L}_n$. The contradiction is reached, so no path joins p_1 and q_2 without intersecting $\mathcal{L}_n$. $\square$

Note that it is possible for a path to join p_2 and q_1

Corollary 5.7. *Suppose v_1 and v_2 are interior vertices in α_1 and α_2 respectively. Then v_1 and v_2 cannot be connected by a path through the interior that does not intersect $\mathcal{L}_n$.*

Proof. If u and v can be connected through the interior without using $\mathcal{L}_n$, then there would be a path joining p_1 and q_2 that does not use $\mathcal{L}_n$. □

6. Characterizing $\mathcal{M}$

Let a path $\alpha \in \mathcal{L}_n \in \mathcal{M}$ be given on a graph G with endpoints p' and q'. As defined in section 2, $\mathcal{Z}(\alpha)$ is the region formed by the union of all cells adjacent to α. We will consider $\mathcal{Z}^+(\alpha)$ to be the portion of $\mathcal{Z}(\alpha)$ that is above α. Note that if a vertex $u \in \mathcal{Z}^+(\alpha)$, then $u \in \alpha$ or $u \in \mathcal{R}(\alpha)$.

Consider $\partial(\mathcal{Z}^+(\alpha))$, the boundary of $\mathcal{Z}^+(\alpha)$. $\partial(\mathcal{Z}^+(\alpha))$ includes α, the portion of the boundary circle immediately clockwise to p and immediately counter-clockwise to q, and possibly more. We will consider $\partial'(\mathcal{Z}^+(\alpha)) = \partial(\mathcal{Z}^+(\alpha)) - \alpha$. We will use the abbreviation ∂' when α is clear from context.

We will consider a sequence U of vertices on ∂', labelled $(p' = u_0, u_1, \ldots, u_n = q')$, labeled clockwise from p' to q'. U is a path, though not through the interior. Some of the u_i's may be boundary vertices of G and some may be interior vertices. Let p_i and q_i be the ith pair of boundary vertices in U such that there is an interior vertex and no boundary vertex between p_i and q_i in U, and p_i is before q_i in U. Note that q_i and p_{i+1} might be the same vertex. Let $p_i = u_{x_i}$ and $q_i = u_{y_i}$, with $x_i < y_i$. By the construction of ∂', $u_{x_i}, u_{x_i+1}, \ldots, u_{y_i}$ form a path α_i between two boundary nodes through the interior of G.

Lemma 6.1. *Let v_i be an interior vertex of α_i as constructed above. Then $\alpha_i = \min(v_i, \alpha)$.*

Proof. By the construction of ∂', α_i does not intersect α. We need to verify that any cell to the right of α_i, oriented from p_i, is necessarily on the side of $\min(v_i, \alpha)$ containing α. The cells to the right of α_i can classifed as one of the following.

- Below α
- $\mathcal{Z}^+(\alpha)$
- To the left of some α_j, oriented from p_j, and $i \neq j$

The cells below α are clearly on the same side of $\min(v_i, \alpha)$ as α. Suppose a cell c in $\mathcal{Z}^+(\alpha)$ is not on the same side of $\min(v_i, \alpha)$ as α. Because c shares a vertex with α, $\min(v_i, \alpha)$ shares a vertex with α, a contradiction.

Suppose a cell c is to the left of some α_j, oriented from p_j, $i \neq j$, and c is on the same side of $\min(v_i, \alpha)$ as α. Then $\min(v_i, \alpha)$ intersects α_j. Then there is a path through the interior joining p_i and q_j. However, such a path contradicts q_i and $p_j \in \mathcal{R}(\alpha)$ by the planarity of G, so c is not on the same side of $\min(v_i, \alpha)$ as α. We conclude $\alpha_i = \min(v_i, \alpha)$. □

Following from Lemma 6.1, we have an alternate characterization of $\mathcal{L}_{n+1}$.

Theorem 6.2. *In the following equation, the α_i's are created as above, for every $\alpha \in \mathcal{L}_n$.*

$$\mathcal{L}_{n+1} = \bigcup_{\alpha \in \mathcal{L}_n} \bigcup_i \alpha_i$$

Proof. By the construction of $\mathcal{L}_{n+1}$, $\alpha_i = \min(v_i, \alpha) \in \mathcal{L}_{n+1}$ for all $\alpha \in \mathcal{L}_n$. Furthermore, the set of all α_i exhausts $\mathcal{R}(\alpha)$, so the above identity holds. □

Corollary 6.3. *The region between $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$ is*

$$\bigcup_{\alpha \in \mathcal{L}_n} \mathcal{Z}^+(\alpha)$$

Proof. It follows from Theorem 6.2. □

If there is an interior node above α, then there is an interior node on the boundary or in the interior of $\mathcal{Z}^+(\alpha)$. This follows because we restricted G not to include any boundary edges between non-consecutive boundary nodes.

