TWO TO ONE NETWORKS

TRACY LOVEJOY

ABSTRACT. The topic of this paper is to generalize our existing notions of two to one networks, that is, networks on which two sets of conductances K give rise to only one response matrix Λ. The outline of a theorem establishing a general class of 2-1 networks will be exposed. Also, and idea relating to adding sets of conductances on a graph, and coverings of loops in Λ space will be examined.

Contents

1. Introduction

I hope that by exposing a clear and general explanation of the algebra that brings about $2-1$ networks other people interested in the topic will be able to pick it up quickly. Much of this work was made possible by close analogy to that found in [1]. In further sections the hope is to increase the understanding of these special networks by examining the locus of degenerate points, probing questions about the topology of our set of response matrices, and outlining computational techniques that I have found to either help or hinder the process.

2. Creating a large set of two to one networks and how

Section One. This section hopes to clearly detail how it comes about that a specific conductor in the network can be determined, through a series of algebraic manipulations, as a quadratic function of the entries in the response matrix. The network shown is the most arbitrary yet known example of a $2-1$ network.

Date: July 14, 2003.

FIGURE 1. Most Arbitrary Yet Known $2-1$ Network

2.1. **The node numbering.** Understanding this figure requires some imagination. First, you are meant to imagine that you are looking at part of a shape with the inner ring a regular polygon of n-vertices. These n nodes are the only interior nodes of the network and are numbered a,b...,n-1,n. Off each side of this polygon are an unspecified number of "petals" (hence the dotted lines after the second line.) The center of these lines have black dots marking the boundary nodes. The numbering of these boundary nodes starts in the inner polygon and grows as you go out to further petals. For example: 1,2,3... or +,++,+++,.... etc.... Because of our node numbering a high degree of rotational symmetry is apparent and this will be used much to our benefit throughout these calculations.

2.2. **Getting two γ's from entries in the response matrix.** Firstly, we will write out three entries in the response matrix, and claim, by rotational symmetry that we needn't write out any more. (Note: σ_k is the sum of the conductors adjacent to node k.) To write these out we will use the fact that the voltage at and interior node is given by the average of the voltages at neighboring nodes weighted by the conductances. Since we are applying a voltage of one at one boundary node and zeros elsewhere the voltage at and interior node is given by simply the ratio of the conductor joining the interior node to the nonzero boundary node, and the sum of conductors around the interior node. Finding the current response becomes trivial at this point.

$$\lambda_{12} = -\frac{\gamma_{1a}\gamma_{2a}}{\sigma_a} - \frac{\gamma_{1n}\gamma_{2n}}{\sigma_n} \tag{1}$$

$$\lambda_{2II} = -\frac{\gamma_{2a}\gamma_{IIa}}{\sigma_a} \qquad \lambda_{1II} = -\frac{\gamma_{1a}\gamma_{IIa}}{\sigma_a} \tag{2}$$

The ratio of the two entries of Λ found in equation (2) will prove to be invaluable to our cause.

$$\frac{\gamma_{1a}}{\gamma_{2a}} = \frac{\lambda_{1II}}{\lambda_{2II}} \tag{3}$$

At this point, on the grounds of rotational symmetry we can write down another such ratio of λ's.

$$\frac{\gamma_{1n}}{\gamma_{2n}} = \frac{\lambda_{1ii}}{\lambda_{2ii}} \tag{4}$$

We are just about to obtain one of the typical equations of $2-1$ networks. We need only to substitute eqs (3) and (4) into equation (1). This gives us

$$\lambda_{12} = \frac{\gamma_{2a}}{\gamma_{IIa}}\lambda_{1II} + \frac{\gamma_{2n}}{\gamma_{iin}}\lambda_{1ii}. \tag{5}$$

By massaging this equation a little we arrive at one of only two equations we really need to get a quadratic for a γ in terms of entries of Λ. We'll call this equation equation I (remember, it did come from equation (1))

$$(I) \qquad \frac{\gamma_{IIa}}{\gamma_{2a}} = \frac{1}{\frac{\lambda_{12}}{\lambda_{1II}} - \frac{\gamma_{2n}}{\gamma_{iin}}\frac{\lambda_{1ii}}{\lambda_{1II}}}$$