7. Unique Permutation

Let $\mathcal{M}$ be constructed as before. The first of the three conditions we need to verify, to conclude $\Lambda(P;Q) \neq 0$, is that $(P;Q)$ can only be formed through one permutation.

Theorem 7.1. *Suppose the sets of boundary nodes P and Q are constructed as described in section 5. Then the only τ such that there exists a set of paths α joining P_i to $Q_{\tau(i)}$ for all i is the τ generated by the construction.*

Proof. We will consider pairings in a $(P;Q)$ connection. We will prove the theorem by induction on the levels. Suppose we start with boundary node q_0 and look for a member of P with which q_0 may be paired. It can be paired with p_0 by the construction of $\mathcal{M}$. Now we will traverse the graph clockwise in search of another boundary node with which q_0 may be paired. Define a counter C on the boundary as follows. $C(p_0) = 0$. As we traverse the graph clockwise, add 1 to C every time we encounter an element of P. Subtract 1 from C every time we encounter an element of Q. Suppose $p^* \in P$, and we can pair q_0 to p^* in $(P;Q)$. Consider the counter-clockwise arc $\widehat{p^*, q}$. Let P^* be the number of elements of P on $\widehat{p^*, q_0}$, and Q^* the number of elements of Q on $\widehat{p^*, q_0}$. $C(p^*) = P^* - Q^*$. If $C(p^*) > 0$, there are elements of P in $\widehat{p^*, q_0}$ that cannot be paired with elements of Q. This is true because there will be more elements of P than Q on $\widehat{p^*, q_0}$, and therefore one of the elements of P will have to cross the path between p^* and q_0, violating the Jordan Curve theorem.

Let v be a boundary node, and $\widehat{v, q_0}$ be the counter-clockwise arc from v to q_0. By construction, every element of Q on $\widehat{v, q_0}$ has a corresponding element of P because we placed the first endpoint of every path clockwise from p_0 in P. Then $C(v) \geq 0$. Suppose $p^* \in P$ and $p^* \neq p_0$. Then $C(p^*) > 0$ because p^* has no corresponding element of Q on $\widehat{p^*, q_0}$. Then a connection $(P;Q)$ cannot join q_0 and p^*. Then p_0 is the only boundary node with which q_0 may be paired.

Suppose we have verified that the endpoints of the paths of the first n levels must be paired in the manner constructed. Label the paths on $\mathcal{L}_{n+1}$: $\alpha_1, \alpha_2, \ldots, \alpha_m$. Label the endpoints of α_i, p_i and q_i so $p_i \in P$ and $q_i \in Q$. By Theorem 5.6 and Corollary 5.7, p_1 can only be paired in $(P;Q)$ with a vertex on $\widehat{(q_1, p_1)}$. Then, we verify p_1 and q_1 are paired by the same argument which showed p_0 and q_0 were paired. Similarly, p_i and q_i are paired for $1 \leq i \leq m$. Then, we see inductively that only one $\tau \in S_k$ will be nontrivial in (2). □

8. Subgraph $H(u, v)$

We need to verify that no interior nodes were skipped in the process of constructing $\mathcal{M}$ on a critical circular planar graph G. Suppose there is an omitted interior node b above $\mathcal{L}_n$ and below $\mathcal{L}_{n+1}$. Consider the regions bounded by $\mathcal{L}_n$, $\mathcal{L}_{n+1}$, and the boundary circle. There may be more than one such region if $\mathcal{L}_n$ consists of more than one path; in that case, consider only the region containing b, and call it R_b. Let $\partial_b^* = \partial(R_b) \cap (V_b \cup \mathcal{L}_{n+1})$.

Lemma 8.1. *An omitted interior node b has a path to at most one point on ∂_b^*.*

Proof. Suppose b has paths to two points on ∂_b^*. Then Figure 3 typifies the four possibilities for these two paths.