We note that this equation has the ratio $\frac{\gamma_{2n}}{\gamma_{iin}}$ in it, but that we can easily write out this ratio because it looks just like Equation I rotated by one petal. This new equation will in turn have the ratio $\frac{\gamma_{=(n-1)}}{\gamma_{ii(n-1)}}$ in it. We can repeat the process descibed above until eventually we have a formula that has only the ratio $\frac{\gamma_{IIa}}{\gamma_{2a}}$ in it. For clarity, the last formula we substitute in will be

$$\frac{\gamma_{++b}}{\gamma_{IIb}} = \frac{1}{\frac{\lambda_{III}}{\lambda_{1++}} - \frac{\gamma_{IIa}}{\gamma_{2a}}\frac{\lambda_{2I}}{\gamma_{I++}}}. \tag{6}$$

The other equation, equation II, comes from one of the equations in (2) and we'll use the first for example. It is at this point, and this is the first and only point we were in becomes relevant, that we must explicitly write out σ_a.

$$\lambda_{2II}(\gamma_{1a} + \gamma_{2a} + \gamma_{3a} + \cdots + \gamma_{Ia} + \gamma_{IIa} + \gamma_{IIIa} + \dots) = -\gamma_{2a}\gamma_{IIa} \tag{7}$$

We get a form of what we call equation II by substituting in ratios like equations (3) and (4) which yields

$$\gamma_{2a}\lambda_{1II} + \gamma_{2a}\lambda_{2II} + \gamma_{2a}\lambda_{3II} + \cdots + \gamma_{IIa}\lambda_{2I} + \gamma_{IIa}\lambda_{2II} + \gamma_{IIa}\lambda_{2III} + \cdots = -\gamma_{2a}\gamma_{IIa}.$$

However, this equation won't look useful until we divide by γ_{IIa} and solve for γ_{2a} to get

$$(II) \quad -\gamma_{2a} = \tfrac{\gamma_{2a}}{\gamma_{IIa}}\lambda_{1II} + \tfrac{\gamma_{2a}}{\gamma_{IIa}}\lambda_{2II} + \tfrac{\gamma_{2a}}{\gamma_{IIa}}\lambda_{3II} + \cdots + \lambda_{2I} + \lambda_{2II} + \lambda_{2III} + \dots$$

and call it equation II!

Equations I and II together give a system of two equations in two unknowns (λ's known and γ_{2a} and γ_{IIa} unknown) that can be solved by the quadratic formula. This claim, I will have to leave without justification for now.

Supposing now that we could see a clear quadratic expression for both of our γ's we would be led to believe that this means there are four possible and equally satisfactory solutions. However, we in fact only have two solutions because one gamma determines all the rest exactly. Firstly, the ratios like equations (3) determine all the other γ's equavalent to the one we've solved for through inversions (eg. knowning γ_{1a} determines γ_{2a}, γ_{3a} and so on). Secondly, the entries like equation (2) tells us the other γ exactly showing us that there are only two sets of the conductances.

While this algebraic argument should be sufficient to quickly work any specific example that fits the picture above, it is not a proof. Even though we should have a quadratic equation,it is quite difficult to explicitly write it down. Even then it is still possible that the lead term could be zero, or that the equation factors leaving only one solution. I challenge the reader (myself included) to come up with a proof that does not rely on the algebraic argument given here but instead upon some kind of geometric or toplogical (hopefully not pathological) intuition.

3. The Locus of Degenerate Points

4. Summing Conductances and the Topology of Response Matric Space

5. MATLAB. A Source of Confusion

Why would a numerical solver find only one solution of and equation which clearly has two?

References

[1] Ernie. '2000 or so.

[2] Burtis, B., and James A. Morrow. "Inverse Problems for Electrical Networks." Series on applied mathematics – Vol. 13. World Scientific, ©2000.

University of Washington, Seattle WA

E-mail address: oedipus@u.washington.edu