Case 1: In the leftmost diagram, the two points on ∂_b^* are in V_b. An additional path would have been constructed on $\mathcal{L}_{n+1}$.

Case 2: In the next diagram, the two points on ∂_b^* are on different paths of $\mathcal{L}_{n+1}$. The two paths on $\mathcal{L}_{n+1}$ would have been modified to make one path.

Case 3: In the third diagram, one point of ∂_b^* is on $\mathcal{L}_{n+1}$ and the other is on V_b. Then the shown path on $\mathcal{L}_{n+1}$ would have been constructed lower.

Case 4: In the fourth diagram, the two on ∂_b^* are on the same path in $\mathcal{L}_{n+1}$. Then the path shown on $\mathcal{L}_{n+1}$ would have been constructed lower.

Therefore, b can have a path to at most one point on ∂_b^*. □

If there is an omitted interior node, G is not critical. The proof of this will occupy the rest of this and the next section.

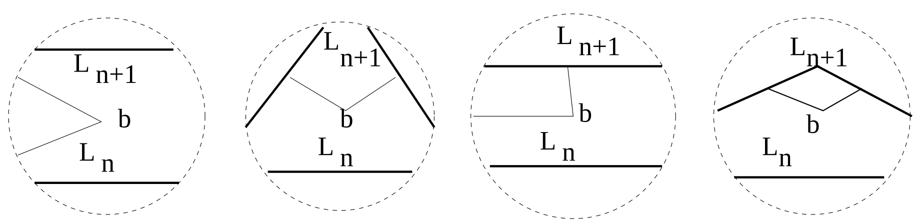

Figure 3. Why interior nodes can't have two paths to the top. New or modified path in each case

Lemma 8.2. *There is no interior node below $\mathcal{L}_1$.*

Proof. Suppose interior node b is below $\mathcal{L}_1$. If b has paths to two distinct points on $\mathcal{L}_1$, then $\mathcal{L}_1$ would have been constructed differently by Lemma 8.1. If $\mathcal{M}$ is built with respect to a boundary spike sr, with s as the boundary node, b cannot connect to s except through r. Therefore, b has no path to the boundary except through $\mathcal{L}_1$. If b has a path to only one point on $\mathcal{L}_1$, then b is an interior pendant and the network is not critical. We conclude that b does not exist. □

Suppose a path α, specified by a sequence of nodes $(v_0, v_1, \ldots, v_n)$, is a path through the interior between boundary nodes with $v_0 \in P$ and $v_n \in Q$. Node v_j is considered *strictly between* v_i and v_k if $i < j < k$ or *between* v_i and v_k if $i \le j \le k$. If $i < j$, v_i is considered *before* v_j in α, and v_j is considered *after* v_i in α. If v_j is strictly between v_i and v_k, and v_i and v_k are joined by an edge v_iv_k, v_iv_k is a *shortcut* in α. The following subgraph will be used to prove several facts.

Construction 8.3. Suppose $\mathcal{M}$ has been constructed in a critical graph G, starting with consecutive boundary nodes p and q or boundary spike sr. Suppose u and v are two vertices, either interior or boundary, on the same path $\alpha \in \mathcal{L}_n$. Then a subgraph $H(u,v)$ will be constructed as follows. First, include all the nodes in α that are between u and v. The nodes of $\mathcal{L}_{n-1}$ are included in two steps. First, add those nodes in $\mathcal{L}_{n-1}$ which connect to a node in $\mathcal{L}_n$ already included in $H(u,v)$. Then, add those nodes in $\mathcal{L}_{n-1}$ which are between those included in the first step. Continue adding levels in this manner until no more levels can be added.

We will define the boundary of $H(u,v)$ as follows: (V_b in G) $\cap H(u,v)$ together with all nodes in $H(u,v)$ that connect with a node in $G - H(u,v)$. All edges are included that join two nodes in $H(u,v)$. Note that $H(u,v)$ is a Simon-subgraph, as defined in [2]. See Figure 4.

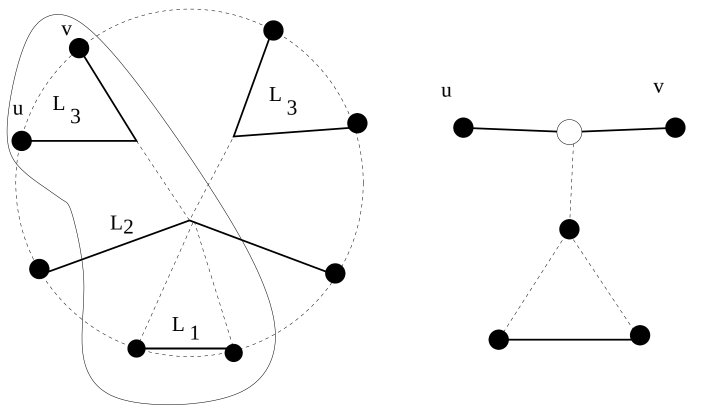

FIGURE 4. A sample subgraph H(u,v). Left Figure: Nodes of H(u,v) circled. Right figure: H(u,v) displayed with levels stacked vertically.

Lemma 8.4. *Suppose u and v are on $\mathcal{L}_n$ in $\mathcal{M}$, and no omitted interior nodes or shortcuts in $\mathcal{M}$ occur before $\mathcal{L}_n$. Then $H(u,v)$ has no more than two boundary nodes on every level in $\mathcal{M}$, except possibly $\mathcal{L}_n$.*

Proof. Suppose w_1, w_2, and w_3 are three nodes on $\mathcal{L}_i$ for some $i < n$ with w_1 before w_2, and w_2 before w_3. We will show w_2 is not a boundary node in $H(u,v)$. Because w_2 is not the endpoint of a path in $\mathcal{L}_i$, w_2 is not on the boundary of G. Because there are no omitted interior nodes in $\mathcal{M}$ before $\mathcal{L}_n$, w_2 has only neighbors that are in $\mathcal{L}_{i-1}$, $\mathcal{L}_i$, or $\mathcal{L}_{i+1}$. The neighbors of w_2 in $\mathcal{L}_i$ are between w_1 or w_3 because there are no shortcuts, so they are included in $H(u,v)$. The neighbors of w_2 in $\mathcal{L}_{i-1}$ are included in $H(u,v)$ by construction. Suppose a and b are the first and last nodes of $\mathcal{L}_{i+1} \cap H(u,v)$ that connect to nodes in $\mathcal{L}_i \cap H(u,v)$. Then they connect to the first and last nodes of $\mathcal{L}_i \cap H(u,v)$; call them c and d. Because w_2 is between c and d, a neighbor $e \in \mathcal{L}_{i+1}$ of w_2 is between a and b. By construction, $e \in H(u,v)$. All neighbors of w_2 are in $H(u,v)$. So, w_2 is an interior node in $H(u,v)$. $\square$

Lemma 8.5. *Suppose $u,\ v \in \alpha \in \mathcal{L}_n \in \mathcal{M}$, and no omitted interior nodes or shortcuts in $\mathcal{M}$ occur before $\mathcal{L}_n$. Then $H(u,v)$ is circular planar.*

Proof. All boundary nodes of $H(u,v)$ are on $\mathcal{L}_n$ or either the first or last of $\mathcal{L}_i \cap H(u,v)$. Therefore, they can be embedded in a disk with the interior of $H(u,v)$ in the interior of the disk. $\square$

By [2], $H(u,v)$ critical if G is critical because $H(u,v)$ is circular planar.

9. No Omitted Interior Nodes

The following three lemmas work together. Inductively, they imply that there are neither shortcuts nor omitted interior nodes in $\mathcal{M}$.

Lemma 9.1. *Suppose $\mathcal{M}$ has been constructed. Suppose on a path $\alpha \in \mathcal{L}_n$, there are two nodes u and v, possibly interior or boundary, that are not consecutive nodes in α. Also suppose there are no shortcuts or omitted interior nodes before $\mathcal{L}_n$. Then u and v are not adjacent.*

Proof. Suppose u is adjacent to v, and there is an edge uv. Without loss of generality, assume u is before v in α. The edge uv must be above $\mathcal{L}_n$, or $\mathcal{L}_n$ would have been constructed differently. Construct the subgraph $H(u,v)$. The existence of edge uv will prevent any node strictly between u and v on $\mathcal{L}_n$ from being adjacent to a node on $\mathcal{L}_{n+1}$ because $H(u,v)$ is planar. So u and v are the only boundary nodes of $H(u,v)$ on $\mathcal{L}_n$. We will show that $H(u,v)$ is not critical, contradicting that G is critical. To do so, we will show that no circular pair $(R;S) = (r_1,\ldots,r_k;s_1,\ldots,s_k)$ in $H(u,v)$ that can be connected through $H(u,v)$ needs to use the edge uv, so uv can be deleted without breaking the connection.

Let m be the first index, if there is such an index, such that $(\widehat{r_m,s_m})$ in $H(u,v)$ contains $(\widehat{v,u})$. Suppose r_i and s_i are in $\mathcal{L}_{a_i}$ and $\mathcal{L}_{b_i}$, not respectively. We will choose $b_i \geq a_i$ for every pair (r_i,s_i). If $f < g < m$, then $b_f < b_g$. This can be seen because $(\widehat{r_f,s_f}) \in (\widehat{r_g,s_g})$, and $(\widehat{v,u}) \notin (\widehat{r_g,s_g})$. Alternately, if $m \leq g < f$, then $b_f < b_g$.

Let β_i be the path joining r_i and s_i in the $(R;S)$ connection. Then β_1 only needs to use the levels c in G such that $c \leq b_1$. To see this, suppose β_1 uses a level above $\mathcal{L}_{b_1}$. Then β_1, constructed from r_1 to s_1, goes above $\mathcal{L}_{b_1}$ at a node x and returns to $\mathcal{L}_{b_1}$ at a node y. A new path β_1' could have been formed that is identical to β_1, except x is joined to y along $\mathcal{L}_{b_1}$ instead of going above $\mathcal{L}_{b_1}$.

Then, we see inductively that β_i only needs to use the levels c in G such that $c \leq b_i$. To see this, observe that if β_{i-1} is constructed before β_i and $i < m$, then the removal of nodes on β_{i-1} will not require that β_i use a node above $\mathcal{L}_{b_i}$. The same is true if β_{i+1} is constructed before β_i and $i \geq m$. Then, no path in $(R;S)$ needs to use a portion of $H(u,v)$ above $\mathcal{L}_n$, and in particular no path in $(R;S)$ needs to use uv, so we can delete uv without breaking any connection in $H(u,v)$. So, we have reached the desired contradiction. $\square$

Construction 9.2. Suppose b is an omitted interior node between $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$. Also suppose b is above α for some $\alpha \in \mathcal{L}_n$. We will construct a new subgraph $H^*(b)$ as follows. First, include b and every node that can be joined to b with a path through the interior that does not intersect $\mathcal{L}_n$ or $\mathcal{L}_{n+1}$. Second, if b has a path to a node a on ∂_b^*, include a in $H^*(b)$. There is no more than one such a by Lemma 8.1. Third, include every node on $\mathcal{L}_n$ that is adjacent to a node included in the first step. Let u and v be the first and last nodes in α that were included in

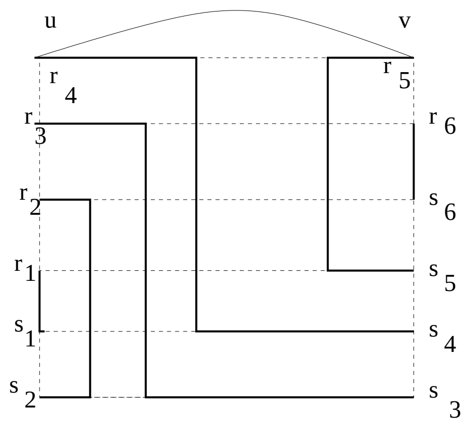

FIGURE 5. A circular pair in $H(u, v)$. The horizontal lines represent levels in G. In this example $m = 5$, and (r_i, s_i) are connected in this order: $i = 1, 2, 3, 4, 6, 5$.

the third step. Fourth, include $H(u, v)$. The boundary nodes and edges of $H^*(b)$ are chosen so that $H^*(b)$ is a Simon subgraph as in [2].

The boundary of $H^*(b)$ is the first and last vertex in $\mathcal{L}_i \cap H^*(b)$ for all $i \leq n$, and a if $a \in H^*(b)$. Therefore, $H^*(b)$ is circular planar.

Lemma 9.3. *Suppose $\mathcal{M}$ has been constructed. Also suppose there are no shortcuts or omitted interior nodes before $\mathcal{L}_n$. Then for every $\alpha \in \mathcal{L}_n$ and for every b above α and below $\mathcal{L}_{n+1}$, min(b, α) exists.*

Proof. Suppose there is an interior node b above some $\alpha \in \mathcal{L}_n$ and below $\mathcal{L}_{n+1}$, and $\min(b, \alpha)$ does not exist. Construct the subgraph $H^*(b)$ as above. We will reach a contradiction by showing $H^*(b)$ is not critical with two cases. In the first case, b has a path to $a \in \partial_b^*$ that does not intersect $\mathcal{L}_n$. In the second case, b does not have a path to ∂_b^* that does not intersect $\mathcal{L}_n$.

Case 1: Suppose a circular pair $(R; S)$ in $H^*(b)$ is given. Using the technique of the previous proof, $(R; S)$ can be constructed in such a way that only at most one path β is in part above $\mathcal{L}_n$, and β has a as an endpoint. If a is the end of a boundary spike, we can contract the boundary spike at a without breaking the connection $(R; S)$. If more than one edge in $H^*(b)$ ends at a, we can delete one of the one of the edges ending in a.

Case 2: The boundary in $H^*(b)$ is the same as the boundary in $H(u, v)$. Again using the argument of the previous theorem, no circular pair in $H^*(b)$ requires any edge above $\mathcal{L}_n$. So, $H^*(b)$ is not critical.

In either case, we see that $H^*(b)$ is not critical, and therefore G is not critical. We have reached the desired contradiction. □

Lemma 9.4. *Suppose $\mathcal{M}$ has been constructed. Also suppose there are no shortcuts or omitted interior nodes before $\mathcal{L}_n$. Then there is no omitted interior node between $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$.*

Proof. Suppose there is an omitted interior node b between $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$. Also suppose b is above some path $\alpha \in \mathcal{L}_n$. By Lemma 9.3, $\min(b, \alpha)$ exists. Then

$\mathrm{rem}(b, \alpha) \geq 2$; otherwise $\min(b, \alpha) \in \mathcal{L}_{n+1}$ by the Theorem 6.2. Because b is between $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$, b is in the boundary of a cell $c \in \mathcal{Z}^+(\alpha)$. That implies $b \in \mathcal{R}(\alpha)$ and $\mathrm{rem}(b, \alpha) = 1$, a contradiction. □

We conclude that every interior node is used in $\mathcal{M}$, and there are no shortcuts.

10. Uniqueness of Paths

Theorem 10.1. *Suppose $\mathcal{M}$ and $(P; Q)$ have been constructed on G as in section 5. Then $\mathcal{M}$ is the only set of paths that will connect $(P; Q)$*

Proof. Suppose we restricted the graph G to G' because of a boundary edge between two non-consecutive bounadary nodes p' and q' as outlined in section 5. G' consists of interior nodes I_1, and I_2 are the interior nodes in G but not in G'. We need to verify that no nodes in I_2 could be used to connect $(P; Q)$. No path in $\mathcal{M}$ can cross $p'q'$, so the only way a path α in $\mathcal{M}$ could use nodes of I_2 is that the endpoints of α are p' and q'. Because α was constructed as a minimal vertex path through a node in I_1, the edge $p'q'$ is a shortcut in α, which is not possible by Lemma 9.1. Therefore, no pair of boundary nodes in $\mathcal{M}$ is (p', q'), and $\mathcal{M}$ could not be reconstructed using nodes in I_2.

Let $\alpha \in \mathcal{L}_n$ with endpoints p_k and q_k be the first path that might be constructed differently. First, α cannot be reconstructed with any extra omitted interior nodes, and no interior node already in use by a path in an earlier level. Also, α has no shortcuts it may use to connect p_k to q_k using the same or fewer interior nodes. Then a modified α must use a higher interior node. Because there are no interior nodes not used in a path, the modified α uses a vertex on a path $\alpha' \in \mathcal{L}_{n+1}$. The necessarily modified α' uses a vertex on a path $\alpha'' \in \mathcal{L}_{n+2}$, and so on to the final level. Suppose there are m levels, so an α^* on $\mathcal{L}_m$ must be constructed differently. There are no shortcuts or extra interior nodes for α^* to use, and no further paths for α^* to borrow from, so α^* cannot be reconstructed. So, we conclude there is no alternate set of paths to join (P, Q) other than the set prescribed by the construction. □

If we build $\mathcal{M}_{pq}$ over a boundary edge pq, deleting pq must break the connection (P, Q) because pq was necessarily used as a path. If we build $\mathcal{M}_p$ over a boundary spike pr, contracting pr breaks the connection (P, Q) because $(P; Q)$ necessarily used what was previously interior node r. Therefore, we have satisfied the conditions to use the boundary edge and boundary spike formulas.

Theorem 10.2. *We can recover any boundary edge or boundary spike on a critical circular planar network.*

11. Contracting Boundary Spikes

We have no problem recovering a boundary edge, recovering a boundary spike, or deleting a boundary edge. Contracting a boundary spike is another matter. If we recover a boundary spike at boundary node p with conductivity ξ, then we adjoin a boundary spike with conductivity $-\xi$ to contract the spike. If $\lambda_{p,p} = \xi$, $\delta = 0$ in equation (11), and we cannot define a new response matrix. A simple example of a network with uncontractable boundary spikes is the top-hat network, for which every conductivity is 1 except that of the edge joining the two interior nodes, whose conductivity is -2. We can define Λ for this network and recover it, but we cannot contract either of the boundary spikes.

Lemma 11.1. *Suppose we cannot contract the boundary spike at node p with conductivity ξ. Also suppose $\lambda_{pq} \neq 0$ for some $p \neq q$. We may adjoin a boundary spike of fixed conductivity ω at q and then contract the boundary spike at p.*

Proof. Choose $\omega \neq -\lambda_{qq}$. Adjoining a boundary spike of conductivity ω at q generates a new response matrix Λ', and has the following effect on λ_{pp}, with $\delta = \lambda_{qq} + \omega$.

$$\lambda'_{pp} = \lambda_{pp} - \frac{\lambda_{pq}^2}{\delta} \tag{14}$$

Because $\lambda'_{pp} \neq \lambda_{pp}$ and ξ has not changed, $\delta' = \lambda'_{pp} - \xi \neq 0$, and it is possible to contract the boundary spike at p. □

If a boundary spike of conductivity ϵ already existed at node q, the process of adjoining a new boundary spike of conductivity ω changes ϵ without creating a new edge. If a boundary spike did not exist at q, then there is a danger that adjoining a new boundary spike will cause the graph to become non-critical. Instead we will adjoin an antenna at q.

Theorem 11.2. *Let a network Γ with response matrix Λ be given. For a given p, there exists a series of boundary antenna adjunctions such that row p of the resulting response matrix Λ^* is not entirely of* 0.

Proof. Suppose q is the boundary node counter-clockwise consecutive to p. Build $\mathcal{M}_{pq}$, and create Γ^* with response $\Lambda^* = \Lambda_{\Gamma^*}$ by adjoining all the boundary antennas necessary in building $\mathcal{M}_{pq}$. P and Q are the endpoints of the paths constructed in $\mathcal{M}_{pq}$. As shown before, $\Lambda(P, Q) \neq 0$. Because $p \in P$, some entry of row p in Λ^* is nonzero. □

Note that no antennas were adjoined to a boundary node clockwise consecutive to p by Lemma 5.5. Also, if $p \in P$, the nonzero entry Λ_{pq^*} guaranteed by the proof is such that $q^* \in Q$. Finally, if p^* is clockwise consecutive to p, then $p^* \notin Q$ by the labeling of P and Q.

12. Continuing Recovery

Suppose we have a critical circular planar network Γ. We can recover any of the boundary edges and boundary spikes of Γ. Suppose, after some steps, the modified Γ' contains no boundary edges, n unknown edges, and the known edges are exactly the boundary spikes. We will describe a process whereby we can recover an additional edge, and then the modified Γ'' have $n-1$ or fewer unknown edges, no boundary edges, and the known edges of Γ'' are exactly the boundary spikes. Furthermore, Γ'' will be critical. Therefore, the process is guaranteed to fully recovered Γ.

We are stuck in recovering Γ with graph G and conductivities γ if Γ has no boundary edges, and we can contract no boundary spike. Consider the graph G' obtained from G by contracting every boundary spike. G' has a boundary edge uv. At least one of u and v is an interior node in G. Then, uv can be made a boundary edge in G by contracting a boundary spike at whichever of u and v is an interior node. If u or v has an antenna, it is sufficient to contract one spike of each antenna to make uv a boundary edge.

Suppose p and q are boundary spikes in G, p is clockwise consecutive to q, and contracting the spikes at p and q will result in uv becoming a boundary edge. Suppose p^* is clockwise consecutive to p and q^* is counter-clockwise consecutive to q. Construct $\mathcal{M}_{p^*p}$ and retain the boundary antennas adjoined in the process. Doing so will not create a boundary antenna at q by Lemma 5.5, and the construction will establish some $p' \neq q$ such that $\Lambda_{pp'} \neq 0$ by Theorem 11.2. If a boundary spike exists at p', adjoin another boundary spike at p'. Otherwise, adjoin a boundary antenna at p'. Then contract the boundary spike at p. Simiarly, construct $\mathcal{M}_{qq^*}$. Doing so will not create a boundary antenna at p and will establish some $q' \neq p$ such that $\Lambda_{qq'} \neq 0$. If a boundary spike exists at q', adjoin another boundary spike at q'. Otherwise, adjoin a boundary antenna at q'. Then contract the boundary spike at q. A new boundary edge pq is created. Recover and delete pq as well as any other boundary edges that may have resulted from this process. Also recover all new boundary spikes resulting from this process.

It is possible that one of p and q is already a boundary vertex. Then it is only necessary to contract one boundary spike. An example of this is illustrated in figure 6.

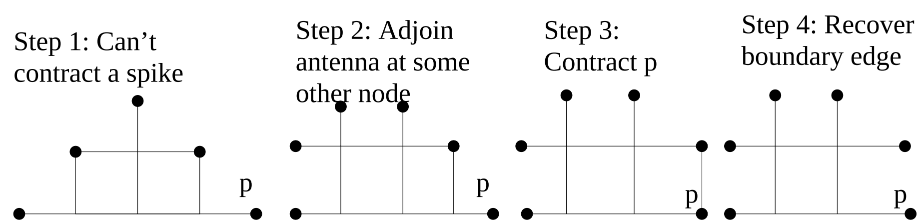

FIGURE 6. Example of branching paths

Lemma 12.1. *Suppose Γ is a critical resistor network with no boundary edges, n unknown edges, and the known edges of Γ are exactly the boundary spikes. Suppose Γ' is obtained from Γ by the above process. Then Γ' is a critical resistor network with no boundary edges, $n-1$ or fewer unknown edges, and the known edges of Γ' are exactly the boundary spikes.*

Proof. Γ' was obtained from Γ by adjoining boundary spikes to pre-existing known boundary spikes, adjoining boundary antennas, and deleting boundary edges. Adjoining a boundary spike to a known boundary spike of conductivity ξ at node p effectively changes the value of ξ without adding a new edge, so every step of this process maintains criticality. Because we recovered a previously unknown edge, the number of unknown edges decreases. By Lemma 5.3, no antennas in Γ' were adjoined to boundary spikes in Γ, so all boundary spikes in Γ remain boundary spikes in Γ'. Every boundary edge is recovered and removed, so Γ' has no boundary edges. Γ' has the desired properties. □

Theorem 12.2. *Suppose Γ is a critical circular planar resistor network with nonzero signed conductivities, and Γ has a defined response matrix Λ. Then Γ is recoverable.*

Proof. The above process reduces the number of unknown edges in Γ every time it is applied. Repeated applications will eventually recover every edge in Γ. □

13. Future Research

The examination of signed conductivities was motivated by a desire to reduce non-circular planar graphs to circular planar graphs. For example, suppose a non-circular planar graph G has the complete graph K_n as a subgraph. It might be possible to replace the K_n with a well-connected graph with n boundary nodes, such as the Towers of Hanoi Σ_n. A sequence of such transformations might make G circular planar. However, doing so might introduce edges with zero conductivity even if the K_n did not have an edge with zero conductivity. Even the $Y - \Delta$ transformation has problems. Suppose a Δ with conductivites $1, -2$, and -2 is replaced with a Y. Then the Y has 0 conductivity for all three edges. Even if the modified network can be recovered, it will be impossible to uniquely perform a $\Delta - Y$ transformation.

However, it seems that problems such as the one outlined above only occur on thin algebraic varieties of conductivities. Perhaps there is a simple way to characterize the space of troublesome conductivities for a $K_n - \Sigma_n$ transformation and work around those difficult values.

Also, perhaps the procedure of building a maximal connection can be adapted for planar graphs that are not circular planar. There is no known analog for the concept of criticality, but the process of constructing a maximal connection might produce such an analog.

References

[1] Curtis, Edward B. & James A. Morrow., *Inverse Problems for Electrical Networks*, Series of Applied Mathematics - Vol. 13. World Scientific, ©2000.

[2] Russell, J.T., *Recoverability of Subgraphs*

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